Filomat 32:16 (2018), 5645–5663
https://doi.org/10.2298/FIL1816645N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

c_0 Can be Renormed to Have the Fixed Point Property for Affine Nonexpansive Mappings

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Abstract. P.K. Lin gave the first example of a non-reflexive Banach space $(X, \|\cdot\|)$ with the fixed point property (FPP) for nonexpansive mappings and showed this fact for $(\ell^1, \|\cdot\|_1)$ with the equivalent norm $\|\cdot\|$ given by

$$\|x\| = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|, \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^1 .$$

We wonder $(c_0, \|\cdot\|_\infty)$ analogue of P.K. Lin's work and we give positive answer if functions are affine nonexpansive. In our work, for $x = (\xi_k)_k \in c_0$, we define

$$|||x||| := \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{\left|\xi_j\right|^p}{2^j} \right)^{\frac{1}{p}} \quad \text{where } \gamma_k \uparrow_k 3,\ \gamma_k \text{ is strictly increasing with } \gamma_k > 2,\ \forall k \in \mathbb{N},$$

then we prove that $(c_0, |||\cdot|||)$ has the fixed point property for affine $|||\cdot|||$-nonexpansive self-mappings.

Next, we generalize this result and show that if $\rho(\cdot)$ is an equivalent norm to the usual norm on c_0 such that

$$\limsup_n \rho\left(\frac{1}{n}\sum_{m=1}^{n} x_m + x\right) = \limsup_n \rho\left(\frac{1}{n}\sum_{m=1}^{n} x_m\right) + \rho(x)$$

for every weakly null sequence $(x_n)_n$ and for all $x \in c_0$, then for every $\lambda > 0$, c_0 with the norm $\|\cdot\|_\rho = \rho(\cdot) + \lambda|||\cdot|||$ has the FPP for affine $\|\cdot\|_\rho$-nonexpansive self-mappings.

1. Introduction

Normed spaces $(X, \|\cdot\|)$ with the property: [♠] [For every closed, bounded, convex (non-empty) subset C of $(X, \|\cdot\|)$, for all nonexpansive mappings $T: C \to C$ [that is, $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$], T has a fixed point in C] became known as spaces with "the fixed point property for nonexpansive mappings" (FPP (n.e.)). Moreover, $(X, \|\cdot\|)$ is said to have "the weak fixed point property for nonexpansive mappings"

2010 *Mathematics Subject Classification*. Primary 46B45; Secondary 47H09, 46B10
Keywords. Nonexpansive mapping, affine mapping, fixed point property, renorming, asymptotically isometric c_0-summing basic sequence, closed bounded convex set
Received: 13 June 2017; Revised: 24 January 2018; Accepted: 27 January 2018
Communicated by Ljubiša D.R. Kočinac
The first author was supported by TUBITAK (The Scientific and Technological Research Council of Turkey) 2232 Grant #114C132 (2014-2017) and Kafkas University BAP (The Scientific Research Projects Coordination Unit) (Project #2017-FM-24 2017-2018)
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(w-FPP (n.e.)) if for every weakly compact, convex (non-empty) subset C of $(X, \|\cdot\|)$, for all nonexpansive mappings $T\colon C \to C$, T has a fixed point in C.

The sequence spaces $(c_0, \|\cdot\|_\infty)$ and $(l^1, \|\cdot\|_1)$ are both nonreflexive and possess the w-FPP(n.e.), but do not have the FPP(n.e.). Researchers do not know whether or not every reflexive Banach space $(X, \|\cdot\|)$ has the fixed point property for nonexpansive maps but this and related questions have been and still are central themes in metric fixed point theory.

For example, Dowling and Lennard [4] showed that every nonreflexive subspace of $L^1[0,1]$ fails the fixed point property and Domínguez Benavides [3] proved that given a reflexive Banach space $(X, \|\cdot\|)$, there exists an equivalent norm $\|\cdot\|_\sim$ on X such that $(X, \|\cdot\|_\sim)$ has the fixed point property for nonexpansive mappings. In 2014, motivated by work in Nezir [17], Lennard and Nezir [14] used the above-described theorem of Domínguez Benavides and the Strong James' Distortion Theorems [5, Theorem 8], to prove the following theorem: [If a Banach space is a Banach lattice, or has an unconditional basis, or is a symmetrically normed ideal of operators on an infinite-dimensional Hilbert space, then it is reflexive if and only if it has an equivalent norm that has the fixed point property for *cascading nonexpansive mappings*]. Moreover, in [15], Lennard and Nezir proved that l^1 cannot be equivalently renormed to have the FPP for semi-strongly asymptotically nonexpansive maps. Moreover, by a similar proof to that of Theorem 10 of Dowling, Lennard and Turett [5], they showed that c_0 cannot be equivalently renormed to have the FPP for strongly asymptotically nonexpansive maps. From this, we conclude that if $(X, \|\cdot\|)$ is a non-reflexive Banach lattice, then $(X, \|\cdot\|)$ fails the fixed point property for $\|\cdot\|$-semi-strongly asymptotically nonexpansive mappings. They strengthened this result to: [If a Banach space is a Banach lattice then it is reflexive if and only if it has the fixed point property for *affine semi-strongly asymptotically nonexpansive mappings*].

In 2008, P.K. Lin [16] showed that ℓ^1 can be renormed to have FPP with the equivalent norm $\|\cdot\|$ given by

$$\|x\| = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|, \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^1 .$$

The analogue of P. K. Lin's work for $(c_0, \|\cdot\|_\infty)$ is still an open question; that is, it is still unknown whether or not there exists any renorming of c_0 such that it can have the FPP (n.e) respect to that equivalent norm.

Moreover, in the Ph.D. Thesis of Carlos Hernández written under supervision of Maria Japón Pineda [9, Theorem 4.2.1] and in their recent paper [10, Theorem 3.2], they invented an equivalent norm on L^1 that has the FPP for all closed bounded sets and all affine nonexpansive mappings. This partially extends Lin's ℓ^1 theorem to L^1.

In the third section of our paper, we show that Banach space c_0 is also in the same category as L^1; that is, c_0 can be renormed to have the fixed point property for affine nonexpansive mappings. Next, we generalize it in our last section by showing that if $\rho(\cdot)$ is an equivalent norm to the usual norm on c_0 such that

$$\limsup_n \rho\left(\frac{1}{n}\sum_{m=1}^{n} x_m + x\right) = \limsup_n \rho\left(\frac{1}{n}\sum_{m=1}^{n} x_m\right) + \rho(x)$$

for every weakly null sequence $(x_n)_n$ and for all $x \in c_0$, then for every $\lambda > 0$, c_0 with the norm $\|\cdot\|_\rho = \rho(\cdot) + \lambda |\!|\!|\cdot|\!|\!|$ has the FPP for affine $\|\cdot\|_\rho$-nonexpansive self-mappings.

2. Preliminaries

Definition 2.1. Let E be a non-empty closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : E \longrightarrow E$ be a mapping.

1. We say T is *affine* if for all $\lambda \in [0,1]$, for all $x, y \in E$, $T\big((1-\lambda)\,x + \lambda\, y\big) = (1-\lambda)\,T(x) + \lambda\,T(y)$.
2. We say T is *nonexpansive* if $\|T(x) - T(y)\| \le \|x - y\|$, *for all* $x, y \in E$.
   Also, we say that E has the *fixed point property for nonexpansive mappings* [FPP(n.e.)] if for all nonexpansive mappings $T : E \longrightarrow E$, there exists $z \in E$ with $T(z) = z$.

Let $(X, \|\cdot\|)$ be a Banach space and $E \subseteq X$. We will denote the closed, convex hull of E by $\overline{\text{co}}(E)$. As usual, $(c_0, \|\cdot\|_\infty)$ is given by $c_0 := \left\{x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0\right\}$ with $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$ for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Also, $\ell^1 := \left\{x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty\right\}$.

Lemma 2.2. ([19]) *c_0 has weak Banach-Saks property ; that is, if $(x_n)_n$ is a sequence in c_0 converging to x in weak topology, then there exists a subsequence $(x_{n_k})_k$ such that its Cesaro average converges in norm to x.*

Lemma 2.3. ([7]) *If $\{x_n\}$ is a sequence in l^1 converging to x in weak-star topology, then for any $y \in l^1$*

$$r(y) = r(x) + \left\|y - x\right\|_1 \text{ where } r(y) = \limsup_n \left\|x_n - y\right\|_1 .$$

3. c_0 Can be Renormed to Have the Fixed Point Property for Affine Nonexpansive Mappings

In this section, proofs of our theorems and lemmas are inspired by the proofs of theorems and lemmas given by Helga Fetter and Berta Gamboa de Buen [6] such that they extend P.K. Lin's work [16]. We implement our ideas and get our desired result with the help of their work.

Lemma 3.1. ([6]) *Let $(X, \|.\|)$ be a Banach space and let $C \subset X$ be a closed convex nonempty (ccne) subset. Assume $T : C \to C$ a fixed point free nonexpansive mapping. Then, there exists a ccne T-invariant set D and $a > 0$ such that for every ccne T-invariant set $D' \subset D$, $diamD' > a$.*

Lemma 3.2. *Let $(X, \|.\|)$ be a Banach space and let $C \subset X$ be a ccne subset. Assume $T : C \to C$ an affine nonexpansive mapping and $(x_n)_n \subset C$ be an approximate fixed point sequence (afps). Consider a function $\Phi : C \to [0, \infty)$ given by*

$$\Phi(y) = \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - y \right\| , \ \forall y \in C .$$

If $d > \inf_{x\in C} \Phi(x)$ and $D = \{x \in C : \ \Phi(x) \le d\}$, then D is a ccne T-invariant set with $diamD \le 2d$.

Lemma 3.3. *Let $(X, \|.\|)$ be a Banach space and let $C \subset X$ be a ccne subset. Assume $T : C \to C$ an affine nonexpansive fixed point free mapping. Let $D \subset C$ and $a > 0$ be as in lemma 3.1. Then*

$$\inf\left\{\limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - y \right\| : (x_n)_n \subset D \ , \ (x_n)_n \textit{is an afps} \ , \ y \in D \right\} \ge \frac{a}{2}.$$

Proof. Assume by contradiction that

$$\inf\left\{\limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - y \right\| : (x_n)_n \subset D \ , \ (x_n)_n \text{is an afps} \ , \ y \in D \right\} < \frac{a}{2}.$$

Then there exists an afps $(x_n)_n \subset D$ such that

$$D' = \left\{ y \in D : \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - y \right\| \le \frac{a}{2} \right\} \ne \emptyset.$$

By Lemma 3.2, D' is ccne, T-invariant and $diamD' \le a$ which is a contradiction. □

Lemma 3.4. *Let $(X, \|.\|)$ be a Banach space and let $C \subset X$ be a ccne subset. Assume $T : C \to C$ an affine nonexpansive fixed point free mapping. Let $D \subset C$ and $a > 0$ be as in Lemma 3.1 so that D is a T-invariant ccne set such that if $D' \subset D$ is a T-invariant ccne set then $diamD' > a$. Then,*

$$\inf_{z\in X}\left\{\limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - z \right\| : (x_n)_n \subset D \ , \ (x_n)_n \textit{is an afps} \right\} \ge \frac{a}{4}.$$

Proof. Using Lemma 3.3, for every $s \in \mathbb{N}$, for every afps $(x_k)_k \subset D$ and for any $z \in X$,

$$\frac{a}{2} \leq \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - \frac{1}{s} \sum_{r=1}^{s} x_r \right\| \leq \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - z \right\| + \left\| z - \frac{1}{s} \sum_{r=1}^{s} x_r \right\|.$$

Thus,

$$\begin{aligned} \frac{a}{2} \leq \limsup_s \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - \frac{1}{s} \sum_{r=1}^{s} x_r \right\| &\leq \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - z \right\| + \limsup_s \left\| z - \frac{1}{s} \sum_{r=1}^{s} x_r \right\| \\ &= 2\limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - z \right\|. \end{aligned}$$

□

Corollary 3.5. *Let $(X, \|\,.\,\|)$ be a Banach space and let $C \subset X$ be a ccne subset. Assume that $T : C \to C$ is an affine nonexpansive fixed point free mapping and that for every T-invariant ccne set $D \subset C$, let $a > 0$ be as in Lemma 3.1 and $diam D > a$. Then,*

$$\inf \left\{ \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - x \right\| : (x_n)_n \subset C \ , \ (x_n)_n \text{is an afps}, \ x_n \underset{w^*}{\longrightarrow} x \right\} \geq \frac{a}{4}.$$

Now we give another lemma but from now on, unless it is stated as a different norm on c_0, the norm $|||\,.\,|||$ on c_0 will denote the equivalent norm as follows: For $x = (\xi_k)_k \in c_0$, define

$$|||x||| := \lim_{p \to \infty} \sup_{k \in \mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{|\xi_j|^p}{2^j} \right)^{\frac{1}{p}} \text{ where } \gamma_k \uparrow_k 3, \ \gamma_k \text{ is strictly increasing with } \gamma_k > 2, \ \forall k \in \mathbb{N}.$$

We can understand easily that the above expression is indeed an equivalent norm on c_0 due to the following facts:

Let $x = (\xi_i)_{i \in \mathbb{N}} \in c_0$. We will consider $x \neq (0, 0, \cdots)$ otherwise proof of the claim is clear.

Then,

$$\exists N \in \mathbb{N} \ni \|x\|_\infty = \sup_{k \in \mathbb{N}} |\xi_k| = \max_{k \in \mathbb{N}} |\xi_k| = |\xi_N|.$$

Due to power mean inequalities formula (see eg. [8]) (using the one given with weighted power means),

$$\begin{aligned} \|x\|_\infty &= \max_{k \leq N} |\xi_k| \\ &= \max \{|\xi_1|, |\xi_2|, \cdots, |\xi_N|\} \\ &= \lim_{p \to \infty} \left(\frac{|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_N|^p}{2^N} \right)^{\frac{1}{p}} \\ &= \lim_{p \to \infty} \left(\sum_{k=1}^{N} \frac{|\xi_k|^p}{2^N} \right)^{\frac{1}{p}}. \end{aligned}$$

Thus,

$$\|x\|_\infty = \lim_{p \to \infty} \left(\sum_{k=1}^{\infty} \frac{|\xi_k|^p}{2^k} \right)^{\frac{1}{p}}$$

Then, it is easy to see that our norm is equivalent to the usual norm on c_0.

Lemma 3.6. *Assume that $(x_n)_n \subset (c_0, \|\cdot\|_\infty)$ and $x_n \underset{w}{\to} 0$. Then, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a sequence $(u_k)_k$ with $u_k = \sum_{i=m_k+1}^{m_{k+1}} a_i e_i$ where $(e_i)_i$ is the canonical basis and $(a_i)_i \subset \mathbb{R}$ such that*

$$\lim_{j\to\infty} \left\| \frac{1}{j} \sum_{k=1}^{j} x_{n_k} - u_j \right\|_\infty = \lim_{j\to\infty} \left|\left|\left| \frac{1}{j} \sum_{k=1}^{j} x_{n_k} - u_j \right|\right|\right| = 0.$$

The proof of the above is straightforward by the proof of the Bessaga-Pełczyński Selection Principle [2, p.46] [1] and by passing to a subsequence of $(x_n)_n$ that is equivalent to the block basic sequence $(u_k)_k$ and is essentially disjointly supported.

One can easily obtain the following partial analogue result to [7, Lemma 1] by the following lemma.

Lemma 3.7. *Let $(x_n)_n$ be a bounded sequence in a Banach space $(X, \|\cdot\|)$. Consider a function $s : X \to [0, \infty)$ given by*

$$s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_k - y \right\| \ , \ \forall y \in X .$$

Then, if X has weak Banach-Saks property and $x \in X$ is the weak limit of the sequence $(x_n)_n$, then there exists a subsequence $(x_{n_k})_k$ whose norm limit is x such that if s is redefined via this subsequence, we have $s(x) = 0$ and $s(y) = \|y - x\|$, $\forall y \in X$ and for any equivalent norm $\|\cdot\|$ on X.

Thus, since c_0 has weak Banach-Saks property [19], the above can be applied.

Remark 3.8. We need to note that exact c_0-analogue of [7, Lemma 1] cannot be done. Indeed, $\exists x \in c_0$ and a sequence $(x_n)_n$ in c_0 such that

$$\limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x \right\|_\infty \neq \limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\|_\infty + \|x\|_\infty.$$

Now consider the following $2^{n-1} \times 2^{n-1}$ matrices E_n, $n \in \mathbb{N}$.

$$E_1 = \begin{bmatrix} 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \cdots$$

i.e., E_n will be the $2^{n-1} \times 2^{n-1}$-matrix whose ij-th entry is 1 if $i \le j$ and 0 otherwise. Using these as "diagonals" and zeroing out all other entries of the matrix, define an infinite matrix

$$E = \begin{bmatrix} E_1 & 0's & 0's & 0's & \cdots \\ 0's & E_2 & 0's & 0's & \cdots \\ 0's & 0's & E_3 & 0's & \cdots \\ 0's & 0's & 0's & E_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

If x_n denotes the n-th row of the infinite matrix E, then (x_n) is a sequence in c_0 that converges weakly to 0.

Note that $\left\| \frac{1}{2^n-1} \sum_{k=1}^{2^n-1} x_k \right\|_\infty = \frac{2^{n-1}}{2^n-1} > \frac{1}{2}$. Then, if $x = e_1$, it is easy to check that

$$1 = \limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x \right\|_\infty \neq \limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\|_\infty + \|x\|_\infty = \frac{1}{2} + 1.$$

But when we work on c_0, one can consider the fact that c_0 has so called m_p property where $p = \infty$ by N. Kalton and D. Werner in [12] and then we could say the followings:

$$\limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x \right\|_\infty = \max\left\{ \limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\|_\infty , \|x\|_\infty \right\} \tag{1}$$

$$= \lim_{p\to\infty} \left(\limsup_n \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\|_\infty^p + \|x\|_\infty{}^p \right)^{\frac{1}{p}} .$$

Lemma 3.9. *Assume that $(x_n)_n \subset (c_0, \|\cdot\|_\infty)$, $x_n \underset{w}{\to} x$ and that*

$$\lim_{j\to\infty} \left|\left|\left| \frac{1}{j} \sum_{n=1}^{j} x_n - x \right|\right|\right| = a \text{ exists. Then, } \lim_{j\to\infty} \left\| \frac{1}{j} \sum_{n=1}^{j} x_n - x \right\|_\infty = a.$$

Proof. Scaling γ_n, we may assume that $\gamma_k \uparrow_k 1$, γ_k is strictly increasing and we can redefine the equivalent norm according to this change. We may also suppose without loss of generality that $x_n \underset{w}{\to} 0$. Let $(x_{n_k})_k$ be a subsequence of $(x_n)_n$ such that $\lim_{j\to\infty} \left\| \frac{1}{j} \sum_{k=1}^{j} x_{n_k} \right\|_\infty$ exists. By Lemma 3.6, we may assume that there is a sequence $(u_k)_k$ with $u_k = \sum_{i=m_k+1}^{m_{k+1}} a_i e_i$ where $(e_i)_i$ is the canonical basis and $(a_i)_i \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} \left\| \frac{1}{j} \sum_{k=1}^{j} x_{n_k} - u_j \right\|_\infty = \lim_{j\to\infty} \left|\left|\left| \frac{1}{j} \sum_{k=1}^{j} x_{n_k} - u_j \right|\right|\right| = 0.$$

Define $y_k = x_{n_k}$ for every $k \in \mathbb{N}$. Then,

$$\lim_{j\to\infty} \left|\left|\left| \frac{1}{j} \sum_{k=1}^{j} y_k \right|\right|\right| = \lim_{j\to\infty} |||u_j||| = a \text{ and } \lim_{j\to\infty} \left\| \frac{1}{j} \sum_{k=1}^{j} y_k \right\|_\infty = \lim_{j\to\infty} \|u_j\|_\infty.$$

By the definition of the equivalent norm $|||\cdot|||$, there exists $l \in \mathbb{N}$ such that

$$|||u_j||| = \gamma_l \lim_{p\to\infty} \left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}} \text{ where } a_i = 0 \text{ in case } i \le m_j.$$

Since $\gamma_n < \gamma_{n+1}$, if $n \le m_j$, we have that

$$\begin{aligned} \gamma_n \lim_{p\to\infty} \left(\sum_{i=n}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}} &= \gamma_n \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}} \\ &\le \gamma_{n+1} \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}} \\ &= \gamma_{n+1} \lim_{p\to\infty} \left(\sum_{i=n+1}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}} \end{aligned}$$

and $m_j + 1 \le l \le m_{j+1}$. Also,

$$\gamma_l \lim_{p\to\infty} \left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}} \ge \gamma_{m_j+1} \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{m_{j+1}} \frac{|a_i|^p}{2^i} \right)^{\frac{1}{p}}$$

and so

$$\frac{\gamma_l - \gamma_{m_j+1}}{\gamma_{m_{j+1}}} \lim_{p\to\infty}\left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} \geq \lim_{p\to\infty}\left(\sum_{i=m_j+1}^{l-1} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}}.$$

Hence, we obtain that

$$\begin{aligned}
\left|\|u_j\|_\infty - |||u_j|||\right| &= \sup_{m_j+1\leq i\leq m_{j+1}} |a_i| - \gamma_l \lim_{p\to\infty}\left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} \\
&= \lim_{p\to\infty}\left(\sum_{i=m_j+1}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} - \gamma_l \lim_{p\to\infty}\left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} \\
&\leq (1-\gamma_l) \lim_{p\to\infty}\left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} + \lim_{p\to\infty}\left(\sum_{i=m_j+1}^{l-1} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} \\
&\leq \left(1 + \frac{\gamma_l - \gamma_{m_j+1}}{\gamma_{m_{j+1}}} - \gamma_l\right) \lim_{p\to\infty}\left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}} \\
&= \left(\frac{\gamma_l}{\gamma_{m_{j+1}}} - \gamma_l\right) \lim_{p\to\infty}\left(\sum_{i=l}^{m_{j+1}} \frac{|a_i|^p}{2^i}\right)^{\frac{1}{p}}.
\end{aligned}$$

Then, taking the limit as $j \to \infty$, we get that $\lim_{j\to\infty} \|u_j\|_\infty - |||u_j||| = 0$ and so

$$\lim_{j\to\infty}\left|\left|\left|\frac{1}{j}\sum_{k=1}^{j} y_k\right|\right|\right| = \lim_{j\to\infty}|||u_j||| = \lim_{j\to\infty}\|u_j\|_\infty = \lim_{j\to\infty}\left\|\frac{1}{j}\sum_{k=1}^{j} y_k\right\|_\infty = a.$$

Since every subsequence of $\frac{1}{j}\sum_{k=1}^{j} x_k$ has in turn a subsequence such that $\lim_{j\to\infty}\left\|\frac{1}{j}\sum_{k=1}^{j} x_{n_k}\right\|_\infty = a$, we get the desired result. Also, the reciprocal can be proved in the same way. □

Lemma 3.10. *Assume that $(x_n)_n \subset (c_0, \|\cdot\|_\infty)$, $x_n \underset{w}{\to} x$ and that*

$$\lim_{s\to\infty}\lim_{j\to\infty}\left|\left|\left|\frac{1}{j}\sum_{n=1}^{j} x_n - \frac{1}{s}\sum_{n=1}^{s} x_n\right|\right|\right| = a \text{ and } \lim_{j\to\infty}\left|\left|\left|\frac{1}{j}\sum_{n=1}^{j} x_n\right|\right|\right|$$

exist. Then, there exists a subsequence $(y_n)_n$ of $(x_n)_n$ such that

$$\lim_{s\to\infty}\lim_{j\to\infty}\left|\left|\left|\frac{1}{j}\sum_{n=1}^{j} y_n - \frac{1}{s}\sum_{n=1}^{s} y_n\right|\right|\right| = a = 2\lim_{j\to\infty}\left|\left|\left|\frac{1}{j}\sum_{n=1}^{j} x_n\right|\right|\right|.$$

Proof. Similarly to the proof of Lemma 3.9, scaling γ_n, we may assume that $\gamma_k \uparrow_k 1$, γ_k is strictly increasing and we can redefine the equivalent norm according to this change and we can let $(x_{n_k})_k$ be a subsequence of $(x_n)_n$ such that $\lim_{j\to\infty}\left\|\frac{1}{j}\sum_{k=1}^{j} x_{n_k}\right\|_\infty$ exists. Furthermore, by Lemma 3.6, we may assume that there is a sequence

$(u_k)_k$ with $u_k = \sum_{i=m_k+1}^{m_{k+1}} a_i e_i$ where $(e_i)_i$ is the canonical basis and $(a_i)_i \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} \left\| \frac{1}{j} \sum_{k=1}^{j} x_{n_k} - u_j \right\|_\infty = \lim_{j\to\infty} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{j} \sum_{k=1}^{j} x_{n_k} - u_j \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert = 0.$$

We may also assume, by passing a subsequence that

$$\lim_{s\to\infty} \lim_{j\to\infty} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{j} \sum_{n=1}^{j} x_n - \frac{1}{s} \sum_{n=1}^{s} x_n \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert \text{ and } \lim_{s\to\infty} \lim_{j\to\infty} |||u_j - u_s|||$$

exist. Define $y_k = x_{n_k}$ for every $k \in \mathbb{N}$. Then, for every $j, s \in \mathbb{N}$,

$$\begin{aligned} |||u_j - u_s||| - \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{j} \sum_{n=1}^{j} y_n - u_j \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert - \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{s} \sum_{n=1}^{s} y_n - u_s \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert &\leq \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{j} \sum_{n=1}^{j} y_n - \frac{1}{s} \sum_{n=1}^{s} y_n \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert \\ &\leq |||u_j - u_s||| + \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{j} \sum_{n=1}^{j} y_n - u_j \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert + \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \frac{1}{s} \sum_{n=1}^{s} y_n - u_s \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert. \end{aligned}$$

and

$$\begin{aligned} \|u_j - u_s\|_\infty - \left\| \frac{1}{j} \sum_{n=1}^{j} y_n - u_j \right\|_\infty - \left\| \frac{1}{s} \sum_{n=1}^{s} y_n - u_s \right\|_\infty &\leq \left\| \frac{1}{j} \sum_{n=1}^{j} y_n - \frac{1}{s} \sum_{n=1}^{s} y_n \right\|_\infty \\ &\leq \|u_j - u_s\|_\infty + \left\| \frac{1}{j} \sum_{n=1}^{j} y_n - u_j \right\|_\infty + \left\| \frac{1}{s} \sum_{n=1}^{s} y_n - u_s \right\|_\infty. \end{aligned}$$

Hence, $\lim_{s\to\infty} \lim_{j\to\infty} |||u_j - u_s||| = a$. Assume that $s > j$. By the definition of the equivalent norm $|||\,.\,|||$, there exists $l \in \mathbb{N}$ with $m_j + 1 \leq l \leq m_{s+1}$ such that

$$|||u_j - u_s||| = \gamma_l \lim_{p\to\infty} \left(\sum_{i=l}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}} \text{ where } a_i' = \begin{cases} 0 & \text{if } i \leq m_j \text{ or } m_{j+1} < i < m_s \\ a_i & \text{if } m_j + 1 \leq i \leq m_{j+1} \\ & \quad \text{or } m_s + 1 \leq i \leq m_{s+1}. \end{cases}$$

Since $\gamma_n < \gamma_{n+1}$, if $n \leq m_j$, we have that

$$\begin{aligned} \gamma_n \lim_{p\to\infty} \left(\sum_{i=n}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}} = \gamma_n \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}} &\leq \gamma_{n+1} \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}} \\ &= \gamma_{n+1} \lim_{p\to\infty} \left(\sum_{i=n+1}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}}. \end{aligned}$$

Hence we obtain that

$$\gamma_l \lim_{p\to\infty} \left(\sum_{i=l}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}} \geq \gamma_{m_j+1} \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{m_{s+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}}$$

and so

$$\frac{\gamma_l - \gamma_{m_j+1}}{\gamma_{m_j+1}} \lim_{p\to\infty} \left(\sum_{i=l}^{m_{j+1}} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}} \geq \lim_{p\to\infty} \left(\sum_{i=m_j+1}^{l-1} \frac{|a_i'|^p}{2^i} \right)^{\frac{1}{p}}.$$

Thus, we have

$$\begin{aligned}
\left|\|u_j - u_s\|_\infty - \left\|\left|u_j - u_s\right|\right\|\right| &= \sup_{m_j+1\le i\le m_{s+1}} |a_i^{'}| - \gamma_l \lim_{p\to\infty}\left(\sum_{i=l}^{m_{s+1}} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}} \\
&= \lim_{p\to\infty}\left(\sum_{i=m_j+1}^{m_{s+1}} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}} - \gamma_l \lim_{p\to\infty}\left(\sum_{i=l}^{m_{s+1}} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}} \\
&\le (1-\gamma_l)\lim_{p\to\infty}\left(\sum_{i=l}^{m_{s+1}} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}} + \lim_{p\to\infty}\left(\sum_{i=m_j+1}^{l-1} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\|u_j - u_s\|_\infty - \left\|\left|u_j - u_s\right|\right\|\right| &\le \left(1 + \frac{\gamma_l - \gamma_{m_j+1}}{\gamma_{m_j+1}} - \gamma_l\right)\lim_{p\to\infty}\left(\sum_{i=l}^{m_{s+1}} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}} \\
&= \left(\frac{\gamma_l}{\gamma_{m_j+1}} - \gamma_l\right)\lim_{p\to\infty}\left(\sum_{i=l}^{m_{s+1}} \frac{\left|a_i^{'}\right|^p}{2^i}\right)^{\frac{1}{p}}.
\end{aligned}$$

Assume that $\|u_j\|_\infty \le d$ (since the sequence is bounded). Then, $0 \le \left|\|u_j - u_s\|_\infty - \left\|\left|u_j - u_s\right|\right\|\right| \le \left(\frac{\gamma_l}{\gamma_{m_j+1}} - \gamma_l\right)2d$ taking the limit as $j \to \infty$ and as $s \to \infty$ next, we get that $\lim\limits_{s\to\infty}\lim\limits_{j\to\infty} \|u_j - u_s\|_\infty - \left\|\left|u_j - u_s\right|\right\| = 0$ and so using Lemma 3.9 and Lemma 3.7 (and considering we had passed to a subsequence already) we obtain that

$$\begin{aligned}
a &= \lim_{s\to\infty}\lim_{j\to\infty}\left\|\left|u_j - u_s\right|\right\| = \lim_{s\to\infty}\lim_{j\to\infty}\|u_j - u_s\|_\infty = \lim_{s\to\infty}\lim_{j\to\infty}\left\|\frac{1}{j}\sum_{k=1}^{j} x_k - \frac{1}{s}\sum_{k=1}^{s} x_k\right\|_\infty \\
&= \lim_{j\to\infty}\left\|\frac{1}{j}\sum_{k=1}^{j} x_k\right\|_\infty + \lim_{s\to\infty}\left\|\frac{1}{s}\sum_{k=1}^{s} x_k\right\|_\infty = 2\lim_{j\to\infty}\left\|\frac{1}{j}\sum_{k=1}^{j} x_k\right\|_\infty = 2\lim_{j\to\infty}\left\|\left|\frac{1}{j}\sum_{k=1}^{j} x_k\right|\right\|.
\end{aligned}$$

□

Lemma 3.11. *Assume that $C \subset (c_0, \|\cdot\|_\infty)$ is a ccne set, $T : C \to C$ is an affine nonexpansive fixed point free mapping, $(x_n)_n \subset C$ is an afps, $x_n \underset{w}{\to} u$, such that*

$$\lim_{j\to\infty}\left\|\frac{1}{j}\sum_{n=1}^{j} x_n - u\right\|_\infty \text{ exists, } \Phi : C \to [0,\infty) \text{ given by}$$

$$\Phi(x) = \limsup_m \left\|\frac{1}{m}\sum_{n=1}^{m} x_n - x\right\|_\infty \text{ and } D = \left\{x : \ \Phi(x) \le \frac{d}{4}\right\} \ne \emptyset \text{ and so } \{x : \ \Phi(x) \le d\} \ne \emptyset.$$

Assume further that $(y_n)_n \subset D$ and $y_n \underset{w}{\to} y$. Then,

$$\limsup_m \left\|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right\|_\infty \le d - \lim_{m\to\infty}\left\|\frac{1}{m}\sum_{n=1}^{m} x_n - u\right\|_\infty.$$

Proof. Using equality (1), for every $s \in \mathbb{N}$,

$$\begin{aligned} \frac{d}{4} &\geq \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - y_s \right\|_\infty = \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - u + u - y_s \right\|_\infty \\ &\geq \frac{1}{2} \lim_{m\to\infty} \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - u \right\|_\infty + \frac{1}{2} \left\| y_s - y + y - u \right\|_\infty . \end{aligned}$$

Therefore, again by equality (1),

$$\begin{aligned} \frac{d}{4} &\geq \limsup_s \limsup_m \frac{1}{s} \sum_{k=1}^{s} \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - y_k \right\|_\infty \\ &\geq \limsup_s \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - \frac{1}{s} \sum_{n=1}^{s} y_n \right\|_\infty \\ &\geq \frac{1}{4} \lim_{m\to\infty} \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - u \right\|_\infty + \frac{1}{4} \left\| y - u \right\|_\infty + \frac{1}{4} \lim_{s\to\infty} \left\| \frac{1}{s} \sum_{n=1}^{s} y_n - y \right\|_\infty . \end{aligned}$$

Thus,

$$\limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} y_n - y \right\|_\infty \leq d - \lim_{m\to\infty} \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - u \right\|_\infty - \left\| y - u \right\|_\infty .$$

□

Now, we prove the corresponding lemma in $(c_0, \vert\vert\vert \cdot \vert\vert\vert)$.

Lemma 3.12. *Assume that $C \subset (c_0, \|\cdot\|_\infty)$ is a ccne set, $T : C \to C$ is an affine nonexpansive fixed point free mapping, $(x_n)_n \subset C$ is an afps, $x_n \underset{w}{\to} u$, such that*

$$\lim_{j\to\infty} \left\vert\left\vert\left\vert \frac{1}{j} \sum_{n=1}^{j} x_n - u \right\vert\right\vert\right\vert \textit{ exists, } (u_n)_n \textit{ is as in lemma } 3.6,$$

$$\lim_{j\to\infty} \left\vert\left\vert\left\vert \frac{1}{j} \sum_{n=1}^{j} x_n - u - u_j \right\vert\right\vert\right\vert = 0, \ \Phi : C \to [0, \infty) \textit{ given by}$$

$$\Phi(x) = \limsup_m \left\vert\left\vert\left\vert \frac{1}{m} \sum_{n=1}^{m} x_n - x \right\vert\right\vert\right\vert \textit{ and } D = \{x : \ \Phi(x) \leq d\} \neq \emptyset.$$

Assume further that $(y_n)_n \subset D$, $y_n \underset{w}{\to} y$ and $\lim_{n\to\infty} \left\vert\left\vert\left\vert \frac{1}{n} \sum_{k=1}^{n} y_k - y \right\vert\right\vert\right\vert$ exists. Then,

$$\lim_m \left\vert\left\vert\left\vert \frac{1}{m} \sum_{n=1}^{m} y_n - y \right\vert\right\vert\right\vert \leq d - \lim_{m\to\infty} \left\vert\left\vert\left\vert \frac{1}{m} \sum_{n=1}^{m} x_n - u \right\vert\right\vert\right\vert.$$

Proof. Assume $\varepsilon > 0$, k such that $\vert\vert\vert y - P_k y \vert\vert\vert < \varepsilon$ and $\vert\vert\vert u - P_k u \vert\vert\vert < \varepsilon$ where P_k is the natural projection. By passing to a subsequence of $(y_n)_n$ we may also assume that there is a sequence $(v_n)_n$ with $v_n = \sum_{i=r_n+1}^{r_{n+1}} b_i e_i$ and

$\lim_{n\to\infty} \left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} y_k - y - v_n \right|\right|\right|$. Let $N > k$ be such that for $n > N$, $\left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} x_k - u - u_n \right|\right|\right| < \varepsilon$ and $\left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} y_k - y - v_n \right|\right|\right| < \varepsilon$. Then, if $m_n > r_{j+1} > N$, then

$$\begin{aligned} \limsup_n \left|\left|\left| u_n + P_k u - v_j - P_k y_k \right|\right|\right| &\leq \limsup_n \begin{pmatrix} \left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} x_n - y_j \right|\right|\right| + \left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} x_n - u - u_n \right|\right|\right| + \left|\left|\left| y_j - y - v_j \right|\right|\right| \\ + |||u - P_k u||| + \left|\left|\left| y - P_k y \right|\right|\right| \end{pmatrix} \\ &\leq d + 4\varepsilon. \end{aligned}$$

Then, if $r_{j+1} \geq k$ and $m_n + 1 > r_{j+1}$ and $r_j + 1 \leq s_j \leq r_{j+1}$ is such that $\left|\left|\left| v_j \right|\right|\right| = \gamma_{s_j} P_{s_j} \|v_j\|_\infty$, we get $\left|\left|\left| u_n + P_k u - v_j - P_k y_k \right|\right|\right| \geq \gamma_{s_j}(\|v_j\|_\infty + \|u_n\|_\infty) = \left|\left|\left| v_j \right|\right|\right| + |||u_n|||$. Hence, since $u_n \underset{w}{\to} 0$, using Lemma 3.9, $d + 4\varepsilon \geq \left|\left|\left| v_j \right|\right|\right| + \gamma_{s_j} \lim_n \|u_n\|_\infty = \left|\left|\left| v_j \right|\right|\right| + \gamma_{s_j} \lim_n |||u_n|||$ and by passing to the limit as $j \to \infty$, we have $d + 4\varepsilon \geq \lim_j \left|\left|\left| v_j \right|\right|\right| + \lim_n |||u_n||| = \lim_j \left|\left|\left| \frac{1}{j} \sum_{k=1}^{j} y_k - y \right|\right|\right| + \lim_n \left|\left|\left| \frac{1}{n} \sum_{m=1}^{n} x_m - u \right|\right|\right|$ and this concludes the proof. □

Theorem 3.13. *Assume that $C \subset (c_0, \|\cdot\|_\infty)$ is a ccne set, $T : C \to C$ is an affine nonexpansive fixed point free mapping, $(x_n)_n \subset C$ is an afps, $x_n \underset{w}{\to} 0$ and that*

$$D = \left\{ x : \limsup_m \left\| \frac{1}{m} \sum_{n=1}^{m} x_n - x \right\|_\infty \leq d \right\} \text{ is assumed not empty.}$$

If

$$c = \inf \left\{ \limsup_m \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} y_n - y \right|\right|\right| : (y_n)_n \subset D, (y_n)_n \text{is an afps with } y_n \underset{w}{\to} y \right\},$$

then

$$\inf \left\{ \limsup_m \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} y_n - z \right|\right|\right| : z \in D, (y_n)_n \subset D \text{ is an afps with } y_n \underset{w}{\to} y \right\} \geq 2c.$$

Proof. By contradiction, assume that there exists $z \in D$ and $(y_n)_n \subset D$ is an afps with $y_n \underset{w}{\to} y$ such that $a = \lim_{m\to\infty} \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} y_n - z \right|\right|\right| < 2c$. Then, by the hypothesis, $\limsup_m \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} y_n - y \right|\right|\right| \geq c$ and

$$z \in D_1 = \left\{ u \in D : \limsup_m \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} y_n - u \right|\right|\right| \leq a \right\} \neq \emptyset$$

and D_1 is a ccne which is T-invariant by Lemma 3.1. Let $(u_n)_n \subset D_1$ be an afps which converges weakly to u. Then, by Lemma 3.12,

$$\limsup_m \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} u_n - u \right|\right|\right| \leq a - \lim_{m\to\infty} \left|\left|\left| \frac{1}{m} \sum_{n=1}^{m} y_n - y \right|\right|\right| < 2c - c = c$$

which is a contradiction. □

Theorem 3.14. *Consider the equivalent norm $|||\cdot|||$ to the usual norm $\|\cdot\|_\infty$ of c_0 given below.*
For $x = (\xi_k)_k \in c_0$, define

$$|||x||| := \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{|\xi_j|^p}{2^j} \right)^{\frac{1}{p}} \text{ where } \gamma_k \uparrow_k 3,\ \gamma_k \text{ is strictly increasing with } \gamma_k > 2,\ \forall k \in \mathbb{N}.$$

Then, $(c_0, |||\cdot|||)$ has the fixed point property for affine $|||\cdot|||$-nonexpansive self-mappings.

Proof.

Define $|||x|||_k := \gamma_k \lim_{p\to\infty}\left(\sum_{j=k}^{\infty}\frac{|\xi_j|^p}{2^j}\right)^{\frac{1}{p}}$, $\forall k \in \mathbb{N}$.

Then, $|||x||| = \sup_{k\in\mathbb{N}} |||x|||_k$. By contradiction, assume that $C \subset (c_0, \|\cdot\|_\infty)$ is a ccne set, $T : C \to C$ is an affine nonexpansive fixed point free mapping and that C satisfies the hypothesis of Corollary 3.5. Thus,

$$b = \inf\left\{\limsup_m \left|\left|\left|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right|\right|\right| : (y_n)_n \subset C \text{ is an afps with } y_n \underset{w}{\to} y\right\} > 0.$$

Let $\alpha = 4\gamma_1 - 8$, $\varepsilon_1 = \frac{\alpha}{12}$, assume that $(x_n)_n \subset C$ is an afps with $x_n \underset{w}{\to} x_0$ and that

$$b \le \lim_n \left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x_0\right|\right|\right| < (1+\varepsilon_1)b.$$

Without loss of generality we may assume $x_0 = 0$ and by Lemma 3.10, taking an appropriate subsequence, $\lim_{n\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} x_i\right|\right|\right|$ exists and

$$\lim_{m\to\infty}\lim_{n\to\infty}\left|\left|\left|\frac{1}{m}\sum_{i=1}^{m} x_i - \frac{1}{n}\sum_{i=1}^{n} x_i\right|\right|\right| = 2\lim_{n\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} x_i\right|\right|\right| < 2(1+\varepsilon_1)b.$$

Let

$$D = \left\{z \in C : \limsup_n \left\|\frac{1}{n}\sum_{i=1}^{n} x_i - z\right\| \le 2(1+\varepsilon_1)b\right\}.$$

Then by the above, D is a ccne set. Now, let

$$c = \inf\left\{\limsup_m \left|\left|\left|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right|\right|\right| : (y_n)_n \subset D \text{ is an afps with } y_n \underset{w}{\to} y\right\}.$$

Note that $c \ge b$ and that if $(y_n)_n \subset D$ is an afps with $y_n \underset{w}{\to} y$ and $\lim_{n\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right|\right|\right|$ exists, by Lemma 3.9, $b \le c \le \lim_{n\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right|\right|\right| = \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right\|_\infty$ and $b \le \lim_{n\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} x_i\right|\right|\right| = \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|_\infty$ and so

$$\begin{aligned}
\frac{1}{\gamma_1}|||y|||_1 &= \|y\|_\infty = \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|_\infty + \|y\|_\infty - \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|_\infty \qquad (2)\\
&\le \lim_{n\to\infty}\left(2\left\|\frac{1}{n}\sum_{i=1}^{n} x_i - y\right\|_\infty - \left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|_\infty\right)\\
&\le \frac{4}{\gamma_1}\lim_{n\to\infty}\lim_{m\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{m}\sum_{i=1}^{m} y_i\right|\right|\right| - \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|_\infty - 2\lim_{m\to\infty}\left\|\frac{1}{m}\sum_{i=1}^{m} y_i - y\right\|_\infty\\
&\le \frac{8}{\gamma_1}(1+\varepsilon_1)b - 3b = b\left(\frac{8}{\gamma_1} - 3 + \frac{8}{\gamma_1}\varepsilon_1\right).
\end{aligned}$$

Now let $x \in D$ satisfy

$$|||x|||_1 \le |||x||| \le (1+\varepsilon_1)b \le (1+\varepsilon_1)c. \qquad (3)$$

Such an element exists since $\lim_{n\to\infty} \left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\| < (1+\varepsilon_1)b$ and so for n large enough $\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\| < (1+\varepsilon_1)b$.

Let $\varepsilon_2 = \frac{\alpha}{3(8+\alpha)}$ and m large enough so that

$$\frac{1}{\gamma_1}|||(I-P_m)x|||_1 = \|(I-P_m)x\|_\infty < c\varepsilon_2. \tag{4}$$

Let $\varepsilon_3 = \frac{\alpha\Phi_m}{4(8+\alpha)(1-\Phi_m)}$ where $\Phi_m = 1-\gamma_m$ and let $(y_n)_n \subset D$ be an afps with $y_n \underset{w}{\to} y$ and $\lim_{n\to\infty} \left\|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right\|$ exists such that

$$\begin{aligned}\frac{1}{\gamma_1}\lim_{n\to\infty}\left|\left|\left|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right|\right|\right|_1 &= \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right\|_\infty = \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right\| \\ &< (1+\varepsilon_3)c.\end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ such that for every $k \geq N$,

$$\frac{1}{\gamma_1}\left\|\frac{1}{k}\sum_{i=1}^{k} y_i - y\right\|_\infty \leq (1+\varepsilon_3)c. \tag{5}$$

Now let $\lambda = \frac{\Phi_m}{2(1-\Phi_m)}$ and set $z = \lambda x + (1-\lambda)\frac{1}{N}\sum_{i=1}^{N} y_i \in D$ (due to convexity of D) and pick $j > N$.

Now note that if $k > m$, then $|||u|||_k \leq \frac{\gamma_k}{\gamma_1}|||u|||_1,\ \forall u \in c_0$ and so using the inequalities 2, 4 and 5, we have

$$\begin{aligned}\left|\left|\left|\frac{1}{j}\sum_{i=1}^{j} y_i - z\right|\right|\right|_k &= \left|\left|\left|(I-P_m)\frac{1}{j}\sum_{i=1}^{j} y_i - z\right|\right|\right|_k = \left|\left|\left|(I-P_m)\frac{1}{j}\sum_{i=1}^{j} y_i - y + y - z\right|\right|\right|_k \\ &\leq \left|\left|\left|\frac{1}{j}\sum_{i=1}^{j} y_i - y\right|\right|\right|_k + (1-\lambda)\left|\left|\left|\frac{1}{N}\sum_{i=1}^{N} y_i - y\right|\right|\right|_k + \lambda\left[|||(I-P_m)x|||_k + |||y|||_k\right] \\ &\leq \left[\gamma_k(1+\varepsilon_3)(2-\lambda) + \lambda\gamma_k\varepsilon_2 + \lambda\gamma_k\left(\frac{8}{\gamma_1} - 3 + \frac{8}{\gamma_1}\varepsilon_1\right)\right]c \\ &\leq \left[2 + 2\varepsilon_3 + \lambda\left(\varepsilon_2 - 4 + \frac{8}{\gamma_1} + \frac{8}{\gamma_1}\varepsilon_1 - \varepsilon_3\right)\right]c \\ &\leq \left[2 - \frac{\alpha\Phi_m^2}{8(8+\alpha)(1-\Phi_m)^2}\right]c.\end{aligned}$$

But if $k \leq m$, then

$$\begin{aligned}\left|\left|\left|\frac{1}{j}\sum_{i=1}^{j} y_i - z\right|\right|\right|_k &\leq \frac{\gamma_k}{\gamma_1}\left|\left|\left|\frac{1}{j}\sum_{i=1}^{j} y_i - y + y - z\right|\right|\right|_1 \\ &\leq \gamma_m\left[\frac{\lambda}{\gamma_1}\left(|||x|||_1 + |||y|||_1\right) + \frac{1}{\gamma_1}\left(\left|\left|\left|\frac{1}{j}\sum_{i=1}^{j} y_i - y\right|\right|\right|_1 + (1-\lambda)\left|\left|\left|\frac{1}{N}\sum_{i=1}^{N} y_i - y\right|\right|\right|_1\right)\right] \\ &\leq \left[2 - 2\Phi_m + (1-\Phi_m)\left\{\lambda\left(\frac{9}{\gamma_1} - 3 + \frac{9}{\gamma_1}\varepsilon_1\right) + 2\varepsilon_3\right\}\right]c \\ &\leq \left[2 - \frac{\Phi_m(20+3\alpha)}{2(8+\alpha)}\right]c.\end{aligned}$$

Hence, $\lim_j \left\| \left\| \frac{1}{j} \sum_{i=1}^{j} y_i - z \right\| \right\| \leq \max \left\{ \left[2 - \frac{\alpha \Phi_m^2}{8(8+\alpha)(1-\Phi_m)^2} \right] c, \left[2 - \frac{\Phi_m(20+3\alpha)}{2(8+\alpha)} \right] c \right\} < 2c$

contradicting with Theorem 3.13. □

4. Family of Equivalent Norms on c_0 with FPP for Affine Nonexpansive Mappings

In this section, we generalize our result from the previous section and firstly we would like to note that the proof of our theorem below is inspired by the proofs of theorems and lemmas given by Carlos A. Hernández-Linares, María A. Japón and Enrique Llorens-Fuster [11] such that they extend P.K. Lin's work [16]. We implement our ideas and get our desired result with the help of their work. In our theorem, the reader will notice that Lemma 3.7 shows that our hypothesis in our theorem below is reasonable. Now, let us see our theorem with its proof and a remark concluding our paper.

Theorem 4.1. *If $\rho(\cdot)$ is an equivalent norm to the usual norm on c_0 such that*

$$\limsup_n \rho\left(\frac{1}{n}\sum_{m=1}^{n} x_m + x\right) = \limsup_n \rho\left(\frac{1}{n}\sum_{m=1}^{n} x_m\right) + \rho(x)$$

for every weakly null sequence $(x_n)_n$ and for all $x \in c_0$, then for every $\lambda > 0$, c_0 with the norm $\|\cdot\|_\rho = \rho(\cdot) + \lambda |||\cdot|||$ has the FPP for affine $\|\cdot\|_\rho$-nonexpansive self-mappings.

Proof. First of all, let us define for any $k \in \mathbb{N}$ and for any $x = (\xi_i)_{i\in\mathbb{N}} \in c_0$,

$$S_k(x) := \lim_{p\to\infty} \left(\sum_{j=k}^{\infty} \frac{|\xi_j|^p}{2^j} \right)^{\frac{1}{p}} \text{ and } \rho_k(x) := \rho(x) + \lambda \gamma_k S_k(x). \tag{6}$$

Note that it is clear that for every $x \in c_0$, $|||x||| = \sup_k S_k(x)$ and due to Lemma 3.7, for every weakly null sequence $(x_n)_n$, there exists a subsequence $(x_{n_l})_l$ such that for every $x \in c_0$ and for all $k \in \mathbb{N}$,

$$\limsup_j \rho_k\left(\frac{1}{j}\sum_{l=1}^{j} x_{n_l} + x\right) = \limsup_j \rho_k\left(\frac{1}{j}\sum_{l=1}^{j} x_{n_l}\right) + \rho_k(x). \tag{7}$$

Now, by contradiction, assume that $\left(c_0, \|\cdot\|_\rho\right)$ fails the fixed point property for affine $\|\cdot\|_\rho$-nonexpansive self-mappings. Let T and D be as in Corollary 3.5 and τ be the weak $(\sigma(l^\infty, l^1))$-topology in c_0 that we will denote by w.

Define

$$M := \inf\left\{ \limsup_m \left\| \frac{1}{m}\sum_{n=1}^{m} x_n - x \right\|_\rho : (x_n)_n \subset D,\ (x_n)_n \text{is an afps},\ x_n \underset{w}{\to} x \right\}.$$

Note that $M > 0$. Assume that $A(D)$ is the set of all $(x_n)_n \subset D$ such that $(x_n)_n$ is an afps converging weakly (w) to some $x \in c_0$ and such that $\lim_j \rho\left(\frac{1}{j}\sum_{k=1}^{j} x_k - u\right)$, $\lim_j \left|\left|\left| \frac{1}{j}\sum_{k=1}^{j} x_k - u \right|\right|\right|$ and $\lim_j S_k\left(\frac{1}{j}\sum_{k=1}^{j} x_k - u\right)$ exist for any $u \in c_0$ and for all $k \in \mathbb{N}$.

Now, due to separability of c_0, we can say that

$$M := \inf\left\{ \lim_m \left\| \frac{1}{m}\sum_{n=1}^{m} x_n - x \right\|_\rho : (x_n)_n \subset A(D),\ x_n \underset{w}{\to} x \right\}.$$

Without loss of generality we can assume that $M = 1$ and from the equivalence of the norms, we can obtain that

$$c := \inf\{|||x||| \ : \ \rho(x) = \lambda\} > 0 \text{ and } d := \inf\left\{|||x||| \ : \ \|x\|_\rho = \lambda\right\} > 0. \tag{8}$$

Choose $\delta_1 > 0$ such that $\frac{1+\delta_1}{1+c} + 2\delta_1 < 1$ an afps $(x_n)_n \subset A(D)$ such that weak-$\lim\limits_n x_n = x$ and $\lim\limits_m \left\|\frac{1}{m}\sum\limits_{n=1}^{m} x_n - x\right\|_\rho < 1 + \delta_1$. Without loss of generality we can assume that $x = 0$. Now, note that $K := \left\{z \in D : \lim\limits_m \left\|\frac{1}{m}\sum\limits_{n=1}^{m} x_n - z\right\|_\rho \leq 2 + 2\delta_1\right\}$ is a nonempty closed, bounded, convex and T-invariant subset and there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $x_n \in K$.

Define

$$Q := \inf\left\{\lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right\|_\rho : (y_n)_n \subset K \cap A(D)\ ,\ y_n \underset{w}{\to} y\right\}$$

and note that

$$1 \leq Q \leq \lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} x_n\right\|_\rho \leq 1 + \delta_1. \tag{9}$$

Define $A(K) := \left\{(y_n)_n \subset A(D) : \ y_n \in K, \forall n \in \mathbb{N}\right\}$ and pick an arbitrary afps $(y_n)_n \subset A(K)$ with $y_n \underset{w}{\to} y$. Then, for every $k \in \mathbb{N}$, without loss of generality, by passing to the appropriate subsequence of $(x_n)_n$ if nexessary, we have

$$\begin{aligned}
2 + 2\delta_1 &\geq \limsup_s \lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} x_n - \frac{1}{s}\sum_{n=1}^{s} y_n\right\|_\rho \\
&\geq \limsup_s \lim_m \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} x_n - \frac{1}{s}\sum_{n=1}^{s} y_n\right) \\
&= \limsup_s \left[\limsup_m \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} x_n\right) + \rho_k\left(\frac{1}{s}\sum_{n=1}^{s} y_n\right)\right] \text{ by (7)} \\
&= \limsup_m \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} x_n\right) + \limsup_s \rho_k\left(\frac{1}{s}\sum_{n=1}^{s} y_n - y\right) + \rho_k(y) \text{ by (7)} \\
&= \lim_m \rho\left(\frac{1}{m}\sum_{n=1}^{m} x_n\right) + \lambda\gamma_k \lim_m S_k\left(\frac{1}{m}\sum_{n=1}^{m} x_n\right) + \lim_s \rho\left(\frac{1}{s}\sum_{n=1}^{s} y_n - y\right) + \lambda\gamma_k \lim_s S_k\left(\frac{1}{s}\sum_{n=1}^{s} y_n - y\right) + \rho_k(y) \\
&= \lim_m \rho\left(\frac{1}{m}\sum_{n=1}^{m} x_n\right) + \lambda\gamma_k \lim_m \left|\left|\left|\frac{1}{m}\sum_{n=1}^{m} x_n\right|\right|\right| + \lim_s \rho\left(\frac{1}{s}\sum_{n=1}^{s} y_n - y\right) + \lambda\gamma_k \lim_s \left|\left|\left|\frac{1}{s}\sum_{n=1}^{s} y_n - y\right|\right|\right| + \rho_k(y) \\
&\geq \gamma_k\left[\lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} x_n\right\|_\rho + \lim_s \left\|\frac{1}{s}\sum_{n=1}^{s} y_n - y\right\|_\rho\right] + \rho_k(y) \\
&\geq 2\gamma_k + \rho_k(y).
\end{aligned}$$

Thus, if $(y_n)_n \subset A(K)$ is an afps with $y_n \underset{w}{\to} y$, we have

$$\rho_k(y) \leq 2(1-\gamma_k) + 2\delta_1 < 2 + 2\delta_1. \tag{10}$$

Now, we choose s such that $\frac{1+\delta_1}{1+c} < 2\delta_1 < s < 1$ and note that from (8) , for all $u \in c_0$, $\rho(u) \leq \frac{\|u\|_\rho}{1+c}$. Thus,

$$\lim_m \rho\left(\frac{1}{m}\sum_{n=1}^{m} x_n\right) \leq \frac{\lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} x_n\right\|_\rho}{1+c} < \frac{1+\delta_1}{1+c}$$

and so there exists $n_0 \in \mathbb{N}$ such that

$$\rho\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) < \frac{\lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} x_n\right\|_\rho}{1+c} < \frac{1+\delta_1}{1+c}$$

But recalling we took $x = 0$,

$$\limsup_k \rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) = \rho\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) + \lambda \limsup_k \gamma_k S_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) = \rho\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right).$$

Therefore, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) < \frac{1+\delta_1}{1+c} \tag{11}$$

and

$$q_k := \frac{1+\delta_1}{1+c} + 2(1-\gamma_k) + 2\delta_1 < s < 1 \leq Q. \tag{12}$$

Since K is a bounded set, $\exists H > 0$ such that $\rho(x) < H$ for all $x \in K$. Thus,

$$\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) < H,\ \forall k \in \mathbb{N}. \tag{13}$$

Now, let us define $s_0 := 1 - M(1-\gamma_{k_0})$. Note that $s_0 < 1$. Define also

$$h := H + 2 + 2\delta_1 \text{ and note that by (9), } h > Q > s_0 Q \tag{14}$$

Now, choose $\alpha \in (0,1)$ that satisfies $\alpha < \frac{2Q(1-s_0)}{h-s_0Q}$; hence, $(2-\alpha)Q + \alpha s = 2Q - \alpha(Q-s) < 2Q$ and $(2-\alpha)s_0Q + \alpha h = 2s_0Q + \alpha(h - s_0Q) < 2s_0Q + 2Q(1-s_0) = 2Q$.

Thus, we can find $\delta_2 > 0$ such that

$$(2-\alpha)(Q+\delta_2) + \alpha s < 2Q \tag{15}$$

and

$$(2-\alpha)s_0(Q+\delta_2) + \alpha h < 2Q. \tag{16}$$

Note that for

$$W := \max\left\{(2-\alpha)(Q+\delta_2) + \alpha s,\ (2-\alpha)s_0(Q+\delta_2) + \alpha h\right\}, W < 2Q. \tag{17}$$

Now, let $(y_n)_n \subset A(K)$ be an afps with $y_n \underset{w}{\to} y$ and $\lim_m \left\| \frac{1}{m} \sum_{n=1}^{m} y_n - y \right\|_\rho < Q + \delta_2$. Then, $\exists N_0 \in \mathbb{N}$ such that $\forall m \geq N_0$,

$$\left\| \frac{1}{m} \sum_{n=1}^{m} y_n - y \right\|_\rho < Q + \delta_2. \tag{18}$$

Also,

$$\begin{aligned}
\lim_m \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) &= \lim_m \rho\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) + \lambda\gamma_k \lim_m S_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) \\
&= \lim_m \rho\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) + \lambda\gamma_k \lim_m \left|\left|\left| \frac{1}{m}\sum_{n=1}^{m} y_n - y \right|\right|\right| \\
&= \lim_m \left\| \frac{1}{m}\sum_{n=1}^{m} y_n - y \right\|_\rho - (1-\gamma_k)\lambda \lim_m \left|\left|\left| \frac{1}{m}\sum_{n=1}^{m} y_n - y \right|\right|\right| \\
&\leq [1-(1-\gamma_k)d] \lim_m \left\| \frac{1}{m}\sum_{n=1}^{m} y_n - y \right\|_\rho \text{ by definition of } d \text{ from (8)} \\
&< [1-(1-\gamma_k)d]\,(Q+\delta_2)
\end{aligned}$$

and we can find $N_1 \geq N_0$ such that for all $m \geq N_1$ and for all $k \leq k_0$

$$\rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) < [1-(1-\gamma_k)d]\,(Q+\delta_2) \leq s_0(Q+\delta_2). \tag{19}$$

Now, define $z_0 := \alpha \frac{1}{n_0}\sum_{n=1}^{n_0} x_n + (1-\alpha)\frac{1}{N_1}\sum_{n=1}^{N_1} y_n$ and note that since K is convex, $z_0 \in K$. Now, we show that $\lim_m \left\| \frac{1}{m}\sum_{n=1}^{m} y_n - z_0 \right\|_\rho \leq W$ and to prove that we will observe for all $k \in \mathbb{N}$ and for all $m \geq N_1$, $\rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - z_0\right) \leq W$.

First, we need to take the equation $\frac{1}{m}\sum_{n=1}^{m} y_n - z_0 = \frac{1}{m}\sum_{n=1}^{m} y_n - y - (1-\alpha)\left(\frac{1}{N_1}\sum_{n=1}^{N_1} y_n - y\right) - \alpha\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n - y\right)$ into consideration.

We will have two cases to see this for $m \geq N_1$.

Case 4.2. $k > k_0$:

$$\begin{aligned}
\rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - z_0\right) &\leq \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) + (1-\alpha)\rho_k\left(\frac{1}{N_1}\sum_{n=1}^{N_1} y_n - y\right) + \alpha\left[\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) + \rho_k(y)\right] \\
&\leq \left\| \frac{1}{m}\sum_{n=1}^{m} y_n - y \right\|_\rho + (1-\alpha)\left\| \frac{1}{N_1}\sum_{n=1}^{N_1} y_n - y \right\|_\rho + \alpha\left[\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) + \rho_k(y)\right] \\
&< (2-\alpha)(Q+\delta_2) + \alpha\left[\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) + \rho_k(y)\right] \textit{ by } (18) \\
&< (2-\alpha)(Q+\delta_2) + \alpha\left[\frac{1+\delta_1}{1+c} + 2(1-\gamma_k) + 2\delta_1\right] \textit{ by } (10) \textit{ and } (11) \\
&= (2-\alpha)(Q+\delta_2) + \alpha q_k < (2-\alpha)(Q+\delta_2) + \alpha s \textit{ by } (12) \\
&\leq W \textit{ from } (17).
\end{aligned}$$

Case 4.3. $k \le k_0$:

$$\begin{aligned}\rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - z_0\right) &\le \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) + (1-\alpha)\rho_k\left(\frac{1}{N_1}\sum_{n=1}^{N_1} y_n - y\right) + \alpha\left[\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) + \rho_k(y)\right] \\ &\le s_0(2-\alpha)(Q+\delta_2) + \alpha\left[\rho_k\left(\frac{1}{n_0}\sum_{n=1}^{n_0} x_n\right) + \rho_k(y)\right] \text{ by } (19) \\ &< s_0(2-\alpha)(Q+\delta_2) + \alpha\left[H + 2 + 2\delta_1\right] \text{ by } (13) \text{ and } (10) \\ &\le s_0(2-\alpha)(Q+\delta_2) + \alpha h \text{ by } (14) \\ &\le W \text{ from } (17).\end{aligned}$$

Then, $\rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - z_0\right) \le W$ for all $k \in \mathbb{N}$ and for all $m \ge N_1$. Thus, $\left\|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right\|_\rho \le W$ for all $m \ge N_1$ and so

$$\limsup_m \left\|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right\|_\rho \le W.$$

Now, we can say that there exists a nonempty subset $K_0 := \left\{z \in K : \limsup_m \left\|\frac{1}{m}\sum_{n=1}^{m} y_n - z\right\|_\rho \le W\right\}$ and an afps $(u_n)_n \subset K_0 \cap A(D)$ with $u_n \underset{w}{\to} u \in c_0$. Then, for every $k \in \mathbb{N}$,

$$\begin{aligned} W &\ge \limsup_s \limsup_m \left\|\frac{1}{m}\sum_{n=1}^{m} y_n - \frac{1}{s}\sum_{n=1}^{s} u_n\right\|_\rho \\ &\ge \limsup_s \limsup_m \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - \frac{1}{s}\sum_{n=1}^{s} u_n\right) \\ &= \lim_m \rho_k\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) + \lim_s \rho_k\left(\frac{1}{s}\sum_{n=1}^{s} u_n - u\right) + \rho_k(y-u) \text{ by } (7) \\ &\ge \lim_m \rho\left(\frac{1}{m}\sum_{n=1}^{m} y_n - y\right) + \lambda\gamma_k \lim_m \left|\!\left|\!\left|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right|\!\right|\!\right| + \lim_s \rho\left(\frac{1}{s}\sum_{n=1}^{s} u_n - u\right) + \lambda\gamma_k \lim_s \left|\!\left|\!\left|\frac{1}{s}\sum_{n=1}^{s} u_n - u\right|\!\right|\!\right|.\end{aligned}$$

Thus, by taking limits as k approaches to infinity, we obtain that

$$W \ge \lim_m \left\|\frac{1}{m}\sum_{n=1}^{m} y_n - y\right\|_\rho + \lim_s \left\|\frac{1}{s}\sum_{n=1}^{s} u_n - u\right\|_\rho \ge Q + Q = 2Q$$

which contradicts the definition of W by (17). Therefore, $\left(c_0, \|\cdot\|_\rho\right)$ has the fixed point property for affine $\|\cdot\|_\rho$-nonexpansive self-mappings. □

Remark 4.4. Since $\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty} \frac{|\xi_k|^p}{2^k}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty} \frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}$, $\forall x = (\xi_i)_{i\in\mathbb{N}} \in c_0$, one can conclude the following result using the equivalent norm constructed by the first author in his recent study [18] that is actually prepared after this paper.

For $x = (\xi_k)_k \in c_0$, define

$$|\!|\!|x|\!|\!|^\sim := \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{\left|\xi_j\right|^p}{j^2}\right)^{\frac{1}{p}} \quad \text{where } \gamma_k \uparrow_k 3,\ \gamma_k \text{ is strictly increasing with } \gamma_k > 2,\ \forall k \in \mathbb{N},$$

then $(c_0, |||\cdot|||^{\sim})$ has the fixed point property for affine $|||\cdot|||^{\sim}$-nonexpansive self-mappings and if $\rho^{\sim}(\cdot)$ is an equivalent norm to the usual norm on c_0 such that

$$\limsup_n \rho^{\sim}\left(\frac{1}{n}\sum_{m=1}^{n} x_m + x\right) = \limsup_n \rho^{\sim}\left(\frac{1}{n}\sum_{m=1}^{n} x_m\right) + \rho^{\sim}(x)$$

for every weakly null sequence $(x_n)_n$ and for all $x \in c_0$, then for every $\lambda > 0$, c_0 with the norm $\|\cdot\|_{\rho^{\sim}} = \rho^{\sim}(\cdot) + \lambda|||\cdot|||^{\sim}$ has the FPP for affine $\|\cdot\|_{\rho^{\sim}}$-nonexpansive self-mappings.

Acknowledgments

We would like to thank the anonymous referee for his/her thorough review and appreciate the comments and suggestions.

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