Filomat 32:16 (2018), 5691–5705
https://doi.org/10.2298/FIL1816691K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Inverse Problem for Euler-Bernoulli Equation with Periodic Boundary Condition

**Fatma Kanca**[a], **Irem Baglan**[b]

[a]*Faculty of Engineering and Architecture, Fenerbahce University, Istanbul, Turkey*
[b]*Department of Mathematics, Kocaeli University, Kocaeli 41380, Turkey*

**Abstract.** In this work the inverse coefficient problem for Euler-Bernoulli equation with periodic boundary and integral addition conditions is investigated. Under some natural regularity and consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the solution are shown by using the generalized Fourier method. Numerical tests using the implicit finite difference scheme combined with an iterative method are presented and discussed. Also an example is presented with figures.

## 1. Introduction

Inverse problems is a research area dealing with inversion of models or data. An inverse problem is a mathematical framework that is used to obtain information about a physical object or system from observed measurements. It is called an inverse problem because it starts with the results and then calculates the causes. This is the inverse of a direct problem, which starts with the causes and then calculates the results. Thus, inverse problems are some of the most important and well-studied mathematical problems in science and mathematics because they provide us about parameters that we cannot directly observe. There are many different applications including, medical imaging, geophysics, computer vision, astronomy, nondestructive testing, and many others. For example, if an acoustic plane wave is scattered by an obstacle, and one observes the scattered field far from the obstacle, or in some exterior region, then the inverse problem is to find the shape and material properties of the obstacle. Such problems are important in identification of flying objects (airplanes missiles, etc.), objects immersed in water (submarines, paces of fish, etc.) and in many other situations. In geophysics one sends an acoustic wave from the surface of the earth and collects the scattered field on the surface for various positions of the source of the field for a fixed frequency, or for several frequencies. The inverse problem is to find the subsurface inhomogeneities. In technology one measures the eigenfrequencies of a piece of a material, and the inverse problem is to find a defect in this material, for example, a hole in a metal. In geophysics the inhomogeneity can be an oil deposit, a cave, a mine. In medicine it may be a tumor or some abnormality in a human body. If one is able to

2010 *Mathematics Subject Classification*. Primary 35K55; Secondary 35K70
*Keywords*. Partial derivative, periodic boundary condition, quasi-linear, mixed problem, Euler-Bernoulli equation, Fourier method, non-linear infinite integral equations
Received: 08 July 2017; Revised: 07 March 2018; Accepted: 10 March 2018
Communicated by Ljubiša D.R. Kočinac
*Email addresses:* fatma.kanca@fbu.edu.tr (Fatma Kanca ), isakinc@kocaeli.edu.tr (Irem Baglan)

find inhomogeneities in a medium by processing the scattered field on the surface, then one does not have to drill a hole in a medium. This, in turn, avoids expensive and destructive evaluation. The practical advantages of remote sensing are what make the inverse problems important.

Mathematical modeling is advantage point to reach a solution in an engineering problem, so the accurate modeling of nonlinear engineering problems is an important step to obtain accurate solutions. Most differential equations of engineering problems do not have exact analytic solutions so approximation and numerical methods must be used. Recently some different methods have been introduced to solving these equations, such as Finite difference method, Finite Element method, Keller Box method, Boundary Element Method, etc.[1, 14, 15].

Mathematical modeling of sound wave distribution problems and also the vibration, buckling and dynamic behavior of various building elements widely used in nano-technology are formulated with following Euler-Bernoulli equations

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = f(x, t, u).$$

Due to the new and exceptionally its electronic and mechanical properties, carbon nanotubes are considered to be one of the most useful material in future. Nowadays, nanotubes are used as atomic force microscopy, nanofillers for composite materials, nanoscale electronic devices and even frictionless nanoactuators, nanomotors, nanobearings and nanosprings [8, 13, 16]. These elements are tackled by different boundary conditions depending on different loading conditions. Therefore, investigation of existence and uniqueness of the solution of Euler-Bernoulli equations with different boundary conditions used in the mathematical modeling of the structural components of nano-materials continues to be a focus of interest amongst mathematicians.

The main goal of this study is to investigate the solution of the unknown function in Euler-Bernoulli equation equation with periodic boundary conditions and integral overdetermination condition. We first obtain the formal solution of this problem using Fourier method. As the next step, we find the existence, uniqueness and continuous dependence of the solution using iteration method. Finally, we investigate the numerical solution of the inverse problem using linearization and finite difference method.

Let $T > 0$ be fixed number and denote by $\Omega := \{0 < x < \pi, 0 < t < T\}$ . Consider the problem of finding a pair of functions $\{r(t), u(x, t)\}$ satisfying the following equations

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - a(t)u = r(t)f(x, t, u), \ (x, t) \in \Omega \tag{1}$$

$$\begin{aligned} u(0, t) &= u(\pi, t) \\ u_x(0, t) &= u_x(\pi, t) \\ u_{xx}(0, t) &= u_{xx}(\pi, t) \\ u_{xxx}(0, t) &= u_{xxx}(\pi, t), \ t \in [0, T] \end{aligned} \tag{2}$$

$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x), \ x \in [0, \pi] \tag{3}$$

$$E(t) = \int_0^\pi u(x, t)dx, \ t \in [0, T] \tag{4}$$

for a quasilinear parabolic equation with the nonlinear source term $F = F(x,t,a,u) = a(t)u(x,t) + r(t)f(x,t,u)$, where $f(x,t,u), \varphi(x), \psi(x)$ and $E(t)$ are known functions which are positive and continuous. Determination of the pair of functions $\{r(t), u(x,t)\}$ is called the inverse problem.

Many problems of modern physics and technology can be effectively described in terms of nonlocal conditions. Also, these problems have many important applications in chemical diffusion, thermoelasticity, heat conduction processes, population dynamics, vibration problems, nuclear reactor dynamics, control theory, medical science, biochemistry and certain biological processes. For example, in the study of the heat conduction with in linear thermoelasticity, [4, 5] investigated a heat equation subject to nonlocal boundary conditions. Over the last years, considerable efforts have been put in to develop either approximate analytical solution and numerical solution to non-local boundary value problems. Cannon et al. [3] implemented implicit finite difference scheme to obtain numerical solution of the one dimensional non-local boundary value problems. [6, 7] studied non-local boundary value problems and concluded that the presence of integral terms in boundary conditions can greatly complicate the application of standard numerical techniques such as finite difference schemes, finite element techniques etc.[2, 11, 12]

The periodic conditions are used on lunar theory [9]. In heat propagation in a thin rod in which the law of variation $E(t)$ of the total quantity of heat in the rod is given in [10].

**Nomenclature**

$\varphi(x)$ Initial function
$r(t)$ Unknown coefficient
$a(t)$ Unknown functions
$E(t)$ Energy
$u(x,t)$ Temperature distribution
$f(x,t,u)$ Source function
$u_0(t), u_{ck}(t), u_{ck}(t)$ Fourier coefficients
$M$ Arbitrary constant
$N_1, N_2, N_3, N_4, N_5, N_6$ Dimensionless constants
$\Omega := \{0 < x < \pi,\ 0 < t < T\}$ Domain of $x, t$

## 2. Solution of the Inverse Problem

**Definition 2.1.** Determination of the pair of functions $\{r(t), u(x,t)\}$ is called the inverse problem.

**Definition 2.2.** $v(t,x) \in C(\overline{\Omega})$ is refered test function that gives following conditions:
$v(T,x) = v_t(T,x) = 0, v(0,t) = v(\pi,t), v_x(0,t) = v_x(\pi,t), v_{xx}(0,t) = v_{xx}(\pi,t), v_{xxx}(0,t) = v_{xxx}(\pi,t), t\epsilon\,[0,T]$.

**Definition 2.3.** $u(x,t) \in C(\overline{\Omega})$ is named generalized solution that gives following equation:

$$\int_0^T\int_0^\pi \left(\left\{\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} - a(t)v\right\}u - r(t)fv\right)dxdt - \int_0^\pi v(x,0)\psi(x)dx + \int_0^\pi v_t(x,0)\varphi(x)dx = 0.$$

Consider the following assumptions:
(A1) $E(t) \in C^2[0,T]$.
(A2)$\varphi(x) \in C^3[0,\pi], \psi(x) \in C^1[0,\pi], E(0) = \int_0^\pi \varphi(x)dx$,
(A3) Let the function $f(x,t,u)$ provide the following conditions in $\overline{\Omega} \times (-\infty,\infty)$
(1)

$$\left|\frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n}\right| \le b(x,t)\,|u - \tilde{u}|\ , n = 0, 1, 2,$$

where $b(x,t) \in L_2(D),\ b(x,t) \geq 0$,

(2) $f(x,t,u) \in C[0,\pi],\ t \in [0,T], \left|f(x,t,u)\right| \leq M$

(3) $\int_0^\pi f(x,t,u)\,dx \neq 0,\ \forall t \in [0,T]$.

By Fourier method, we obtain

$$\begin{aligned}
u_0 &= \varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)F(\xi,\tau,a,u)d\xi d\tau,\\
u_{ck} &= \varphi_{ck}\cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2}\sin(2k)^2 t + \frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi F(\xi,\tau,a,u)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau,\\
u_{sk} &= \varphi_{sk}\cos(2k)^2 t + \frac{\psi_{sk}}{(2k)^2}\sin(2k)^2 t + \frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi F(\xi,\tau,a,u)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau.
\end{aligned}$$

Let $F(x,t,a,u) = a(t)u(x,t) + r(t)f(x,t,u)$.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left[\varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\left(a(\tau)u(\xi,\tau) + r(t)f(\xi,\tau,u)\right)d\xi d\tau\right]\\
&+ \sum_{k=1}^\infty \cos 2kx\left[\varphi_{ck}\cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2}\sin(2k)^2 t\right]\\
&+ \sum_{k=1}^\infty \cos 2kx\left[\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(a(\tau)u(\xi,\tau) + r(t)f(\xi,\tau,u)\right)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau\right]\\
&+ \sum_{k=1}^\infty \sin 2kx\left[\varphi_{sk}\cos(2k)^2 t + \frac{\psi_{sk}}{(2k)^2}\sin(2k)^2 t\right]\\
&+ \sum_{k=1}^\infty \sin 2kx\left[\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(a(\tau)u(\xi,\tau) + r(t)f(\xi,\tau,u)\right)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau\right],
\end{aligned} \tag{5}$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(x)dx, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(x)\cos 2kxdx, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin 2kxdx$,

$\psi_0 = \frac{2}{\pi}\int_0^\pi \psi(x)dx, \psi_{ck} = \frac{2}{\pi}\int_0^\pi \psi(x)\cos 2kxdx, \psi_{sk} = \frac{2}{\pi}\int_0^\pi \psi(x)\sin 2kxdx$,

$f_0(t,u) = \frac{2}{\pi}\int_0^\pi f(x,t,u)dx,\ f_{ck}(t,u) = \frac{2}{\pi}\int_0^\pi f(x,t,u)\cos 2kxdx,\ f_{sk}(t,u) = \frac{2}{\pi}\int_0^\pi f(x,t,u)\sin 2kxdx,\ k = 1,2,3,\ldots$ .

Under the condition (A1)-(A3), differentiating (4), we obtain

$$\int_0^\pi u_{tt}(x,t)dx = E''(t), 0 \leq t \leq T. \tag{6}$$

From (5) and (6)

$$r(t)=\frac{E''(t)-a(t)E(t)}{\int\limits_0^{\pi} f(\xi,t,u)\,d\xi}. \tag{7}$$

**Definition 2.4.** Denote the set
Let $\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),k=1,...,n\}$ is satisfied that
$\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)<\infty$, by $\mathbf{B}_1$.
$\|u(t)\|=\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)$, be the norm where $\mathbf{B}_1$ is Banach space.

**Theorem 2.5.** *If the conditions (A1)-(A3) be ensured, then the Euler-Bernoulli problem has a unique solution.*

*Proof.* An iteration for (5) :

$$u_0^{(N+1)}(t) = u_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^{\pi}(t-\tau)\left(a(\tau)u^{(N)}(\xi,\tau)+r^{(N)}(\tau)f(\xi,\tau,u^{(N)})\right)d\xi d\tau, \tag{8}$$

$$u_{ck}^{(N+1)}(t) = u_{ck}^{(0)}(t)+\frac{2}{\pi(2k)^2}\int\limits_0^t\int\limits_0^{\pi}\left(a^{(N)}(\tau)u^{(N)}(\xi,\tau)+r^{(N)}(\tau)f(\xi,\tau,u^{(N)})\right)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau,$$

$$u_{sk}^{(N+1)}(t) = u_{sk}^{(0)}(t)+\frac{2}{\pi(2k)^2}\int\limits_0^t\int\limits_0^{\pi}\left(a^{(N)}(\tau)u^{(N)}(\xi,\tau)+r^{(N)}(\tau)f(\xi,\tau,u^{(N)})\right)\sin(2k)^2(t-\tau)\sin 2k\xi\, d\xi d\tau,$$

$$u_0^{(0)}(t)=\varphi_0+\psi_0 t,\ u_{ck}^{(0)}(t)=\varphi_{ck}\cos(2k)^2t+\frac{\psi_{ck}}{(2k)^2}\sin(2k)^2t,\ u_{sk}^{(0)}(t)=\varphi_{sk}\cos(2k)^2t+\frac{\psi_{sk}}{(2k)^2}\sin(2k)^2t.$$

$$r^{(N+1)}(t)=\frac{E''(t)-a(t)E(t)}{\int\limits_0^{\pi} f(\xi,t,u^{(N)})\,d\xi},$$

From condition of the theorem, $u^{(0)}(t)\in\mathbf{B}_1$, $t\in[0,T]$.

$$u_0^{(1)}(t)=u_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^{\pi}(t-\tau)\left(a(\tau)u^{(0)}(\xi,\tau)+r^{(0)}(\tau)f(\xi,\tau,u^{(0)})\right)d\xi d\tau$$

Adding and subtracting $\int\limits_0^t\int\limits_0^{\pi} r^{(0)}(\tau)f(\xi,\tau,0)d\xi d\tau$,after applying Cauchy, Bessel, Lipschitz inequalities consecutively, we get

$$\begin{aligned}\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| \le\ & \left|\varphi_0\right|+T\left|\psi_0\right|+2\sqrt{\frac{T^3}{3\pi}}\left\|u^{(0)}(t)\right\|_{B_1}\|a(t)\|_{C[0,T]}\\ &+2\sqrt{\frac{T^3}{3\pi}}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)}+2\sqrt{\frac{T^3}{3\pi}}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|f(x,t,0)\right\|_{L_2(D)}.\end{aligned}$$

$$u_{ck}^{(1)}(t) = u_{ck}^{(0)}(t) + \frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(a(\tau)u^{(0)}(\xi,\tau) + r^{(0)}(\tau)f(\xi,\tau,u^{(0)})\right)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau.$$

Adding and subtracting $\int_0^t\int_0^\pi r^{(0)}(\tau)f(\xi,\tau,0)d\xi d\tau$, after applying Cauchy, Bessel, Lipschitz, Hölder inequalities, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| \le{}& \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\pi^2}{24}\sum_{k=1}^{\infty}\left|\psi_{ck}\right| \\ &+\frac{\pi\sqrt{T}}{12}\left\|u^{(0)}(t)\right\|_{B_1}\|a(t)\|_{C[0,T]} + \frac{\pi\sqrt{T}}{12}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)} \\ &+\frac{\pi\sqrt{T}}{12}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|f(x,t,0)\right\|_{L_2(D)},\end{aligned}$$

and from the same approaches,

$$\begin{aligned}\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right| \le{}& \sum_{k=1}^{\infty}\left|\varphi_{sk}\right| + \frac{\pi^2}{24}\sum_{k=1}^{\infty}\left|\psi_{sk}\right| \\ &+\frac{\pi\sqrt{T}}{12}\left\|u^{(0)}(t)\right\|_{B_1}\|a(t)\|_{C[0,T]} + \frac{\pi\sqrt{T}}{12}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)} \\ &+\frac{\pi\sqrt{T}}{12}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|f(x,t,0)\right\|_{L_2(D)},\end{aligned}$$

Finally we have the following inequalities:

$$\begin{aligned}\left\|u^{(1)}(t)\right\|_{\mathbf{B}_1} ={}& \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right|\right) \\ \le{}& \frac{\left|\varphi_0\right|}{2} + \sum_{k=1}^{\infty}\left(\left|\varphi_{ck}\right| + \left|\varphi_{sk}\right|\right) + \frac{\pi^2}{24}\sum_{k=1}^{\infty}\left(\left|\psi_{ck}\right| + \left|\psi_{sk}\right|\right) \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_{B_1}\|a(t)\|_{C[0,T]} \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)} \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|f(x,t,0)\right\|_{L_2(D)}.\end{aligned}$$

From the conditions of the theorem $u^{(1)}(t) \in \mathbf{B}_1$.

Same estimations for the step $N$,

$$\begin{aligned}\left\|u^{((N+1)}(t)\right\|_{\mathbf{B}_1} &= \max_{0\le t\le T}\frac{\left|u_0^{(N)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{sk}^{(N)}(t)\right|\right)\\ &\le \frac{\left|\varphi_0\right|}{2}+\sum_{k=1}^{\infty}\left(\left|\varphi_{ck}\right|+\left|\varphi_{sk}\right|\right)+\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left(\left|\psi_{ck}\right|+\left|\psi_{sk}\right|\right)\\ &\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(N)}(t)\right\|_{B_1}\|a(t)\|_{C[0,T]}\\ &\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|r^{(N)}(t)\right\|_{C[0,T]}\left\|u^{(N)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)}\\ &\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|r^{(N)}(t)\right\|_{C[0,T]}\left\|f(x,t,0)\right\|_{L_2(D)}.\end{aligned}$$

According to $u^{(N)}(t)\in\mathbf{B}_1$ and from the conditions of the theorem, we have $u^{(N+1)}(t)\in\mathbf{B}_1$,

$$\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),\ k=1,2,...\}\in\mathbf{B}_1.$$

For $N\to\infty$, $u^{(N+1)}(t)$, $r^{(N+1)}$ are converged.

After applying Cauchy, Bessel, Lipschitz, Hölder inequalities consecutively, we have

$$\begin{aligned}A &= ((2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_{B_1}\|a(t)\|_{C[0,T]}+\\ &\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(D)}+\left\|f\right\|_{L_2(D)}).\end{aligned}$$

$$\left\|r^{(1)}(t)-r^{(0)}(t)\right\|_{C[0,T]}\le\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|u^{(1)}(t)-u^{(0)}\right\|_{B_1}\|b(x,t)\|_{L_2(D)},$$

where

$$\left|\int_0^{\pi}f\left(\xi,t,u\right)d\xi\right|\le M_0.$$

$$\begin{aligned}\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1} &\le (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{C[0,T]}\|a(t)\|_{C[0,T]}\\ &\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|r^{(1)}(t)\right\|_{C[0,T]}\left\|u^{(1)}-u^{(0)}\right\|_{B_1}\\ &\quad +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|r^{(1)}-r^{(0)}\right\|_{C[0,T]}\left\|f\right\|_{L_2(D)},\end{aligned}$$

$$\left\|r^{(2)}(t)-r^{(1)}(t)\right\|_{C[0,T]}\le\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|u^{(2)}(t)-u^{(1)}\right\|_{B_1}\|b(x,t)\|_{L_2(D)},$$

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1} \leq \frac{1}{M_*}(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})M_1\left\|u^{(1)}-u^{(0)}\right\|_{B_1}\|b(x,t)\|_{L_2(D)},$$

where

$$\begin{aligned}
M_* &= 1-(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\,\|a(t)\|_{C[0,T]}\\
M_1 &= \left\|r^{(1)}(t)\right\|_{B_1}+\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|f\right\|_{L_2(D)}\\
M_2 &= \left\|r^{(2)}(t)\right\|_{B_1}+\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|f\right\|_{L_2(D)}\\
&\;\;.\\
&\;\;.\\
&\;\;.\\
M_N &= \left\|r^{(N)}(t)\right\|_{B_1}+\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|f\right\|_{L_2(D)}
\end{aligned}$$

For the step $N$ :

$$\left\|r^{(N+1)}(t)-r^{(N)}(t)\right\|_{C[0,T]} \leq \frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|u^{(N+1)}(t)-u^{(N)}\right\|_{B_1}\|b(x,t)\|_{L_2(D)}.$$

$$\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{B_1} \leq (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{(M_1.M_2...M_N)}{M_*}\frac{A}{\sqrt{N!}}\|b(x,t)\|^N_{L_2(D)}. \tag{9}$$

$u^{(N+1)} \to u^{(N)}$ , $N \to \infty$, then $r^{(N+1)} \to r^{(N)}$, $N \to \infty$.

Let us show the following limits

$$\lim_{N\to\infty} u^{(N+1)}(t) = u(t),\ \lim_{N\to\infty} r^{(N+1)}(t) = r(t).$$

$$\begin{aligned}
\left\|u-u^{(N+1)}\right\|_{B_1} \leq\; & (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u(t)-u^{(N+1)}(t)\right\|_{C[0,T]}\|a(t)\|_{C[0,T]}\\
&+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{C[0,T]}\|a(t)\|_{C[0,T]}\\
&+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u-u^{(N+1)}\right\|_{B_1}\|r(t)\|_{C[0,T]}\|b(x,t)\|_{L_2(D)}\\
&+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|u^{(N+1)}-u^{(N)}\right\|_{B_1}\|r(t)\|_{C[0,T]}\|b(x,t)\|_{L_2(D)}\\
&+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|r(t)-r^{(N+1)}\right\|_{C[0,T]}\left\|f\right\|_{L_2(D)},
\end{aligned} \tag{10}$$

$$\left\|r(t)-r^{(N+1)}(t)\right\|_{C[0,T]} \leq \frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|u(t)-u^{(N+1)}\right\|_{B_1}\|b(x,t)\|_{L_2(D)}.$$

Let us consider (9) in (10) and apply Gronwall's inequality to (10) and taking maximum of both side of the last inequalities, we find the following inequalities

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}_1}^2 \leq \\ & 2\left[(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{A.M_1M_2...M_N}{M_*\sqrt{N!}}\|b(x,t)\|_{L_2(D)}^N\right]^2 \\ & \times\exp 2\left[(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left(\|r(t)\|_{B_1}+\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|f\right\|_{L_2(D)}\right)\right]^2.\end{aligned}$$

We obtain $u^{(N+1)}\to u$, $r^{(N+1)}\to r$, $N\to\infty$.

For the uniqueness, we assume that the problem (1)-(4) has two solution pair $(u,r)$, $(v,q)$. Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to $|u(t)-v(t)|$ and $\left|r(t)-q(t)\right|$, therefore we have

$$\begin{aligned}\|u(t)-v(t)\|_{B_1} \leq\ & (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\|u(t)-v(t)\|_{B_1}\|a(t)\|_{C[0,T]} \\ & +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|r(t)-q(t)\right\|_{C[0,T]}\left\|f\right\|_{L_2(D)} \\ & +(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left(\int_0^t\int_0^\pi r^2(\tau)b^2(\xi,\tau)\left|u(\tau)-v(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}},\end{aligned}$$

$$\left\|r(t)-q(t)\right\|_{C[0,T]} \leq \frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\|u(t)-v(t)\|_{B_1}\|b(x,t)\|_{L_2(D)},$$

$$\begin{aligned}\|u(t)-v(t)\|_{B_1} \leq\ & \frac{1}{M_*}(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6}) \\ & \left(\|r(t)\|_{C[0,T]}+\frac{\left|E''(t)-a(t)E(t)\right|}{M_0}\left\|f\right\|_{L_2(D)}\right)\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|u(\tau)-v(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}},\end{aligned}$$

applying Gronwall's inequality to the last equations, we have
$u(t)=v(t)$. Hence $r(t)=q(t)$.
The theorem is proved. □

## 3. Continuous Dependence of (r,u) Upon the Data

**Theorem 3.1.** *Under assumptions (A1)-(A3) the solution $(r,u)$ of the problem (1)-(4) depends continuously upon the data $\varphi, E$.*

*Proof.* Let $\Phi=\{\varphi,\ \psi,\ f\}$ and $\overline{\Phi}=\left\{\overline{\varphi},\ \overline{\psi},\ f\right\}$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Suppose that there exist positive constant $N_1$ such that

$$\|E\|_{C^1[0,T]}\leq N_1,\left\|\overline{E}\right\|_{C^1[0,T]}\leq N_1$$

Let us denote $\|\Phi\| = (\|\varphi\|_{C^3[0,\pi]} + \|\psi\|_{C^3[0,\pi]} + \|f\|_{C^{3,0}(\overline{D})})$. Let $(r, u)$ and $(\overline{r}, \overline{u})$ be solutions of inverse problems (1)-(4) corresponding to the data $\Phi = \{\varphi,\ \psi,\ f\}$and $\overline{\Phi} = \left\{\overline{\varphi},\ \overline{\psi},\ f\right\}$ respectively. According to (5), we have

$$\begin{aligned} u - \overline{u} \;\; = \;\; & \frac{(\varphi_0 - \overline{\varphi_0})}{2} + \frac{\left(\psi_0 - \overline{\psi_0}\right)t}{2} + \sum_{k=1}^{\infty} \cos 2kx\ (\varphi_{ck} - \overline{\varphi_{ck}}) \cos(2k)^2 t + \sum_{k=1}^{\infty} \sin 2kx\, (\varphi_{sk} - \overline{\varphi_{sk}}) \sin(2k)^2 t \\ & + \frac{1}{2}\left(\frac{2}{\pi} \int_0^t \int_0^\pi (t-\tau)\left(u(\tau) - \overline{u(\tau)}\right) a(\tau) d\xi d\tau\right) \\ & + \frac{1}{2}\left(\frac{2}{\pi} \int_0^t \int_0^\pi (t-\tau) r(\tau) \left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] d\xi d\tau\right) \\ & + \frac{1}{2}\left(\frac{2}{\pi} \int_0^t \int_0^\pi (t-\tau)\left(r(\tau) - \overline{r(\tau)}\right) f(\xi,\tau,\overline{u}(\xi,\tau)) d\xi d\tau\right) \\ & + \sum_{k=1}^{\infty} \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi \left(u(\tau) - \overline{u(\tau)}\right) a(\tau) \sin(2k)^2(t-\tau) \cos 2k\xi d\xi d\tau \\ & + \sum_{k=1}^{\infty} \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi r(\tau) \left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] \sin(2k)^2(t-\tau) \cos 2k\xi d\xi d\tau \\ & + \sum_{k=1}^{\infty} \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi \left(r(\tau) - \overline{r(\tau)}\right) f(\xi,\tau,\overline{u}(\xi,\tau)) \sin(2k)^2(t-\tau) \cos 2k\xi d\xi d\tau \\ & + \sum_{k=1}^{\infty} \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi \left(u(\tau) - \overline{u(\tau)}\right) a(\tau) \sin(2k)^2(t-\tau) \sin 2k\xi d\xi d\tau \\ & + \sum_{k=1}^{\infty} \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi r(\tau) \left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] \sin(2k)^2(t-\tau) \sin 2k\xi d\xi d\tau \\ & + \sum_{k=1}^{\infty} \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi \left(r(\tau) - \overline{r(\tau)}\right) f(\xi,\tau,\overline{u}(\xi,\tau)) \sin(2k)^2(t-\tau) \sin 2k\xi d\xi d\tau, \end{aligned}$$

By using same estimations as in Theorem 2.5, we obtain

$$\begin{aligned} \left\|u(t) - \overline{u}(t)\right\|_{B_1} \;\; \leq \;\; & \frac{\left\|\varphi - \overline{\varphi}\right\| + \left\|\psi - \overline{\psi}\right\|}{M_*} + (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \|r(t)\| \left\|u(t) - \overline{u}(t)\right\|_{B_1} \\ & + (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) M \left\|r(t) - \overline{r}(t)\right\|_{C[0,T]}, \end{aligned}$$

$$\left\|r(t) - \overline{r}(t)\right\|_{C[0,T]} \leq \frac{\left\|E - \overline{E}\right\|}{M_0} \|b(x,t)\|_{L_2(D)} \left\|u(t) - \overline{u}(t)\right\|_{B_1} + N_2,$$

where

$$N_2 = 2MN_1\left\|a(t)\right\| + 2MN_1.$$

$$\left\|u(t)-\overline{u}(t)\right\|_{B_1} \le \left\|\Phi-\overline{\Phi}\right\| + N_3\left(\int_0^t\int_0^\pi \left|u-\overline{u}\right|^2 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}, \tag{11}$$

where

$$N_3 = \frac{1}{M_*}\left(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}\right)\left\|r(t)\right\|.$$

Applying Gronwall's inequality to (11), we obtain

$$\begin{aligned}\left\|u(t)-\overline{u}(t)\right\|_{B_1}^2 &\le 2\left\|\Phi-\overline{\Phi}\right\|^2 \\ &\quad\times \exp 2N_5^2\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right),\end{aligned}$$

where

$$\begin{aligned}\left\|\Phi-\overline{\Phi}\right\| &= \frac{\left\|\varphi-\overline{\varphi}\right\| + \left\|\psi-\overline{\psi}\right\|}{M_*} \\ &\quad+\frac{1}{M_*}\left(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}\right)\frac{\left\|E-\overline{E}\right\|}{M_0}M \\ N_5 &= \max\{N_3, N_4\}.\end{aligned}$$

For $\Phi \to \overline{\Phi}$ then $u \to \overline{u}$. Hence $r \to \overline{r}$. □

## 4. Numerical Method for the Problem (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\frac{\partial^2 u^{(n)}}{\partial t^2} + \frac{\partial^4 u^{(n)}}{\partial x^4} - a(t)u^{(n)} = r(t)f(x,t,u^{(n-1)}),\ (x,t)\in\Omega, \tag{12}$$

$$\begin{aligned} u^{(n)}(0,t) &= u^{(n)}(\pi,t) \\ u_x^{(n)}(0,t) &= u_x^{(n)}(\pi,t) \\ u_{xx}^{(n)}(0,t) &= u_{xx}^{(n)}(\pi,t) \\ u_{xxx}^{(n)}(0,t) &= u_{xxx}^{(n)}(\pi,t), t\in[0,T], \end{aligned} \tag{13}$$

$$u^{(n)}(x,0) = \varphi(x), u_t^{(n)}(x,0) = \psi(x)\ , x\in[0,\pi], \tag{14}$$

$$E(t) = \int_0^\pi u^{(n)}(x,t)dx,\ t \in [0,T] \tag{15}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then the problem (12)-(15) can be written as a linear problem:

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} - a(t)v = r(t)\widetilde{f}(x,t),\ (x,t) \in \Omega, \tag{16}$$

$$\begin{aligned} v(0,t) &= v(\pi,t) \\ v_x(0,t) &= v_x(\pi,t) \\ v_{xx}(0,t) &= v_{xx}(\pi,t) \\ v_{xxx}(0,t) &= v_{xxx}(\pi,t),\ t \in [0,T], \end{aligned} \tag{17}$$

$$v(x,0) = \varphi(x), v_t(x,0) = \psi(x),\ x \in [0,\pi], \tag{18}$$

$$E(t) = \int_0^\pi v(x,t)dx,\ t \in [0,T] \tag{19}$$

We use finite-difference approximation for discretizing problem (16)-(19):

$$\frac{1}{\tau^2}\left(v_i^{j+1} - 2v_i^j + v_i^{j-1}\right) + \frac{1}{h^4}\left(v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j\right) = a^j v_i^j + r^{(t)} j \widetilde{f}_i^j$$

$$v_i^0 = \phi_i,\ \frac{1}{\tau}\left(v_i^1 - v_i^0\right) = \psi_i \tag{20}$$

$$v_0^j = v_{N_x+1}^j, \tag{21}$$

$$v_1^j = v_{N_x+2}^j, \tag{22}$$

$$v_{-1}^j = v_{N_x}^j, \tag{23}$$

$$v_2^j - v_{-2}^j = v_{N_x+3}^j - v_{N_x-1}^j, \tag{24}$$

The domain $[0,\pi] \times [0,T]$ is divided into an $N_x \times N_t$ mesh with the spatial step size $h = \pi/N_x$ in $x$ direction and the time step size $\tau = T/N_t$, respectively.

Grid points $x_i$, $t_j$ are defined by
$x_i = ih; i = 0; 1; 2; ...; N_x;$
$t_j = j\tau; j = 0; 1; 2; ...; N_t;$
$v_i^j = u(x_i, t_j)$, $\widetilde{f}_i^j = \widetilde{f}(x_i, t_j)$, $a^j = a(t_j)$, $r^j = r(t_j)$.
Let us integrate the equation (12) respect to $x$ from 0 to $\pi$ and use (13) and (15), we obtain

$$r(t) = \frac{[E''(t) - a(t)E(t)]}{\int_0^\pi \widetilde{f}(x,t)dx}. \tag{25}$$

The finite difference approximation of (25) is

$$r^j = \frac{\left[\left(\left(E^{j+1} - 2E^j + E^{j-1}\right)/\tau^2\right) - a^j E^j\right]}{\left(\int_0^\pi \widetilde{f}_i^j dx\right)}.$$

where $E^j = E(t_j)$, $a^j = a(t_j)$, $j = 0, 1, ..., N_t$. We mention that the integral is numerically calculated using Simpson's rule of integration.

In order to illustrate the behavior of our numerical method, an example is considered.

**Example 4.1.** This example investigates finding the exact solution

$$\{r(t),\ u(x,t)\} = \{\exp(t),\ (1 + \sin 2x)\exp(t)\}.$$

for the given functions

$$\begin{aligned} \varphi(x) &= (1 + \sin 2x), E(t) = \pi \exp(t), \\ f(x,t,u) &= (1 - \exp(3t))\, u + 16 \sin 2x \exp(t). \end{aligned}$$

The step sizes are $h = 0.0393$, $\tau = 0.005$.

The comparisons between the exact solution and the numerical finite difference solution are shown in Figures 1 and 2 when $T = 2$.

## References


[1] A.A. Ahmadi Asoor, P. Valipour, S.E. Ghasemi, D.D. Ganji, Mathematical Modelling of Carbon Nanotube with FluidFlow using Keller Box Method: A Vibrational Study, Int. J. Appl. Comput. Math. 3 (2016) 1689–1701.

[2] I. Baglan, F. Kanca, An inverse coefficient problem for a quasilinear parabolic equation with periodic boundary and integral overdetermination condition, Math. Methods Appl. Sci. 38 (2015) 851–867.

[3] J.R. Cannon, Y. Lin, Determination of parameter p(t) in Hölder classes for some semilinear parabolic equations, Inverse Problems Sci. Engin. 4 (2001) 595–606.

[4] W.A. Day, Heat Conduction Within Linear Thermoelasticity, Springer-Verlag, New York, 1985.

[5] W.A. Day, A decreasing property of solutions of a parabolic equation with applications to thermoelasticity, Quart. Appl. Math. 41 (1987) 468–475.

[6] M. Dehghan, Implicit locally one-dimensional methods for two-dimensional diffusion with a nonlocal boundary condition, Math. Comput. Simul. 49 (1999) 331–349.

[7] M. Dehghan, Parameter determination in a partial differential equation from the overspecified data, Math. Comput. Model. 41 (2005) 197–213.

[8] X.Q. He, S. Kitipornchai, K.M. Liew, Buckling analysis of multi-walled carbon nanotubes: a continuum model accounting for van der Waals interaction, J.e Mech. Physics Solids 53 (2005) 303–326.

[9] G.M. Hill, On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon, Acta Math. 8 (1886) 1–36.

[10] N.I. Ionkin, Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, Difference Equations 13 (1997) 204–211.

[11] F. Kanca, I. Baglan, An inverse coefficient problem for a quasilinear parabolic equation with nonlocal boundary conditions, Boundary Value Problems 2013:213 (2013) doi:10.1186/1687-2770-2013-213.

[12] F. Kanca, I. Baglan, An inverse problem for a quasilinear parabolic equation with nonlocal boundary and overdetermination conditions, J. Ineq. Appl. 76 (2014) doi:10.1186/1029-242X-2014-76.
[13] T. Natsuki, Q.Q. Ni, M. Endo, Wave propagation in single-and double-walled carbon nano tubes filled with fluids, J. Appl. Physics 101 (2007) 034319.
[14] M. Sheikholeslami, D.D. Ganji, Nanofluid convective heat transfer using semi analytical and numerical approaches, J. Taiwan Inst. Chemical Engin. 65 (2016) 43–77.
[15] M. Sheikholeslami, D.D. Ganji, M.M. Rashidi, Magnetic field effect on unsteady nanofluid flowand heat transfer using Buongiorno model, J. Magnetism Magnetic Materials 416 (2016) 164–173.
[16] Y. Yan, X.Q. He, L.X. Zhang, C.M. Wang, Dynamic behavior of triple-walled carbon nano-tubes conveying fluid, J. Sound Vibration 319 (2010) 1003–1018.

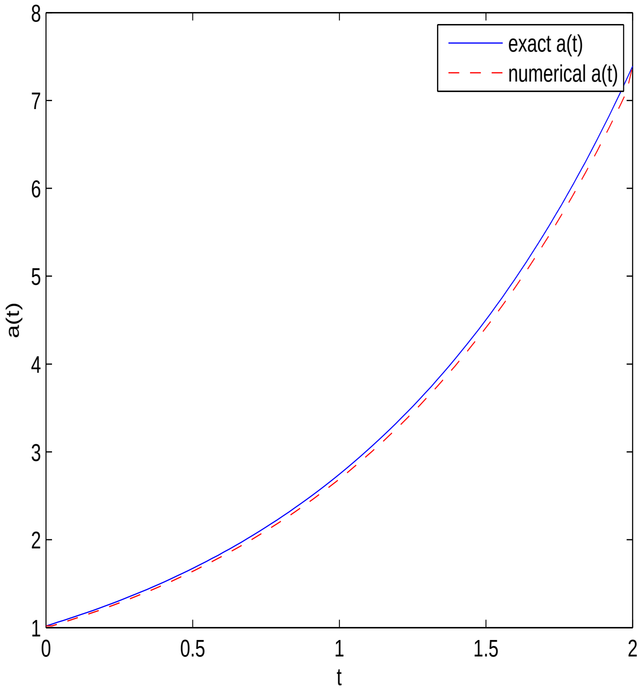


Figure 1: The exact and approximate solutions of r(t). The approximate solution is shown with dashed line.

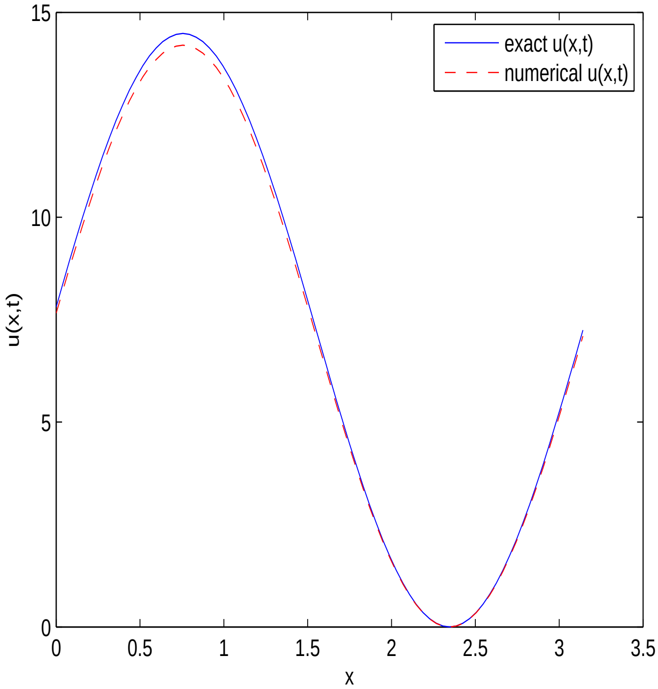


Figure 2: The exact and approximate solutions of u(x,2). The approximate solution is shown with dashed line.