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On Some New Inequalities via GG-Convexity and $GA-$Convexity

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Abstract. In this paper, we established some integral inequalities for functions whose derivatives of absolute values are $GG-$convex and $GA-$convex.

1. Introduction

We will start with the definition of convexity.

Definition 1.1. The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on I, if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. We say that f is concave if $-f$ is convex.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function where $a, b \in I$ with $a < b$. Then the following double inequality hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

This inequality is well-known in the literature as Hermite-Hadamard inequality that gives us upper and lower bounds for the mean-value of a convex function. If f is concave function both of the inequalities in above hold in reversed direction.

Anderson et al. mentioned mean function in [2] as following:

Definition 1.2. A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if
(1) $M(x, y) = M(y, x)$,
(2) $M(x, x) = x$,
(3) $x < M(x, y) < y$, whenever $x < y$,
(4) $M(ax, ay) = aM(x, y)$ for all $a > 0$.

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Based on the definition of mean function, let us recall special means (see [2])
1. Arithmetic Mean: $M(x,y) = A(x,y) = \frac{x+y}{2}$.
2. Geometric Mean: $M(x,y) = G(x,y) = \sqrt{xy}$.
3. Harmonic Mean: $M(x,y) = H(x,y) = 1/A\left(\frac{1}{x}, \frac{1}{y}\right)$.
4. Logarithmic Mean: $M(x,y) = L(x,y) = (x-y)/(\log x - \log y)$ for $x \neq y$ nd $L(x,x) = x$.
5. Identric Mean: $M(x,y) = I(x,y) = (1/e)\left(x^x/y^y\right)^{1/(x-y)}$ for $x \neq y$ nd $I(x,x) = x$.
In [2], Anderson et al. also gave a definition that include several different classes of convex functions as the following:

Definition 1.3. Let $f : I \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y))$$

for all $x, y \in I$.

In [10], Niculescu mentioned the following considerable definitions:

Definition 1.4. The $GG-$convex functions are those functions $f : I \to J$ (acting on subintervals of $(0,\infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1}$$

Definition 1.5. The class of all GA-convex functions is constituted by all functions $f : I \to \mathbb{R}$ (acting on subintervals of $(0,\infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leq (1-\lambda) f(x) + \lambda f(y). \tag{2}$$

In [3], the authors proved the following lemma and establish new inequalities.

Lemma 1.6. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ then the following identity holds:*

$$\begin{aligned} & bf(b) - af(a) - \int_a^b f(u)du \\ = \; & (\ln x - \ln a)\int_0^1 \left(x^{2t}a^{2(1-t)}\right) f'\left(x^t a^{1-t}\right)dt - (\ln x - \ln b)\int_0^1 \left(x^{2t}b^{2(1-t)}\right) f'\left(x^t b^{1-t}\right)dt \end{aligned}$$

for all $x \in [a,b]$.

For recent results, generalizations, improvements see the papers [1–11].
The main aim of this paper is to prove some new integral inequalities for $GG-$convex and $GA-$convex functions by using a new integral identity.

2. New Inequalities for $GG-$Convex Functions

We need the following integral identity to get our new results.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta, \mu \in I^\circ$ with $\eta < \mu$. If $f' \in L([\eta,\mu])$ for all $\xi \in [\eta,\mu]$, then the following equality holds:*

$$\begin{aligned} & \mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi \\ = \; & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right) f'\left(\mu^\tau \xi^{1-\tau}\right)d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right) f'\left(\xi^\tau \eta^{1-\tau}\right)d\tau. \end{aligned}$$

Proof. Let

$$I_1 = \int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right) f'\left(\mu^{\tau}\xi^{1-\tau}\right) d\tau$$

and

$$I_2 = \int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right) f'\left(\xi^{\tau}\eta^{1-\tau}\right) d\tau.$$

We notice that

$$\begin{aligned} I_1 &= \int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right) f'\left(\mu^{\tau}\xi^{1-\tau}\right) d\tau \\ &= \frac{1}{\ln\mu - \ln\xi} \int_0^1 \left(\mu^{2\tau}\xi^{2(1-\tau)}\right) f'\left(\mu^{\tau}\xi^{1-\tau}\right) d\left(\mu^{\tau}\xi^{1-\tau}\right). \end{aligned}$$

By the change of the variable $\psi = \mu^{\tau}\xi^{1-\tau}$ and integrating by parts, we have

$$I_1 = \frac{1}{\ln\mu - \ln\xi}\left[\mu^2 f(\mu) - \xi^2 f(\xi) - 2\int_{\xi}^{\mu} \psi f(\psi) d\psi\right].$$

Conformably, we have

$$I_2 = \frac{1}{\ln\xi - \ln\eta}\left[\xi^2 f(\xi) - \eta^2 f(\eta) - 2\int_{\eta}^{\xi} \psi f(\psi) d\psi\right].$$

Multiplying I_1 by $(\ln\mu - \ln\xi)$, I_2 by $(\ln\xi - \ln\eta)$ and adding the results we get the desired identity. □

Our first result is given in the following theorem.

Theorem 2.2. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta,\mu \in I^\circ$ with $\eta < \mu$ and $f' \in L([\eta,\mu])$. If $\left|f'\right|$ is $GG-$ convex on $[\eta,\mu]$, for all $\xi \in [\eta,\mu]$ then the following inequality holds:*

$$\begin{aligned} &\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_{\eta}^{\mu} \psi f(\psi) d\psi\right| \\ \leq\ & (\ln\mu - \ln\xi)\, L\left(\mu^3\left|f'(\mu)\right|, \xi^3\left|f'(\xi)\right|\right) + (\ln\xi - \ln\eta)\, L\left(\xi^3\left|f'(\xi)\right|, \eta^3\left|f'(\eta)\right|\right). \end{aligned}$$

Proof. From Lemma 2.1, using the property of the modulus and $GG-$ convexity of $\left|f'\right|$, we can write

$$\begin{aligned} &\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_{\eta}^{\mu} \psi f(\psi) d\psi\right| \\ \leq\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^{\tau}\xi^{1-\tau}\right)\right| d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^{\tau}\eta^{1-\tau}\right)\right| d\tau \\ \leq\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'(\mu)\right|^{\tau}\left|f'(\xi)\right|^{1-\tau} d\tau \\ & + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'(\eta)\right|^{\tau}\left|f'(\mu)\right|^{1-\tau} d\tau \\ =\ & (\ln\mu - \ln\xi)\,\xi^3\left|f'(\xi)\right|\int_0^1 \left(\frac{\mu^3\left|f'(\mu)\right|}{\xi^3\left|f'(\xi)\right|}\right)^{\tau} d\tau + (\ln\xi - \ln\eta)\,\eta^3\left|f'(\eta)\right|\int_0^1 \left(\frac{\xi^3\left|f'(\xi)\right|}{\eta^3\left|f'(\eta)\right|}\right)^{\tau} d\tau. \end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 2.3. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta,\mu \in I^\circ$ with $\eta < \mu$ and $f' \in L([\eta,\mu])$. If $\left|f'\right|^q$ is GG− convex on $[\eta,\mu]$ for all $\xi \in [\eta,\mu]$, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu - \ln\xi)\left(L\left(\mu^{3p},\xi^{3p}\right)\right)^{\frac{1}{p}}\left(L\left(\left|f'(\mu)\right|^q,\left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
& + (\ln\xi - \ln\eta)\left(L\left(\xi^{3p},\eta^{3p}\right)\right)^{\frac{1}{p}}\left(L\left(\left|f'(\xi)\right|^q,\left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}},
\end{aligned}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GG-$ convexity of $\left|f'\right|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu - \ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau + (\ln\xi - \ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu - \ln\xi)\left(\int_0^1 \mu^{3\tau p}\xi^{3(1-\tau)p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
& + (\ln\xi - \ln\eta)\left(\int_0^1 \xi^{3\tau p}\eta^{3(1-\tau)p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
\leq\ & (\ln\mu - \ln\xi)\left(\xi^{3p}\int_0^1\left(\frac{\mu^{3p}}{\xi^{3p}}\right)^\tau d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(\mu)\right|^{q\tau}\left|f'(\xi)\right|^{(1-\xi)q}d\tau\right)^{\frac{1}{q}} \\
& + (\ln\xi - \ln\eta)\left(\eta^{3p}\int_0^1\left(\frac{\xi^{3p}}{\eta^{3p}}\right)^\tau d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(\xi)\right|^{q\tau}\left|f'(\eta)\right|^{(1-\xi)q}d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 2.4. *Under the assumptions of Theorem 2.3, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu - \ln\xi)\left(L\left(\mu^{3q}\left|f'(\mu)\right|^q,\xi^{3q}\left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
& + (\ln\xi - \ln\eta)\left(L\left(\xi^{3q}\left|f'(\xi)\right|^q,\eta^{3q}\left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}},
\end{aligned}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GG-$ convexity of $\left|f'\right|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
= \;& (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq \;& (\ln\mu-\ln\xi)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \mu^{3\tau q}\xi^{3(1-\tau)q}\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \xi^{3\tau p}\eta^{3(1-\tau)q}\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
\leq \;& (\ln\mu-\ln\xi)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \mu^{3\tau q}\xi^{3(1-\tau)q}\left|f'(\mu)\right|^{q\tau}\left|f'(\xi)\right|^{(1-\tau)q} d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \xi^{3\tau p}\eta^{3(1-\tau)q}\left|f'(\xi)\right|^{q\tau}\left|f'(\eta)\right|^{(1-\tau)q} d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

If we calculate the integrals above, we get the desired result. □

Theorem 2.5. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta,\mu \in I^\circ$with $\eta < \mu$ and $f' \in L([\eta,\mu])$. If $\left|f'\right|^q$ is GG− convex on $[\eta,\mu]$ for all $\xi \in [\eta,\mu]$, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq \;& (\ln\mu-\ln\xi)\left(L(\mu^3,\xi^3)\right)^{1-\frac{1}{q}}\left(L\left(\mu^3\left|f'(\mu)\right|^q,\xi^3\left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(L(\xi^3,\eta^3)\right)^{1-\frac{1}{q}}\left(L\left(\xi^3\left|f'(\xi)\right|^q,\eta^3\left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}},
\end{aligned}$$

for $q \geq 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GG-$ convexity of $\left|f'\right|^q$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
= \;& (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq \;& (\ln\mu-\ln\xi)\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau a^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
\leq \;& (\ln\mu-\ln\xi)\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'(\mu)\right|^{q\tau}\left|f'(\xi)\right|^{(1-\tau)q} d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'(\xi)\right|^{q\tau}\left|f'(\eta)\right|^{(1-\tau)q} d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

We get the desired result by a simple calculation. □

3. New Inequalities for $GA-$Convex Functions

In this section, we obtain some inequalities for $GA-$ convex functions.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta,\mu \in I^\circ$ with $\eta < \mu$ and $f' \in L([\eta,\mu])$. If $\left|f'\right|$ is $GA-$ convex on $[\eta,\mu]$ for all $\xi \in [\eta,\mu]$, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq\ & \frac{\left|f'(\mu)\right|}{3}\left[\mu^3 - L(\xi^3,\mu^3)\right] + \frac{\left|f'(\xi)\right|}{3}\left[L(\xi^3,\mu^3) - L(\eta^3,\xi^3)\right] + \frac{\left|f'(\eta)\right|}{3}\left[L(\eta^3,\xi^3) - \eta^3\right].
\end{aligned}$$

Proof. From Lemma 2.1, using the property of the modulus, $GA-$ convexity of $\left|f'\right|$, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left[\tau\left|f'(\mu)\right| + (1-\tau)\left|f'(\xi)\right|\right]d\tau \\
& + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left[\tau\left|f'(\xi)\right| + (1-\tau)\left|f'(\eta)\right|\right]d\tau
\end{aligned}$$

We get the desired result by a simple calculation. □

Theorem 3.2. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta,\mu \in I^\circ$ with $\eta < \mu$ and $f' \in L([\eta,\mu])$. If $\left|f'\right|^q$ is $GA-$ convex on $[\eta,\mu]$ for all $\xi \in [\eta,\mu]$, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu - \ln\xi)^{1-\frac{1}{q}} L^{1-\frac{1}{q}}\left(\xi^3,\mu^3\right)\left(\frac{\left|f'(\mu)\right|^q\left[\mu^3 - L(\xi^3,\mu^3)\right] + \left|f'(\xi)\right|^q\left[L(\xi^3,\mu^3) - \xi^3\right]}{3}\right)^{\frac{1}{q}} \\
& + (\ln\xi - \ln\eta)^{1-\frac{1}{q}} L^{1-\frac{1}{q}}\left(\eta^3,\xi^3\right)\left(\frac{\left|f'(\xi)\right|^q\left[\xi^3 - L(\eta^3,\xi^3)\right] + \left|f'(\eta)\right|^q\left[L(\eta^3,\xi^3) - \eta^3\right]}{3}\right)^{\frac{1}{q}},
\end{aligned}$$

for $q \geq 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GA-$ convexity of $\left|f'\right|^q$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu - \ln\xi)\left(\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left[\tau\left|f'(\mu)\right|^q + (1-\tau)\left|f'(\xi)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
& + (\ln\xi - \ln\eta)\left(\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left[\tau\left|f'(\xi)\right|^q + (1-\tau)\left|f'(\eta)\right|^q\right]d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

We get the desired result by a simple calculation. □

Remark 3.3. In Theorem 3.2, if we choose $q = 1$, Theorem 3.2 reduces to Theorem 3.1.

Theorem 3.4. *Under the assumptions of Theorem 2.3, but now for $|f'|^q$ GA−convex, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq\ & (\ln\mu - \ln\xi)\left(L\left(\mu^{\frac{3q}{q-1}}, \xi^{\frac{3q}{q-1}}\right)\right)^{1-\frac{1}{q}} \left(A\left(\left|f'(\mu)\right|^q, \left|f'(\xi)\right|^q\right)\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(L\left(\xi^{\frac{3q}{q-1}}, \eta^{\frac{3q}{q-1}}\right)\right)^{1-\frac{1}{q}} \left(A\left(\left|f'(\xi)\right|^q, \left|f'(\eta)\right|^q\right)\right)^{\frac{1}{q}},
\end{aligned}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GA-$ convexity of $\left|f'\right|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
=\ & (\ln\mu - \ln\xi)\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau + (\ln\xi - \ln\eta)\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\leq\ & (\ln\mu - \ln\xi)\left(\int_0^1 \left(\mu^{3\tau}\xi^{3(1-\tau)}\right)^p d\tau\right)^{1-\frac{1}{q}} \left(\int_0^1 \tau\left|f'(\mu)\right|^q + (1-\tau)\left|f'(\xi)\right|^q d\tau\right)^{\frac{1}{q}} \\
&+ (\ln\xi - \ln\eta)\left(\int_0^1 \left(\xi^{3\tau}\eta^{3(1-\tau)}\right)^p d\tau\right)^{1-\frac{1}{q}} \left(\int_0^1 \tau\left|f'(\xi)\right|^q + (1-\tau)\left|f'(\eta)\right|^q d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

We get the desired result by simple calculations. □

Theorem 3.5. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° where $\eta, \mu \in I^\circ$with $\eta < \mu$ and $f' \in L([\eta,\mu])$. If $\left|f'\right|^q$ is GA− convex on $[\eta,\mu]$ for all $\xi \in [\eta,\mu]$, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\leq\ & \frac{(\ln\mu - \ln\xi)^{1-\frac{1}{q}}}{q^{\frac{1}{q}}}\left(K_q(\mu,\xi)\right)^{\frac{1}{q}} + \frac{(\ln\xi - \ln\eta)^{1-\frac{1}{q}}}{q^{\frac{1}{q}}}\left(K_q(\xi,\eta)\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
K_q(\mu,\xi) &= \frac{\left|f'(\mu)\right|^q\left[\mu^{3q} - L(\xi^{3q},\mu^{3q})\right] + \left|f'(\xi)\right|^q\left[L(\xi^{3q},\mu^{3q}) - \xi^{3q}\right]}{3} \\
K_q(\xi,\eta) &= \frac{\left|f'(\xi)\right|^q\left[\xi^{3q} - L(\eta^{3q},\xi^{3q})\right] + \left|f'(\eta)\right|^q\left[L(\eta^{3q},\xi^{3q}) - \eta^{3q}\right]}{3}
\end{aligned}$$

for all $\psi \in [\eta,\mu]$ and $q \geq 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GA-$ convexity of $\left|f'\right|^q$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
=\;& (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\le\;& (\ln\mu-\ln\xi)\left(\int_0^1 d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\mu^{3\tau}\xi^{3(1-\tau)}\right]^q\left[\tau\left|f'(\mu)\right|^q+(1-\tau)\left|f'(\xi)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1 d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\xi^{3\tau}\eta^{3(1-\tau)}\right]^q\left[\tau\left|f'(\xi)\right|^q+(1-\tau)\left|f'(\eta)\right|^q\right]d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

A simple calculation gives the desired result. □

Theorem 3.6. *Let $f: I\subseteq\mathbb{R}_+=(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I° where $\eta,\mu\in I^\circ$with $\eta<\mu$ and $f'\in L([\eta,\mu])$. If $\left|f'\right|^q$ is $GA-$ convex on $[\eta,\mu]$ for all $\xi\in[\eta,\mu]$, the following inequality holds:*

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
\le\;& \frac{(\ln\mu-\ln\xi)^{1-\frac{1}{q}}}{(p)^{\frac{1}{q}}}\left[L\left(\mu^{\frac{3q-3p}{q-1}},\xi^{\frac{3q-3p}{q-1}}\right)\right]^{\frac{q-1}{q}}\left(K_q(\mu,\xi)\right)^{\frac{1}{q}} \\
&+\frac{(\ln\xi-\ln\eta)^{1-\frac{1}{q}}}{(p)^{\frac{1}{q}}}\left[L\left(\xi^{\frac{3q-3p}{q-1}},\eta^{\frac{3q-3p}{q-1}}\right)\right]^{\frac{q-1}{q}}\left(K_q(\xi,\eta)\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
K_q(\mu,\xi) &= \frac{\left|f'(\mu)\right|^q\left[\mu^{3q}-L(\xi^{3q},\mu^{3q})\right]+\left|f'(\xi)\right|^q\left[L(\xi^{3q},\mu^{3q})-\xi^{3q}\right]}{3} \\
K_q(\xi,\eta) &= \frac{\left|f'(\xi)\right|^q\left[\xi^{3q}-L(\eta^{3q},\xi^{3q})\right]+\left|f'(\eta)\right|^q\left[L(\eta^{3q},\xi^{3q})-\eta^{3q}\right]}{3}
\end{aligned}$$

for all $\psi\in[\eta,\mu]$ and $q\ge 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GA-$ convexity of $\left|f'\right|^q$ and power-mean integral inequality, we can write

$$\begin{aligned}
&\left|\mu^2 f(\mu)-\eta^2 f(\eta)-2\int_\eta^\mu \psi f(\psi)d\psi\right| \\
=\;& (\ln\mu-\ln\xi)\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)\left|f'\left(\mu^\tau\xi^{1-\tau}\right)\right|d\tau+(\ln\xi-\ln\eta)\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)\left|f'\left(\xi^\tau\eta^{1-\tau}\right)\right|d\tau \\
\le\;& (\ln\mu-\ln\xi)\left(\int_0^1\left(\mu^{3\tau}\xi^{3(1-\tau)}\right)^{\frac{q-p}{q-1}}d\tau\right)^{\frac{q-1}{q}}\left(\int_0^1\left[\mu^{3\tau}\xi^{3(1-\tau)}\right]^p\left[\tau\left|f'(\mu)\right|^q+(1-\tau)\left|f'(\xi)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
&+(\ln\xi-\ln\eta)\left(\int_0^1\left(\xi^{3\tau}\eta^{3(1-\tau)}\right)^{\frac{q-p}{q-1}}d\tau\right)^{\frac{q-1}{q}}\left(\int_0^1\left[\xi^{3\tau}\eta^{3(1-\tau)}\right]^p\left[\tau\left|f'(\xi)\right|^q+(1-\tau)\left|f'(\eta)\right|^q\right]d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

We get the desired result by a simple calculation. □

4. Applications to Special Means

Proposition 4.1. *For $0 < \eta < \xi < \mu$, one can obtain:*

$$\begin{aligned} & \left| e^{\mu}\left(\mu^2 - 2\mu + 2\right) - e^{\eta}\left(\eta^2 - 2\eta + 2\right) \right| \\ \leq \; & (\ln\mu - \ln\xi) L\left(\mu^3 e^{\mu}, \xi^3 e^{\xi}\right) + (\ln\xi - \ln\eta) L\left(\xi^3 e^{\xi}, \eta^3 e^{\eta}\right). \end{aligned}$$

Proof. Let $f(\psi) = e^{\psi}$ for $\psi \in \mathbb{R}_+$. Then, $\left|f'(\psi)\right| = e^{\psi}$ is a $GG-$convex function on $\mathbb{R}_+$ and by using the inequality that is given in Theorem 2.2, we get

$$\begin{aligned} & \left| e^{\mu}\mu^2 - e^{\eta}\eta^2 - 2\left[e^{\mu}(\mu - 1) - e^{\eta}(\eta - 1)\right] \right| \\ \leq \; & (\ln\mu - \ln\xi) L\left(\mu^3 e^{\mu}, \xi^3 e^{\xi}\right) + (\ln\xi - \ln\eta) L\left(\xi^3 e^{\xi}, \eta^3 e^{\eta}\right). \end{aligned}$$

□

Proposition 4.2. *For $0 < \eta < \xi < \mu$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$\begin{aligned} & \left| \mu^2 q e^{\frac{\mu}{q}} - \eta^2 q e^{\frac{\eta}{q}} - 2\left[\left(q^2 \mu e^{\frac{\mu}{q}} - q^3 e^{\frac{\mu}{q}}\right) - \left(q^2 \eta e^{\frac{\eta}{q}} - q^3 e^{\frac{\eta}{q}}\right)\right] \right| \\ \leq \; & (\ln\mu - \ln\xi) \left(L\left(\mu^{3p}, \xi^{3p}\right)\right)^{\frac{1}{p}} \left(L\left(e^{\mu}, e^{\xi}\right)\right)^{\frac{1}{q}} \\ & + (\ln\xi - \ln\eta) \left(L\left(\xi^{3p}, \eta^{3p}\right)\right)^{\frac{1}{p}} \left(L\left(e^{\xi}, e^{\eta}\right)\right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Let $f(\psi) = q e^{\frac{\psi}{q}}$ for $\psi \in \mathbb{R}_+$. Then, $\left|f'(\psi)\right|^q = e^{\psi}$ is a $GG-$convex function on $\mathbb{R}_+$ and by using the inequality that is given in Theorem 2.3, we get

$$\begin{aligned} & \left| \mu^2 q e^{\frac{\mu}{q}} - \eta^2 q e^{\frac{\eta}{q}} - 2\left[\left(q^2 \mu e^{\frac{\mu}{q}} - q^3 e^{\frac{\mu}{q}}\right) - \left(q^2 \eta e^{\frac{\eta}{q}} - q^3 e^{\frac{\eta}{q}}\right)\right] \right| \\ \leq \; & (\ln\mu - \ln\xi) \left(L\left(\mu^{3p}, \xi^{3p}\right)\right)^{\frac{1}{p}} \left(L\left(e^{\mu}, e^{\xi}\right)\right)^{\frac{1}{q}} \\ & + (\ln\xi - \ln\eta) \left(L\left(\xi^{3p}, \eta^{3p}\right)\right)^{\frac{1}{p}} \left(L\left(e^{\xi}, e^{\eta}\right)\right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. □

Proposition 4.3. *Under assumptions of Proposition 2, we have:*

$$\begin{aligned} & \left| \mu^2 q e^{\frac{\mu}{q}} - \eta^2 q e^{\frac{\eta}{q}} - 2\left[\left(q^2 \mu e^{\frac{\mu}{q}} - q^3 e^{\frac{\mu}{q}}\right) - \left(q^2 \eta e^{\frac{\eta}{q}} - q^3 e^{\frac{\eta}{q}}\right)\right] \right| \\ \leq \; & (\ln\mu - \ln\xi) \left(L\left(\mu^{3p} e^{\mu}, \xi^{3p} e^{\xi}\right)\right)^{\frac{1}{q}} + (\ln\xi - \ln\eta) \left(L\left(\xi^{3p} e^{\xi}, \eta^{3p} e^{\eta}\right)\right)^{\frac{1}{q}}, \end{aligned}$$

Proof. Let $f(\psi) = q e^{\frac{\psi}{q}}$ for $\psi \in \mathbb{R}_+$. Then, $\left|f'(\psi)\right|^q = e^{\psi}$ is a $GG-$convex function on $\mathbb{R}_+$ and by using the inequality that is given in Theorem 2.4, we get

$$\begin{aligned} & \left| \mu^2 q e^{\frac{\mu}{q}} - \eta^2 q e^{\frac{\eta}{q}} - 2\left[\left(q^2 \mu e^{\frac{\mu}{q}} - q^3 e^{\frac{\mu}{q}}\right) - \left(q^2 \eta e^{\frac{\eta}{q}} - q^3 e^{\frac{\eta}{q}}\right)\right] \right| \\ \leq \; & (\ln\mu - \ln\xi) \left(L\left(\mu^{3p} e^{\mu}, \xi^{3p} e^{\xi}\right)\right)^{\frac{1}{q}} + (\ln\xi - \ln\eta) \left(L\left(\xi^{3p} e^{\xi}, \eta^{3p} e^{\eta}\right)\right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. □

Proposition 4.4. *For $0 < \eta < \xi < \mu$, $s > 0$, we obtain:*

$$\begin{aligned} & 3(\mu - \eta)\left[L_{s+2}(\eta, \mu)\right]^{s+2} \\ \leq \; & \left(\mu^{s+3} - \eta^{s+3}\right) + L(\xi^3, \mu^3)\left[\xi^s - \mu^s\right] + L(\eta^3, \xi^3)\left[\eta^s - \xi^s\right]. \end{aligned}$$

Proof. If we set $f(x) = \frac{x^{s+1}}{s+1}$ for $x \in \mathbb{R}_+$ and $s > 0$. Then, it is clear that $\left|f'(x)\right| = x^s$ is a $GA-$convex function on $\mathbb{R}_+$ and from the inequality that is given in Theorem 3.1, we can write

$$\begin{aligned} & \left|\mu^2 f(\mu) - \eta^2 f(\eta) - 2\int_a^b \psi f(\psi) d\psi\right| \\ = & \left|\frac{\mu^{s+3}}{s+1} - \frac{\eta^{s+3}}{s+1} - 2\left[\frac{\mu^{s+3} - \eta^{s+3}}{(s+1)(s+3)}\right]\right| \\ = & (\mu - \eta)\left[L_{s+2}(\eta, \mu)\right]^{s+2}. \end{aligned}$$

For the right hand side of the inequality, we get

$$\begin{aligned} & \frac{\left|f'(\mu)\right|}{3}\left[\mu^3 - L(\xi^3, \mu^3)\right] + \frac{\left|f'(\xi)\right|}{3}\left[L(\xi^3, \mu^3) - L(\eta^3, \xi^3)\right] + \frac{\left|f'(\eta)\right|}{3}\left[L(\eta^3, \xi^3) - \eta^3\right] \\ = & \frac{\mu^{s+3}}{3} - \frac{\mu^s L(\xi^3, \mu^3)}{3} + \frac{\xi^s}{3}\left[L(\xi^3, \mu^3) - L(\eta^3, \xi^3)\right] + \frac{\eta^s}{3} L(\eta^3, \xi^3) - \frac{\eta^{s+3}}{3}. \end{aligned}$$

By simplifying the above results, we get the proof. □

Proposition 4.5. *For $0 < \eta < \xi < \mu$, $q \geq 1$, $s > 0$ and $sq \neq 1$, we get:*

$$\begin{aligned} & (\mu - \eta)\left[L_{s+2}(\eta, \mu)\right]^{s+2} \\ \leq & (\ln\mu - \ln\xi)^{1-\frac{1}{q}} L^{1-\frac{1}{q}}\left(\xi^3, \mu^3\right)\left(\frac{\mu^{sq+3} - \mu^{sq} L(\xi^3, \mu^3) + \xi^{sq} L(\xi^3, \mu^3) - \xi^{sq+3}}{3}\right)^{\frac{1}{q}} \\ & + (\ln\xi - \ln\eta)^{1-\frac{1}{q}} L^{1-\frac{1}{q}}\left(\eta^3, \xi^3\right)\left(\frac{\xi^{sq+3} - \xi^{sq} L(\eta^3, \xi^3) + \eta^{sq} L(\eta^3, \xi^3) - \eta^{sq+3}}{3}\right)^{\frac{1}{q}}. \end{aligned}$$

Proof. If we set $f(x) = \frac{x^{s+1}}{s+1}$ for $x \in \mathbb{R}_+$ and $s > 0$. Then, it is clear that $\left|f'(x)\right|^q = x^{sq}$ is a $GA-$convex function on $\mathbb{R}_+$. Therefore, by applying Theorem 3.2, we obtain

$$\begin{aligned} & (\mu - \eta)\left[L_{s+2}(\eta, \mu)\right]^{s+2} \\ \leq & (\ln\mu - \ln\xi)^{1-\frac{1}{q}} L^{1-\frac{1}{q}}\left(\xi^3, \mu^3\right)\left(\frac{\mu^{sq+3} - \mu^{sq} L(\xi^3, \mu^3) + \xi^{sq} L(\xi^3, \mu^3) - \xi^{sq+3}}{3}\right)^{\frac{1}{q}} \\ & + (\ln\xi - \ln\eta)^{1-\frac{1}{q}} L^{1-\frac{1}{q}}\left(\eta^3, \xi^3\right)\left(\frac{\xi^{sq+3} - \xi^{sq} L(\eta^3, \xi^3) + \eta^{sq} L(\eta^3, \xi^3) - \eta^{sq+3}}{3}\right)^{\frac{1}{q}} \end{aligned}$$

which is the desired result. □

Proposition 4.6. *For $0 < \eta < \xi < \mu$, $q > 1$ and $s > 0$, we get:*

$$\begin{aligned} & (\mu - \eta)\left[L_{s+2}(\eta, \mu)\right]^{s+2} \\ \leq & (\ln\mu - \ln\xi)\left(L\left(\mu^{\frac{3q}{q-1}}, \xi^{\frac{3q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left(A\left(\mu^{sq}, \xi^{sq}\right)\right)^{\frac{1}{q}} \\ & + (\ln\xi - \ln\eta)\left(L\left(\xi^{\frac{3q}{q-1}}, \eta^{\frac{3q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left(A\left(\xi^{sq}, \eta^{sq}\right)\right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Similarly, by choosing $f(x) = \frac{x^{s+1}}{s+1}$ for $x \in \mathbb{R}_+$ and $s > 0$. By taking into account that $\left|f'(x)\right|^q = x^{sq}$ is a $GA-$convex function on $\mathbb{R}_+$. By applying Theorem 3.4, we get the result. We omit the details. □

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