Some Results on Generalized Derivations and \((\sigma, \tau)\)–Lie Ideals

Evrim Güven

Kocaeli University, Faculty of Art and Sciences, Department of Mathematics, Kocaeli, Turkey

Abstract. Let \(R\) be a prime ring with characteristic not 2 and \(\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma\) automorphisms of \(R\). Let \(h : R \to R\) be a nonzero left(resp.right)-generalized \((\alpha, \beta)\)–derivation, \(b \in R\) and \(V \neq 0\) a left \((\sigma, \tau)\)–Lie ideal of \(R\). The main object in this article is to study the situations. (1) \(h(l) \in C_{\sigma,\mu}(V)\), (2) \(bh(l) \in C_{\sigma,\mu}(V)\) or \(h(l)b \in C_{\sigma,\mu}(V)\), (3) \(h(V) = 0\), (4) \(h(V)b = 0\) or \(bhh(V) = 0\).

1. Introduction

Let \(R\) be a ring and \(\sigma, \tau\) two mappings of \(R\). For each \(r, s \in R\) we set \([r, s]_{\sigma,\tau} = rs - \tau(s)r\) and \((r, s)_{\sigma,\tau} = rs - \tau(s)r\). Let \(U\) be an additive subgroup of \(R\). If \([U, R] \subset U\) then \(U\) is called a Lie ideal of \(R\). The definition of \((\sigma, \tau)\)–Lie ideal of \(R\) is introduced in [7] as follows: (i) \(U\) is called a right \((\sigma, \tau)\)–Lie ideal of \(R\) if \([U, R]_{\sigma,\tau} \subset U\), (ii) \(U\) is called a left \((\sigma, \tau)\)–Lie ideal if \([R, U]_{\sigma,\tau} \subset U\). (iii) \(U\) is called a \((\sigma, \tau)\)–Lie ideal if \(U\) is both right and left \((\sigma, \tau)\)–Lie ideal of \(R\). Every Lie ideal of \(R\) is a \((1, 1)\)–Lie ideal of \(R\), where \(1 : R \to R\) is identity map. If \(R = \{ a^{x,y} \mid x, y \text{ are integers} \}, U = \{ a^{x,y} \mid x \text{ is integer} \}, \sigma a^{x,y} = (\sigma^{x,y})\) and \(\tau a^{x,y} = (\tau^{x,y})\) then \(U\) is right \((\sigma, \tau)\)–Lie ideal but not a Lie ideal of \(R\).

A derivation \(d\) is an additive mapping on \(R\) which satisfies \(d(rs) = d(r)s + rd(s), \forall r, s \in R\). The notion of generalized derivation was introduced by Brešar [2] as follows. An additive mapping \(h : R \to R\) is called a generalized derivation if there exists a derivation \(d : R \to R\) such that \(h(xy) = h(x)y + xd(y)\), for all \(x, y \in R\).

An additive mapping \(d : R \to R\) is said to be a \((\sigma, \tau)\)–derivation if \(d(rs) = d(r)\sigma(s) + \tau(r)d(s)\) for all \(r, s \in R\). Every derivation \(d : R \to R\) is a \((1, 1)\)–derivation. Chang [3] gave the following definition. Let \(R\) be a ring, \(\sigma, \tau\) automorphisms of \(R\) and \(d\) a \((\sigma, \tau)\)–derivation of \(R\). An additive mapping \(h : R \to R\) is said to be a right generalized \((\sigma, \tau)\)–derivation of \(R\) associated with \(d\) if \(h(xy) = h(x)\sigma(y) + \tau(x)d(y)\), for all \(x, y \in R\) and \(h\) is said to be a left generalized \((\sigma, \tau)\)–derivation of \(R\) associated with \(d\) if \(h(xy) = d(x)\sigma(y) + \tau(x)h(y)\), for all \(x, y \in R\). \(h\) is said to be a generalized \((\sigma, \tau)\)–derivation of \(R\) associated with \(d\) if it is both a left and right generalized \((\sigma, \tau)\)–derivation of \(R\) associated with \(d\).

According to Chang’s definition, every \((\alpha, \tau)\)–derivation \(d : R \to R\) is a generalized \((\sigma, \tau)\)–derivation associated with \(d\) and every derivation \(d : R \to R\) is a \((1, 1)\)–derivation associated with \(d\). A generalized \((1, 1)\)–derivation is simply called a generalized derivation. The definition of generalized derivation given in Bresar [2] is a right generalized derivation associated with derivation \(d\) according to Chang’s definition.

The mapping defined by \(h(r) = [r, a]_{\sigma,\tau}, \forall r \in R\) is a right-generalized derivation associated with derivation \(d(r) = [r, \sigma(a)], \forall r \in R\) and left-generalized derivation associated with derivation \(d_l(r) = [r, \tau(a)], \forall r \in R\).
The mapping \( h(r) = (a, r), \forall r \in R \) is a left-generalized \((\sigma, \tau)\)-derivation associated with \((\sigma, \tau)\)-derivation \( d_1(r) = [a, r], \forall r \in R \) and right-generalized \((\sigma, \tau)\)-derivation associated with \((\sigma, \tau)\)-derivation \( d(r) = -[a, r], \forall r \in R \).

The following result is given in [5]. Let \( U \) be a nonzero left \((\sigma, \tau)\)-Lie ideal of \( R \) and \( d : R \to R \) a nonzero \((\alpha, \beta)\)-derivation. If \( d(U) = 0 \) then \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \). We generalized this result as follows.

Let \( h : R \to R \) be a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d : R \to R \) and \( V \) a nonzero left \((\sigma, \tau)\)-Lie ideal of \( R \). If \( h\lambda(v) = 0 \) then \( \sigma(v) + \tau(v) \in Z, \forall v \in V \). Kyoo-Hong Park and Yong-Soo Jung [9] proved the following. Let \( R \) be a prime ring with characteristic different from two and \( d \) be a nonzero \((\alpha, \beta)\)-derivation of \( R \). Let \( U \) be a left \((\sigma, \tau)\)-Lie ideal. If \( d[R, U]_{\sigma, \tau} = 0 \) then \( \sigma(u) + \tau(u) \in Z \) for all \( u \in U \). Replacing \( d \) with a nonzero left-generalized \((\alpha, \beta)\)-derivation \( h : R \to R \) and a nonzero ideal \( I \) with \( R \) we generalized this result.

In this paper, we give some other results about left (resp. right)-generalized \((\alpha, \beta)\)-derivation on left \((\sigma, \tau)\)-Lie ideals of \( R \).

Throughout the paper, \( R \) will be a prime ring with characteristic not 2 and \( \sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma \) automorphisms of \( R \). We set \( C_{\sigma, \tau} = \{r \in R | \sigma(r)c = \tau(r)c, \forall r \in R \} \), and shall use the following relations frequently:

\[
\begin{align*}
[r, s]_{\sigma, \tau} &= r[s, r] + [r, t] s + r[s, t],
[r, s]_{\lambda, \mu} &= r[s, r] + [r, t] s + r[s, t],
[r, s]_{\sigma, \tau} &= r[s, r] + [r, t] s + r[s, t].
\end{align*}
\]

2. Results

Lemma 2.1. ([1, Lemma 1]) Let \( R \) be a prime ring and \( d : R \to R \) a \((\sigma, \tau)\)-derivation. If \( U \) is a right ideal of \( R \) and \( d(U) = 0 \) then \( d = 0 \).

Lemma 2.2. ([8, Lemma 4]) If a prime ring contains a nonzero commutative right ideal then \( R \) is commutative.

Lemma 2.3. ([6, Theorem 2]) Let \( V \) be a noncentral left \((\sigma, \tau)\)-Lie ideal of \( R \). Then there exist a nonzero ideal \( M \) of \( R \) such that \( [R, M]_{\sigma, \tau} \subset U \) and \( [R, M]_{\sigma, \tau} \not\subset C_{\sigma, \tau} \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in V \).

Lemma 2.4. ([4, Lemma 7]) Let \( h : R \to R \) be a nonzero right-generalized \((\sigma, \tau)\)-derivation associated with a nonzero \((\sigma, \tau)\)-derivation \( d \) and let \( I \) be a nonzero ideal of \( R \). If \( a, b \in R \) such that \( [ah(l), b]_{\lambda, \mu} = 0 \) then \([a, \mu(b)]a = 0 \) or \( da^{-1}\lambda(b) = 0 \).

Lemma 2.5. Let \( I \) be a nonzero ideal of \( R \) and \( a, b, c \in R \).

(i) If \( b[I, a]_{\lambda, \mu} = 0 \) then \( a \in Z \) or \( b = 0 \).

(ii) If \( c[I, a]_{\lambda, \mu} b = 0 \) then \( c = 0 \) or \([b, \gamma\lambda(a)]b = 0 \).

Proof. (i) If \( b[I, a]_{\lambda, \mu} = 0 \) then we have for all \( r \in R, x \in I \)

\[ 0 = b[y[x, a]_{\lambda, \mu}] + b[y[x, a]_{\lambda, \mu}] + b[y[x, a]_{\lambda, \mu}] = b[y[x, a]_{\lambda, \mu}] + b[y[x, a]_{\lambda, \mu}] \]

That is

\[ b[y[I]_{\lambda, \mu}] = 0. \]

Since \( y(I) \) is a nonzero ideal of \( R \) then we obtain \( b = 0 \) or \([R, \lambda(a)] = 0 \). That is \( b = 0 \) or \( a \in Z \).

(ii) If \( c[I, a]_{\lambda, \mu} b = 0 \) then we get for all \( x \in I \)

\[ 0 = c[y[x, a]^{-1}(b), a]_{\lambda, \mu} b = c[y[x, a]^{-1}(b), \lambda(a)] b + c[y[x, a]_{\lambda, \mu} b b \]

\[ = c[y[x, a]^{-1}(b), \lambda(a)] b. \]

That is

\[ c[I]_{\lambda, \mu} b = 0. \]

This gives that \( c = 0 \) or \([b, \gamma\lambda(a)]b = 0 \). □
Lemma 2.6. Let \( h : R \longrightarrow R \) be a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d : R \longrightarrow R \) and \( I \) be a nonzero ideal of \( R \). If \( b \in R \) such that \( hy[I, b]_{\lambda,\mu} = 0 \) then \( b \in Z \) or \( d\gamma\mu(b) = 0 \).

Proof. If \( hy[I, b]_{\lambda,\mu} = 0 \) then we get for all \( x \in I \)

\[
0 = hy[\mu(b)x, b]_{\lambda,\mu} = hy(\mu(b)[x, b])_{\lambda,\mu} = h(\gamma\mu(b)\gamma[x, b])_{\lambda,\mu} = dy\mu(b)\gamma[x, b]_{\lambda,\mu}.
\]

That is

\[ dy\mu(b)\gamma[I, b]_{\lambda,\mu} = 0. \]

Using Lemma 2.5 (i) and the last relation we obtain that \( dy\mu(b) = 0 \) or \( b \in Z \). □

Theorem 2.7. Let \( h : R \longrightarrow R \) be a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d : R \longrightarrow R \) and \( I, M \) be nonzero ideals of \( R \). Let \( V \) be a left \((\alpha, \tau)\)-Lie ideal of \( R \).

(i) If \( hy[I, M]_{\lambda,\mu} = 0 \) then \( R \) is commutative.

(ii) If \( h\lambda(V) = 0 \) then \( \alpha(v) + \tau(v) \in Z, \forall v \in V \).

(iii) If \( hy[I, V]_{\lambda,\mu} = 0 \) then \( \alpha(v) + \tau(v) \in Z \) for all \( v \in V \).

Proof. (i) If \( hy[I, m]_{\lambda,\mu} = 0, \forall m \in M \) then using Lemma 2.6, for any \( m \in M \) we get

\[ m \in Z \text{ or } dy\mu(m) = 0. \]

Let \( K = \{m \in M \mid m \in Z\} \) and \( L = \{m \in M \mid dy\mu(m) = 0\} \). Then \( K \) and \( L \) are additive proper subgroups of \( M \) moreover \( M = K \cup L \). Then it must be \( M = K \) or \( M = L \). We have \( M \subset Z \) or \( dy\mu(M) = 0 \). Since \( \gamma\mu(M) \) is a nonzero ideal of \( R \) and \( d \) is a nonzero derivation then using Lemma 2.1, \( dy\mu(M) \neq 0 \). Hence we obtain \( M \subset Z \). This gives that \( R \) is commutative from Lemma 2.2.

(ii) If \( V \subset Z \) then \( \alpha(v) + \tau(v) \in Z \) for all \( v \in V \). If \( V \not\subset Z \) then using Lemma 2.3, there exist a nonzero ideal \( m \) of \( R \) such that

\[
([R, M]_{\lambda,\tau} \subset V \text{ and } [R, M]_{\lambda,\tau} \not\subset C_{\alpha,\tau}) \text{ or } (\alpha(v) + \tau(v) \in Z, \forall v \in V).
\]

If \( [R, M]_{\lambda,\tau} \subset V \) then \( h\lambda(V) = 0 \) implies that \( h\lambda[R, M]_{\lambda,\tau} = 0 \). If we use (i) then we have \( R \) is commutative and so \( V \subset Z \). This contradicts with \( V \not\subset Z \). Hence finally we obtain that \( \alpha(v) + \tau(v) \in Z, \forall v \in V \) for any cases.

(iii) If \( hy[I, V]_{\lambda,\mu} = 0 \) then for any \( v \in V \), we have \( v \in Z \) or \( dy\mu(v) = 0 \) from Lemma 2.6. Considering as in the proof of (i) we get \( V \subset Z \) or \( dy\mu(V) = 0 \). It is clear that \( V \subset Z \) gives that \( \alpha(v) + \tau(v) \in Z, \forall v \in V \). On the other hand, since \( d \) is a nonzero \((\alpha, \beta)\)-derivation then \( d \) is a nonzero left (and right)-generalized \((\alpha, \beta)\)-derivation associated with \( d \). Hence, if \( dy\mu(V) = 0 \) then we obtain that \( \alpha(v) + \tau(v) \in Z, \forall v \in V \) by (ii). □

Corollary 2.8. ([9, Corollary 5]) Let \( R \) be a prime ring with characteristic different from two, let \( d \) be a nonzero \((\theta, \varphi)\)-derivation of \( R \) and let \( U \) be a left \((\alpha, \tau)\)-Lie ideal. If \( d[R, U]_{\lambda,\tau} = 0 \) then \( \alpha(u) + \tau(u) \in Z \) for all \( u \in U \).

Corollary 2.9. Let \( V \) be a left \((\alpha, \tau)\)-Lie ideal of \( R \). If \( a \in R \) such that \( \alpha(L(V))_{a,\beta} = 0 \) then \( a \in C_{a,\beta} \) or \( \alpha(v) + \tau(v) \in Z, \forall v \in V \).

Proof. The mapping defined by \( g(r) = (a, r)_{a,\beta}, \forall r \in R \) is a left-generalized \((\alpha, \beta)\)-derivation associated with \((\alpha, \beta)\)-derivation \( d(r) = [a, r]_{a,\beta}, \forall r \in R \). If \( g = 0 \) then we have \( d = 0 \) and so \( a \in C_{a,\beta} \). Let \( g \neq 0 \) and \( d \neq 0 \). If \( \alpha(L(V))_{a,\beta} = 0 \) then we can write \( g\lambda(V) = 0 \). Using Theorem 2.7 (ii) we obtain that \( \alpha(v) + \tau(v) \in Z \) for all \( v \in V \). □

Lemma 2.10. Let \( h : R \longrightarrow R \) be a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d \) and \( I \) be a nonzero ideal of \( R \). If \( a, b \in R \) such that \( h\lambda[I, a]_{\lambda,\tau}b = 0 \) then \( d\lambda\tau(a) = 0 \) or \( b, \alpha\lambda\sigma(a)b = 0 \).
Proof. If $h \lambda [I, a]_{\lambda, \beta}, b = 0$ then we have for all $x \in I$

$$0 = h \lambda [\tau(a)x, a]_{\lambda, \beta} = h(\lambda \tau(a)\lambda[x, a]_{\lambda, \beta})b$$

$$= d\lambda \tau(a)\lambda[a, x]_{\lambda, \beta} + \beta \lambda \tau(a)h\lambda[x, a]_{\lambda, \beta}b = d\lambda \tau(a)\lambda[x, a]_{\lambda, \beta}b.$$

That is $d\lambda \tau(a)\lambda[I, a]_{\lambda, \beta}b = 0$.

Using Lemma 2.5 (ii) we obtain that $d\lambda \tau(a) = 0$ or $[b, a\lambda \sigma(a)]b = 0$. \hfill \Box

**Lemma 2.11.** Let $V$ be a left $(\sigma, \tau)$–Lie ideal of $R$ and $a, b \in R$. If $[a, \lambda(V)]_{\alpha, \beta}b = 0$ then $a \in C_{\alpha, \beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.

**Proof.** If $V \subset Z$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. If $V \not\subset Z$ then using Lemma 2.3, there exist a nonzero ideal $M$ of $R$ such that

$$([R, M]_{\alpha, \beta} \subset V) \text{ and } ([R, M]_{\alpha, \beta} \not\subset C_{\alpha, \beta} \text{ or } \sigma(v) + \tau(v) \in Z, \forall v \in V).$$

Let $[R, M]_{\alpha, \beta} \subset V$. The mapping $d(v) = [a, r]_{\alpha, \beta}, \forall r \in R$ is a $(\alpha, \beta)$–derivation and so left (and right)-generalized $(\alpha, \beta)$–derivation with $d$. If $d = 0$ then $a \in C_{\alpha, \beta}$ is obtained. Let $d \neq 0$.

If $[a, \lambda(V)]_{\alpha, \beta}b = 0$ then we have $d\lambda(V)b = 0$ and so $d\lambda[R, M]_{\alpha, \beta}b = 0$. Using Lemma 2.10, this gives that for any $m \in M$

$$d\lambda \tau(m) = 0 \text{ or } [b, a\lambda \sigma(m)]b = 0.$$

Considering as in the proof of Theorem 2.7 (i) we obtain that

$$d\lambda \tau(M) = 0 \text{ or } [b, a\lambda \sigma(M)]b = 0.$$ 

Since $\lambda \tau(M)$ is a nonzero ideal of $R$ and $d$ is a nonzero derivation then using by Lemma 2.1, $d\lambda \tau(M) \neq 0$.

On the other hand, if $[b, a\lambda \sigma(M)]b = 0$ then we get for all $m, m_1 \in M$

$$0 = [b, a\lambda \sigma(mm_1)]b = a\lambda \sigma(m)[b, a\lambda \sigma(m_1)]b + [b, a\lambda \sigma(m)]a\lambda \sigma(m_1)b$$

$$= [b, a\lambda \sigma(m)]a\lambda \sigma(m_1)b.$$

That is

$$[b, a\lambda \sigma(M)]a\lambda \sigma(M)b = 0.$$

Since $a\lambda \sigma(M)$ is a nonzero ideal of $R$ then we have $b \in Z$. Finally we obtain that $a \in C_{\alpha, \beta}$ or $b \in Z$ or for all $v \in V$, $\sigma(v) + \tau(v) \in Z$ for all case. \hfill \Box

**Corollary 2.12.** Let $V$ be a left $(\sigma, \tau)$–Lie ideal of $R$ and $a, b \in R$. If $[a, \lambda(V)]b = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.

**Proof.** Taking $\alpha = \beta = 1$ an identity map of $R$ in Lemma 2.11 we get the required result. \hfill \Box

**Theorem 2.13.** Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)$–derivation associated with $(\alpha, \beta)$–derivation $d$ and let $V$ be a left $(\alpha, \tau)$–Lie ideal of $R$. If $b \in R$ such that $h\lambda(V)b = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

**Proof.** If $h\lambda(V)b = 0$ then we have $h\lambda[R, V]_{\alpha, \beta}b = 0$. Using Lemma 2.10, for any $v \in V$, we obtain that $d\lambda \tau(v) = 0$ or $[b, a\lambda \sigma(v)]b = 0$. Let $K = \{v \in V \mid d\lambda \tau(v) = 0\}$ and $L = \{v \in V \mid [b, a\lambda \sigma(v)]b = 0\}$. Considering as in the proof of Theorem 2.7 (i), we get

$$d\lambda \tau(V) = 0 \text{ or } [b, a\lambda \sigma(V)]b = 0.$$

Since $d$ is a $(\alpha, \beta)$–derivation then left-generalized $(\alpha, \beta)$–derivation with $d$. If $d\lambda \tau(V) = 0$ then using Theorem 2.7 (ii), we have $\sigma(v) + \tau(v) \in Z, \forall v \in V$. On the other hand if $[b, a\lambda \sigma(V)]b = 0$ then using Corollary 2.12, we obtain that $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. \hfill \Box
Corollary 2.14. Let $h : R \longrightarrow R$ be a nonzero left-generalized derivation associated with derivation $d$ and let $V$ be a left $(\alpha, \tau)$-Lie ideal of $R$. If $b \in R$ such that $h(V)b = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.

Proof. Taking $\alpha = \beta = \lambda = 1$ an identity map of $R$ in Theorem 2.13, we get the required result. \hfill \blacksquare

Remark 2.15. Let $M$ be a nonzero ideal of $R$. If $b[M, \lambda(M)] = 0$ then $b \in Z$.

Proof. If $b[M, \lambda(M)] = 0$ then for all $m \in M, r \in R$

$$0 = b[b, \lambda(m)] = b[b, \lambda(m)] + b(b, \lambda(m)) = b\lambda(m)[b, \lambda(r)].$$

That is $b\lambda(M)[b, R] = 0$. This gives that $b \in Z$. \hfill \blacksquare

Lemma 2.16. Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha, \beta)$-derivation associated with a nonzero $(\alpha, \beta)$-derivation $d : R \longrightarrow R$. Let $I$ be a nonzero ideal of $R$ and $a, b \in R$. If $bh(I, a)_{\alpha, \beta} = 0$ then $b[h, \beta\lambda\tau(a)] = 0$ or $d\lambda\sigma(a) = 0$.

Proof. If $bh(I, a)_{\alpha, \beta} = 0$ then $b[hI, a]_{\alpha, \beta} = 0$ then for all $x \in I$

$$0 = bh\lambda(x\sigma(a), a)_{\alpha, \beta} = bh\lambda(x, a)_{\alpha, \beta} + bh\lambda(\alpha\lambda\sigma(a)) + bh\lambda(a, a)_{\alpha, \beta}d\lambda\sigma(a).$$

That is

$$bh\lambda(x, a)_{\alpha, \beta}d\lambda\sigma(a) = 0, \forall x \in I. \tag{1}$$

Replacing $x$ by $\lambda^{-1}\beta^{-1}(b)x$ in (1) we get for all $x \in I$

$$0 = bh\lambda[\lambda^{-1}\beta^{-1}(b)x, a]_{\alpha, \beta}d\lambda\sigma(a)$$

$$= bh\lambda[\lambda^{-1}\beta^{-1}(b), \alpha\lambda\sigma(a)] + bh\lambda[\lambda^{-1}\beta^{-1}(b), \beta\lambda]\alpha\lambda\sigma(a)$$

$$= bh\lambda[\lambda^{-1}\beta^{-1}(b), \beta\lambda]\alpha\lambda\sigma(a).$$

That is

$$bh\lambda[\lambda^{-1}\beta^{-1}(b), \beta\lambda]\alpha\lambda\sigma(a) = 0.$$

Since $\beta\lambda(I)$ an ideal of $R$ then we have $bh\lambda[\lambda^{-1}\beta^{-1}(b), \beta\lambda]\alpha\lambda\sigma(a) = 0$ or $d\lambda\sigma(a) = 0$. That is $b[b, \beta\lambda\tau(a)] = 0$ or $d\lambda\sigma(a) = 0$. \hfill \blacksquare

Corollary 2.17. Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha, \beta)$-derivation associated with a nonzero $(\alpha, \beta)$-derivation $d$ and $M$ be a nonzero ideal of $R$. If $b \in R$ such that $bh\lambda[R, M]_{\alpha, \beta} = 0$ then $b \in Z$.

Proof. If $bh\lambda[R, M]_{\alpha, \beta} = 0, \forall m \in M$ then using Lemma 2.16 we have for any $m \in M$

$$bh\lambda[R, m]_{\alpha, \beta} = 0.$$

Let $K = \{m \in M : b[b, \beta\lambda\tau(m)] = 0\}$ and $L = \{m \in M : d\lambda\sigma(m) = 0\}$. Considering as in the proof of Theorem 2.7 (i) we obtain $b[b, \beta\lambda\tau(M)] = 0$ or $d\lambda\sigma(M) = 0$. Since $d$ is a nonzero derivation then $d\lambda\sigma(M) = 0$ from Lemma 2.1. If $b[b, \beta\lambda\tau(M)] = 0$ then using Remark 2.15, $b \in Z$ is obtained. \hfill \blacksquare

Lemma 2.18. Let $V$ be a left $(\alpha, \tau)$-Lie ideal of $R$ and $a, b \in R$. If $b[a, \lambda(V)]_{\alpha, \beta} = 0$ then $a \in C_{\alpha, \beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.

Proof. If $V \subset Z$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. If $V \not\subset Z$ then using Lemma 2.3, there exist a nonzero ideal $M$ of $R$ such that

$$([R, M]_{\alpha, \tau} \subset V \text{ and } [R, M]_{\alpha, \tau} \not\subset C_{\alpha, \tau}) \text{ or } \sigma(v) + \tau(v) \in Z, \forall v \in V.$$
Let $[R,M]_{\alpha,\tau} \subset V$. The mapping $d(r) = [a,r]_{\alpha,\beta}$, $\forall r \in R$ is a $(\alpha,\beta)$–derivation and so right (and left)–generalized $(\alpha,\beta)$–derivation associated with $d$. If $d = 0$ then $a \in C_{\alpha,\beta}$ is obtained. Assume that $d$ is a nonzero derivation.

If $[a,\lambda(V)]_{\alpha,\beta} = 0$ then we have $bd\lambda(V) = 0$ and so $bd\lambda[R,M]_{\alpha,\tau} = 0$. This gives that by Lemma 2.16, for any $m \in M$

$$d\lambda_{\alpha}(m) = 0 \text{ or } b[b,\beta \lambda_{\alpha}(m)] = 0.$$  

Considering as in the proof of Theorem 2.7 (i), we obtain that

$$d\lambda_{\alpha}(M) = 0 \text{ or } b[b,\beta \lambda_{\alpha}(M)] = 0.$$  

Since $\lambda_{\alpha}(M)$ is a nonzero ideal of $R$ and $d$ is a nonzero derivation then $d\lambda_{\alpha}(M) \neq 0$ by Lemma 2.1. On the other hand, if $b[b,\beta \lambda_{\alpha}(M)] = 0$ then using Remark 2.15, we have $b \in Z$.

Finally we obtain that $a \in C_{\alpha,\beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$ for all case.

**Corollary 2.19.** Let $V$ be a left $(\alpha,\tau)$–Lie ideal of $R$ and $a, b \in R$. If $b[a,\lambda(V)] = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.

**Proof.** Taking $\alpha = \beta = 1$ an identity map of $R$ in Lemma 2.18, we get the required result. \qed

**Theorem 2.20.** Let $h : R \rightarrow R$ be a nonzero right–generalized $(\alpha,\beta)$–derivation associated with nonzero $(\alpha,\beta)$–derivation $d$ and $a,b \in R$. Let $V$ be a left $(\alpha,\tau)$–Lie ideal and $I \neq 0$ an ideal of $R$.

(i) If $bh\lambda(V) = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.

(ii) If $bh\lambda[I,V]_{\alpha,\tau} = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.

**Proof.** (i) If $V \subset Z$ then it is clear that $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$. If $V \notin Z$ then using Lemma 2.3, there exist a nonzero ideal $M$ of $R$ such that $([R,M]_{\alpha,\tau} \subset V$ and $[R,M]_{\alpha,\tau} \notin C_{\alpha,\tau}$) or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$. If $[R,M]_{\alpha,\tau} \subset V$ then $bh\lambda(V) = 0$ means that $bh\lambda[R,M]_{\alpha,\tau} = 0$. Using Corollary 2.17, we get $b \in Z$.

Finally we obtain that $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$ for any case.

(ii) If $bh\lambda[I,V]_{\alpha,\tau} = 0$, $\forall v \in V$ then using Lemma 2.16, we have for any $v \in V$

$$b[b,\beta \lambda_{\alpha}(v)] = 0 \text{ or } d\lambda_{\alpha}(v) = 0.$$  

Let $K = \{v \in V \mid b[b,\beta \lambda_{\alpha}(v)] = 0\}$ and $L = \{v \in V \mid d\lambda_{\alpha}(v) = 0\}$. Considering as the proof of Theorem 2.7 (i), we obtain

$$b[b,\beta \lambda_{\alpha}(v)] = 0 \text{ or } d\lambda_{\alpha}(v) = 0.$$  

Since $d$ is a $(\alpha,\beta)$–derivation then $d$ is a left–generalized $(\alpha,\beta)$–derivation with $d$. Hence $d\lambda_{\alpha}(V) = 0$ implies that $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$ from Theorem 2.7 (ii). On the other hand,

if $b[b,\beta \lambda_{\alpha}(v)] = 0$ then using Corollary 2.19, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. \qed

**Corollary 2.21.** Let $h : R \rightarrow R$ be a nonzero right–generalized derivation associated with nonzero derivation $d$ and $a,b \in R$. Let $V$ be a left $(\alpha,\tau)$–Lie ideal and $I \neq 0$ an ideal of $R$. If $bh(V) = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.

**Proof.** Taking $\alpha = \beta = \lambda = 1$ an identity map of $R$ in Theorem 2.20 (i), we get the required result. \qed

**Theorem 2.22.** Let $V$ be a nonzero left $(\sigma,\tau)$–Lie ideal of $R$ and $a,b \in R$.

(i) If $b[V,a]_{\alpha,\beta} = 0 \text{ or } [V,a]_{\alpha,\beta} b = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.

(ii) If $(a,V)_{\alpha,\beta} = 0$ then $a \in C_{\alpha,\beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.

(iii) If $(V,a)_{\alpha,\beta} b = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$, $\forall v \in V$.
Proof. (i) Let us consider the mapping defined by \( h(r) = [r, a]_{\alpha, \beta} \), \( \forall r \in R \). Then \( h \) is a right(left resp.)-generalized derivation associated with derivation \( d_1(r) = [r, a(\alpha)] \), \( \forall r \in R \) and derivation \( d_2 = [r, \beta(a)] \), \( \forall r \in R \) respectively. Because

\[
\begin{align*}
    h(rs) &= [rs, a]_{\alpha, \beta} = r[s, \alpha(a)] + [r, a]_{\alpha, \beta} s = h(r)s + rd_1(s), \forall r, s \in R, \\
    h(rs) &= [rs, a]_{\alpha, \beta} = r[s, \alpha(a)] + [r, \beta(a)] s = d_2(r)s + rh(s), \forall r, s \in R.
\end{align*}
\]

If \( h = 0 \) then \( d_1 = 0 \) and \( d_2 = 0 \) and so \( a \in Z \) is obtained. Let \( h \neq 0 \), \( d_1 \neq 0 \) and \( d_2 \neq 0 \).

If \( b [V, a]_{\alpha, \beta} = 0 \) then we have \( bh(V) = 0 \). Since \( h \) is a right-generalized derivation associated with derivation \( d_1 \) then using Corollary 2.21, we get \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \).

Similarly if \( [V, a]_{\alpha, \beta} b = 0 \) then we write \( h(V)b = 0 \). Since \( h \) is a left-generalized derivation associated with derivation \( d_2 \) then using Corollary 2.14 we have \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \). Finally we obtain that \( a \in Z \) or \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \) for two case.

(ii) Let \( g(r) = (a, r)_{\alpha, \beta, \gamma}, \forall r \in R \) and \( d_3(r) = [a, r]_{\alpha, \beta}, \forall r \in R \). Then \( g \) is a left-generalized \((\alpha, \beta)\)-derivation associated with \((\alpha, \beta)\)-derivation \( d_3 \). Because for all \( r, s \in R \)

\[
g(rs) = (a, rs)_{\alpha, \beta} = [r(s, a), s]_{\alpha, \beta} + [a, r]_{\alpha, \beta} s = d_3(r)(s) + \beta(r) g(s).
\]

If \( g = 0 \) then we have \( d_3 = 0 \) and so \( a \in C_{\alpha, \beta} \) is obtained. Let \( g \neq 0 \) and \( d_3 \neq 0 \).

If \((a, V)_{\alpha, \beta} b = 0 \) then we have \( g(V)b = 0 \). Since \( g \) is a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with \((\alpha, \beta)\)-derivation \( d_3 \) then considering Theorem 2.13, we get \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \). Finally we obtain that \( a \in C_{\alpha, \beta} \) or \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \) for two case.

(iii) The mapping \( f(r) = (r, a)_{\alpha, \beta, \gamma}, \forall r \in R \) is a left-generalized derivation associated with derivation \( d_4(r) = [r, \beta(a)], \forall r \in R \). Because for all \( r, s \in R \)

\[
f(rs) = (rs, a)_{\alpha, \beta} = (r, s)_{\alpha, \beta} - [r, \beta(a)]s = d_4(r)s + rf(s).
\]

If \( f = 0 \) then \( d_4 = 0 \) and so \( a \in Z \) is obtained.

Let \( f \neq 0 \) and \( d_4 \neq 0 \). If \((V, a)_{\alpha, \beta} b = 0 \) then we have \( f(V)b = 0 \). This gives that \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \) by Corollary 2.14. \( \square \)

**Lemma 2.23.** Let \( h : R \longrightarrow R \) be a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d : R \longrightarrow R \). Let \( V \) be a nonzero left \((\alpha, \tau)\)-Lie ideal of \( R \) and \( I \) be a nonzero ideal of \( R \). If \( a, b \in R \) such that \( [h(l)b, a]_{\alpha, \mu} = 0 \) then \( d \beta^{-1}(\mu(a)) = 0 \) or \( b[l, \mu(a)] = 0 \).

**Proof.** If \( [h(x)b, a]_{\alpha, \mu} = 0, \forall x \in I \) then we have

\[
h[xb, \mu(a)] + [h(x), a]_{\alpha, \mu} b = 0, \forall x \in I.
\]

Replacing \( x \) by \( sx, s \in R \) in (2) and using (2), we get for all \( x \in I, s \in R \)

\[
0 = h(sx)[b, \mu(a)] + [h(sx), a]_{\alpha, \mu} b
= d(s)[a(x)b, \mu(a)] + \beta(s)h(x)[b, \mu(a)] + [d(s)a(x) + \beta(s)h(x), a]_{\alpha, \mu} b
= d(s)[a(x)b, \mu(a)] + \beta(s)h(x)[b, \mu(a)] + d(s)[a(x), \mu(a)] b
+ [d(s), a]_{\alpha, \mu} [a(x)b, \mu(a)] + [\beta(s), \mu(a)]h(x)b
= d(s)[a(x)b, \mu(a)] + [d(s)[a(x), \mu(a)] b + [d(s), a]_{\alpha, \mu} a(x)b + [\beta(s), \mu(a)]h(x)b].
\]

If we take \( \beta^{-1}(\mu(a)) \) instead of \( s \) and say that \( k = d \beta^{-1}(\mu(a)) \) then the last relation gives that

\[
k[\alpha(x)b, \mu(a)] + k[\alpha(x), \mu(a)] b + [k, a]_{\alpha, \mu} a(x)b = 0, \forall x \in I.
\]

Replacing \( x \) by \( x^{-1}(b) \) in (3) we have for all \( x \in I \)

\[
0 = k\alpha(x)b[\mu, \mu(a)] + k[\alpha(x), \mu(a)] b + [k, a]_{\alpha, \mu} a(x)b = k[\alpha(x)b, \mu(a)] + [k, a]_{\alpha, \mu} a(x)b = \alpha(x)b[\mu, \mu(a)] + [k, a]_{\alpha, \mu} a(x)b
\]

and so \( k[\mu, \mu(a)] = 0 \).

Since \( \alpha(l) \) is a nonzero ideal of \( R \) then we obtain \( d \beta^{-1}(\mu(a)) = 0 \) or \( b[l, \mu(a)] = 0 \). \( \square \)
**Theorem 2.24.** Let $h : R \rightarrow R$ be a nonzero left-generalized $(\alpha, \beta)$–derivation associated with a nonzero $(\alpha, \beta)$–derivation $d : R \rightarrow R$. Let $V$ be a nonzero left $(\sigma, \tau)$–Lie ideal of $R$ and $I$ be a nonzero ideal of $R$. If $b \in R$ such that $h(l)b \subset C_{\lambda, \mu}(V)$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

**Proof.** If $h(l)b \subset C_{\lambda, \mu}(V)$ then we have $[h(x)b, v]_{\lambda, \mu} = 0, \forall v \in V, x \in I$. Using Lemma 2.23, we get $d\beta^{-1}\mu(v) = 0$ or $[b, \lambda(v)] = 0$.

Let $K = \{v \in V \mid [b, \lambda(v)] = 0\}$ and $L = \{v \in V \mid d\beta^{-1}\mu(v) = 0\}$. Considering as in the proof of Theorem 2.7 (i) we obtain

$$d\beta^{-1}\mu(V) = 0 \text{ or } b[B, \lambda(V)] = 0.$$  

If $d\beta^{-1}\mu(V) = 0$ then using Theorem 2.7 (ii), we have $\sigma(v) + \tau(v) \in Z, \forall v \in V$. On the other hand using Corollary 2.19, $b[B, \lambda(V)] = 0$ means that $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

**Corollary 2.25.** Let $V$ be a nonzero left $(\alpha, \tau)$–Lie ideal of $R$ and $a, b \in R$. If $(a, I)_{\beta, \mu} \subset C_{\lambda, \mu}(V)$ then $a \in C_{\alpha, \tau}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

**Proof.** The mapping defined by $h(r) = (a, r)_{\beta, \mu}, \forall r \in R$ is a left-generalized $(\alpha, \tau)$–derivation associated with $(\sigma, \tau)$–derivation $d(r) = [a, r]_{\beta, \mu}, \forall r \in R$. If $h = 0$ then $d = 0$ and so we have $a \in C_{\alpha, \tau}$. Let $h \neq 0$ and $d \neq 0$.

If $a, b \lambda, \mu \subset C_{\lambda, \mu}(V)$ then we can write $h(l)b \subset C_{\lambda, \mu}(V)$. Using Theorem 2.24, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

**Theorem 2.26.** Let $h : R \rightarrow R$ be a nonzero left-generalized $(\alpha, \beta)$–derivation associated with nonzero $(\alpha, \beta)$–derivation $d : R \rightarrow R$ and $I$ be a nonzero ideal of $R$. Let $V$ be a nonzero left $(\alpha, \tau)$–Lie ideal of $R$. If $h(l) \subset C_{\lambda, \mu}(V)$ then $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

**Proof.** If $h(l) \subset C_{\lambda, \mu}(V)$ then we have $[h(x), v]_{\lambda, \mu} = 0, \forall x \in I, v \in V$. Let us consider the following relations

$$[rs, t]_{\lambda, \mu} = r[s, \lambda(t)] + [r, t]_{\lambda, \mu}s, \forall r, s, t \in R$$

and so

$$[rs, t]_{\lambda, \mu} = r[s, t]_{\lambda, \mu} + [r, \mu(t)]s, \forall r, s, t \in R.$$  

Using the hypothesis and the above relations we get for all $v \in V, x \in I, s \in R$

$$0 = [h(sx), v]_{\lambda, \mu} = [ds]a(x) + \beta(s)h(x), v]_{\lambda, \mu}$$

and so

$$[ds]a(x), \lambda(v)] + [ds]v, \lambda, \mu a(x) + \beta(s)h(x) = 0, \forall v \in V, x \in I, s \in R.$$  

If we take $\beta^{-1}\mu(v)$ instead of $s$ in (4) and say that $k(v) = d\beta^{-1}\mu(v)$ then we obtain

$$k(v)[a(x), \lambda(v)] + [k(v), v]_{\lambda, \mu}a(x) = 0, \forall v \in V, x \in I.$$  

Replacing $x$ by $xr, r \in R$ in (5) and using (5) we have for all $v \in V, x \in I, r \in R$

$$0 = k(v)a(x)[a(r), \lambda(v)] + k(v)[a(x), \lambda(v)]a(r) + [k(v), v]_{\lambda, \mu}a(x)a(r)$$

and so

$$k(v)a(x)[a(r), \lambda(v)] = 0, \forall v \in V.$$  

(6)
For any \( v \in V \), the relation (6) gives that \( d \beta^{-1} \mu(v) = 0 \) or \( v \in Z \).

Let \( K = \{ v \in V \mid d \beta^{-1} \mu(v) = 0 \} \) and \( L = \{ v \in V \mid v \in Z \} \). Considering as in the proof of Theorem 2.7 (i), we get \( d \beta^{-1} \mu(V) = 0 \) or \( V \subseteq Z \).

If \( V \subseteq Z \) then \( \sigma(v) + \tau(v) \in Z, \forall v \in V \). On the other hand, since \( d \) is a \((\alpha, \beta)\)-derivation and so a left-generalized \((\alpha, \beta)\)-derivation associated with \( d \) then \( d \beta^{-1} \mu(V) = 0 \) means that \( \sigma(v) + \tau(v) \in Z, \forall v \in V \) by Theorem 2.7 (ii). \( \square \)

**Corollary 2.27.** Let \( V \) be a nonzero left \((\alpha, \tau)\)-Lie ideal of \( R \) and \( b \in R \). If \( (b, I)_{\alpha,\beta} \subseteq C_{\lambda,\mu}(V) \) then \( b \in C_{\alpha,\beta} \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \).

**Proof.** Let us consider the mappings defined by \( h(r) = (b, r)_{\alpha,\beta}, \forall r \in R \) and \( d(r) = [b, r]_{\alpha,\beta}, \forall r \in R \). Since

\[
h(rs) = (b, rs)_{\alpha,\beta} = \tau(r)(b, s)_{\alpha,\beta} + [b, r]_{\alpha,\beta} \sigma(s) = d(r)\sigma(s) + \tau(r)h(s), \forall r, s \in R.
\]

Then \( h \) is a left-generalized \((\alpha, \beta)\)-derivation associated with \((\alpha, \beta)\)-derivation \( d \). If \( h = 0 \) then \( d = 0 \) and so \( b \in C_{\alpha,\beta} \) is obtained.

If \( (b, I)_{\alpha,\beta} \subseteq C_{\lambda,\mu}(V) \) then we have \( h(I) \subseteq C_{\lambda,\mu}(V) \). Let \( h \neq 0 \) and \( d \neq 0 \). Using Theorem 2.26, we obtain \( \sigma(v) + \tau(v) \in Z, \forall v \in V \). \( \square \)

**Theorem 2.28.** Let \( h : R \rightarrow R \) be a nonzero right-generalized \((\alpha, \beta)\)-derivation associated with \((\alpha, \beta)\)-derivation \( 0 \neq d : R \rightarrow R \). Let \( V \) be a nonzero left \((\alpha, \tau)\)-Lie ideal of \( R \) and \( I \) be a nonzero ideal of \( R \). If \( b \in R \) such that \( bh(I) \subseteq C_{\lambda,\mu}(V) \) then \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \).

**Proof.** If \( bh(I) \subseteq C_{\lambda,\mu}(V) \) then \([bh(I), v]_{\lambda,\mu} = 0, \forall v \in V \). Using Lemma 2.4, for any \( v \in V \), we obtain that \([b, \mu(v)]b = 0 \) or \( da^{-1} \lambda(V) = 0 \).

Let \( K = \{ v \in V \mid [b, \mu(v)]b = 0 \} \) and \( L = \{ v \in V \mid da^{-1} \lambda(v) = 0 \} \). Considering as in the proof of Theorem 2.7 (i), we have

\[
[b, \mu(V)]b = 0 \text{ or } da^{-1} \lambda(V) = 0.
\]

If \( da^{-1} \lambda(V) = 0 \) then \( \sigma(v) + \tau(v) \in Z, \forall v \in V \) is obtained by Theorem 2.7 (ii). On the other hand \([b, \mu(V)]b = 0 \) means that \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \) by Corollary 2.12. \( \square \)

**Corollary 2.29.** Let \( I \) be a nonzero ideal of \( R \) and \( a, b \in R \). If \( b(a, I)_{\alpha,\tau} \subseteq C_{\lambda,\mu}(V) \) then \( a \in C_{\alpha,\tau} \) or \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \).

**Proof.** The mapping defined by \( h(r) = (a, r)_{\alpha,\tau}, \forall r \in R \) is a right-generalized \((\alpha, \tau)\)-derivation associated with \((\alpha, \tau)\)-derivation \( d(r) = [a, r]_{\alpha,\tau}, \forall r \in R \). If \( h = 0 \) then \( d = 0 \) and so we have \( a \in C_{\alpha,\tau} \). Assume that \( h \) and \( d \) are nonzero.

If \( b(a, I)_{\alpha,\tau} \subseteq C_{\lambda,\mu}(V) \) then we have \( bh(I) \subseteq C_{\lambda,\mu}(V) \). Since \( h \) is right-generalized \((\alpha, \tau)\)-derivation then using Theorem 2.2, we get \( b \in Z \) or \( \sigma(v) + \tau(v) \in Z, \forall v \in V \). \( \square \)

**References**

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