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Δ_p^m-Statistical Convergence of Order α

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Abstract. In this work, we generalize the concepts of statistically convergent sequence of order α and statistical Cauchy sequence of order α by using the generalized difference operator Δ^m. We prove that a sequence is Δ_p^m-statistically convergent of order α if and only if it is Δ_p^m-statistically Cauchy of order α.

1. Introduction

Throughout we denote the space of all complex sequences by w and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1, 2, 3, ...\}$, the set of positive integers.

In 1981, the difference sequence spaces $X(\Delta)$ were introduced by Kızmaz [16] for $X = \ell_\infty$, c and c_0 and the notion was generalized by Et and Çolak [10]. Out of these, using the generalized difference operator Δ^m, Ioan [17] introduced and discussed the concept of $p-$convex sequences. Later on, Karakaş and Altin [15] defined and studied some basic topological and algebraic properties of the sequence spaces $X\left(\Delta_p^m\right)$ for $X = \ell_\infty, c, c_0$, where $p, m \in \mathbb{N}$, $\Delta_p x = (px_k - x_{k+1})$, and $\Delta_p^m x = \left(\Delta_p^m x_k\right) = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} p^{m-v} x_{k+v}$. In the case $x \in X\left(\Delta_p^m\right)$ (for $X = \ell_\infty, c$ and c_0), we call $\Delta_p^m -$ *bounded*, $\Delta_p^m -$ *convergent* and $\Delta_p^m -$ *zero*, respectively. Let X be any sequence space, if $x \in X(\Delta^m)$ then there exists one and only one $y = (y_k) \in X$ such that

$$\begin{aligned} x_k &= \sum_{i=1}^{k-m} (-1)^m \binom{k-i-1}{m-1} y_i = \sum_{i=1}^{k} (-1)^m \binom{k+m-i-1}{m-1} y_{i-m}, \\ y_{1-m} &= y_{2-m} = \cdots = y_0 = 0 \end{aligned} \tag{1}$$

for sufficiently large k, for instance $k > 2m$. We use this fact to formulate (2), (3) and (4). Recently the difference sequence spaces have been studied by many researchers [1],[2],[8],[15],[19],[26].

The idea of statistical convergence goes back to the first edition of monograph of Zygmund [27]. This notion has firstly been defined for real and complex sequences by Steinhaus [23] and Fast [12]. Schoenberg [21] has defined from a sequence- to- sequence summability method called $D-$convergence which, implies

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statistical convergence. Later on, it has been studied by Bhuania *et al*. [3], Connor [4], Çolak [5], Çolak and Altin [6], Et et al. [9, 11, 22], Fridy [13], Gadjiev and Orhan [14], Moricz [18], Šalát [20], Tripathy [25], Dutta and Tripathy [7], and many others.

The concept of statistical convergence depends on the density of subsets of the set $\mathbb{N}$. The natural density of a subset A of $\mathbb{N}$ is defined by $\delta(A) = \lim_n \frac{1}{n} |\{k \leq n : k \in A\}|$, if the limit exists, where $|.|$ is cardinality of set A.

A sequence $x = (x_k)$ of complex numbers is said to be statistically convergent to some number L if, for every positive number ε, $\delta(\{k \in \mathbb{N}\text{: } |x_k - L| \geq \varepsilon\})$ has natural density zero. The number L is called the statistical limit of (x_k) and written as $S - \lim x_k = L$. We denote the space of all statistically convergent sequences by S.

2. Some Properties of $\Delta_p^m (X)$

In this section, we give some topological properties of $\Delta_p^m (X)$ and some inclusion relations.

Theorem 2.1. *The sequence spaces $\ell_\infty\left(\Delta_p^m\right), c\left(\Delta_p^m\right)$ and $c_0\left(\Delta_p^m\right)$ are BK−spaces with norm*

$$\|x\|_1 = \sum_{i=1}^{m} \left|x_i\right| + \left\|\Delta_p^m x\right\|_\infty .$$

Proof. The proof is similar to the proof of Theorem 1.1 of Et and Çolak [10]. □

Theorem 2.2. *Let X be a vector space and let $A \subset X$. If A is a convex set, then $\Delta_p^m (A)$ is a convex set in $\Delta_p^m (X)$.*

Proof. Can be established using standard techniques, so omitted. □

Theorem 2.3. *The following statements hold:*
i) $\ell_\infty \subset \ell_\infty\left(\Delta_p^m\right)$ and the inclusion is strict,
ii) $c\left(\Delta_p^m\right) \subset \ell_\infty\left(\Delta_p^m\right)$ and the inclusion is strict,
iii) $c(\Delta) \subset c\left(\Delta_p^m\right)$ and the inclusion is strict,
iv) The sequence space $\ell_\infty(\Delta)$ is different from the sequence space $\ell_\infty\left(\Delta_p^m\right)$ and $\ell_\infty(\Delta) \cap \ell_\infty\left(\Delta_p^m\right) \neq \emptyset$.

Proof. i) Let $x \in \ell_\infty$. Then

$$\begin{aligned}\left|\Delta_p^m x\right| &= \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} p^{m-v} x_{k+v}\right| \\ &\leq \binom{m}{0} p^m |x_k| + \binom{m}{1} p^{m-1} |x_{k+1}| + \binom{m}{2} p^{m-2} |x_{k+2}| + ... \binom{m}{m-1} p\, |x_{k+v}| < M\end{aligned}$$

for some $M > 0$; i.e. , $\left(\Delta_p^m x_k\right) \in \ell_\infty$ and so $x \in \ell_\infty(\Delta_p^m)$. Hence $\ell_\infty \subset \ell_\infty\left(\Delta_p^m\right)$.

To show that the inclusion is strict, let us consider the sequence $x = (x_k)$ with $x_k = p^k - \sum_{i=1}^{k} p^i$ so that $\Delta_p^m x = \left(p(p-1)^{m-1}, p(p-1)^{m-1}, p(p-1)^{m-1}, ...\right)$. Then we obtain $\left(\Delta_p^m x_k\right) \in \ell_\infty$ but $(x_k) \notin \ell_\infty$.

ii) Let $x \in c\left(\Delta_p^m\right)$. Then, we have $\left(\Delta_p^m x\right) \in c \subset \ell_\infty$, that is, $x \in \ell_\infty\left(\Delta_p^m\right)$. Therefore, $c\left(\Delta_p^m\right) \subset \ell_\infty\left(\Delta_p^m\right)$. To show that the inclusion is strict, define a sequence $x = (x_k)$ such that

$$x_k = (0, p, 0, p, 0, ...),$$

then $x \in \ell_\infty\left(\Delta_p^m\right) \backslash c\left(\Delta_p^m\right)$.

iii) If we choose $(x_k) = (p, 2p, 3p, 4p, ...)$, then we obtain $x \in c(\Delta)$ but $x \notin c\left(\Delta_p^m\right)$.

iv) If we choose $(x_k) = (1, 2, 3, ...)$, then $x \in \ell_\infty(\Delta)$, but $x \notin \ell_\infty\left(\Delta_p^m\right)$. Let us take the sequence $x = (x_k)$ such that $x_k = p^k - \sum_{i=1}^{k} p^i$. Then, we get $x \notin \ell_\infty(\Delta)$ but $x \in \ell_\infty\left(\Delta_p^m\right)$. Since all constant sequences belong to both $\ell_\infty(\Delta)$ and $\ell_\infty\left(\Delta_p^m\right)$, the spaces $\ell_\infty(\Delta)$ and $\ell_\infty\left(\Delta_p^m\right)$ are overlapping. $\square$

3. Main Results

In this section, we introduce and examine the concepts of Δ_p^m-statistically convergent sequence of order α and Δ_p^m-statistically Cauchy sequence of order α.

Definition 3.1. Let $x = (x_k) \in w$ and $0 < \alpha \le 1$ be given. The sequence $x = (x_k)$ is said to be Δ_p^m-statistically convergent of order α if there exists a complex number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \le n : \left|\Delta_p^m x_k - L\right| \ge \varepsilon\right\}\right| = 0$$

for every $\varepsilon > 0$. In this case we write $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k \to L$. The set of Δ_p^m-statistically convergent sequences of order α will be denoted by $S^\alpha\left(\Delta_p^m\right)$. In case of $L = 0$, we shall write $S_0^\alpha\left(\Delta_p^m\right)$.

Theorem 3.2. *Let $0 < \alpha \le 1$. If a sequence $x = (x_k)$ is Δ_p^m-statistically convergent of order α, then $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k$ is unique.*

Proof. Suppose that $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L_1$ and $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L_2$. Given $\varepsilon \ge 0$, consider the following sets:

$$K_1(\varepsilon) = \left\{k \in \mathbb{N} : \left|\Delta_p^m x_k - L_1\right| \ge \frac{\varepsilon}{2}\right\}$$

and

$$K_2(\varepsilon) = \left\{k \in \mathbb{N} : \left|\Delta_p^m x_k - L_2\right| \ge \frac{\varepsilon}{2}\right\}.$$

Therefore, we obtain $\delta^\alpha(K_1(\varepsilon)) = 0$ since $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L_1$ and $\delta^\alpha(K_2(\varepsilon)) = 0$ since $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L_2$. Now, let $K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon)$. Thus, we get $\delta^\alpha(K(\varepsilon)) = 0$ which implies $\mathbb{N}/\delta^\alpha(K(\varepsilon)) = 0$. Now let $K^c(\varepsilon) = \mathbb{N}/K(\varepsilon)$, then we get

$$\begin{aligned} |L_1 - L_2| &\le \left|L_1 - \Delta_p^m x_k\right| + \left|\Delta_p^m x_k - L_2\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, we have $|L_1 - L_2| = 0$, i.e. $L_1 = L_2$.

From Theorem 3.2 we see that the Δ_p^m-statistical convergence of order α is well defined for $0 < \alpha \le 1$. However, for $\alpha > 1$ it is not well defined, since $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k$ is not uniquely defined. To show it, let $x = (x_k)$ be defined as

$$x_k = \begin{cases} 1, & k = 2n \ (n = 1, 2, 3...) \\ 0, & k \ne 2n \text{ otherwise} \end{cases}.$$

Then we have

$$\Delta_p x_k = \begin{cases} p, & k = 2n \ (n = 1, 2, 3...) \\ 0, & k \ne 2n \end{cases}$$

for $m = 1$. Then both

$$\lim_{n\to\infty}\left|\left\{k \le n : \left|\Delta_p^m x_k - p\right| \ge \varepsilon\right\}\right| \le \lim_n \frac{n}{2n^\alpha} = 0$$

and

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \left|\Delta_p^m x_k - 0\right| \ge \varepsilon\right\}\right| \le \lim_n \frac{n}{2n^\alpha} = 0$$

for $\alpha > 1$, so that $x = (x_k)$ is Δ_p^m-statistically convergent of order α both to p and 0.

Since the $\alpha-$density of a finite set is zero, every Δ_p^m-convergent sequence is Δ_p^m-statistically convergent of order α, but the converse is not true in general as can be seen in the following example.

Let $x = (x_k)$ be defined as

$$x_k = \begin{cases} p, & k = n^2 \ (n = 1, 2, 3...) \\ 0, & \text{otherwise} \end{cases}.$$

Then we obtain

$$\Delta_p x_k = \begin{cases} p^2, & k = n^2 \ (n = 1, 2, 3...) \\ -p, & k + 1 = n^2 \\ 0, & \text{otherwise} \end{cases},$$

for $m = 1$. It is easy to see that $x = (x_k)$ is Δ_p-statistically convergent of order α for $\alpha > \frac{1}{2}$, but is not convergent. □

Theorem 3.3. *Let $0 < \alpha \le \beta \le 1$. Then $S^\alpha\left(\Delta_p^m\right) \subseteq S^\beta\left(\Delta_p^m\right)$ and the inclusion is strict for at least those α and β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$.*

Proof. The inclusion part of proof is trivial. To show the inclusion $S^\alpha\left(\Delta_p^m\right) \subseteq S^\beta\left(\Delta_p^m\right)$ is strict choose $m = 1$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} p, & k = n^3 \ (n = 1, 2, 3...) \\ 0, & k \ne n^3 \end{cases}$$

Then we have

$$\Delta_p x_k = \begin{cases} p^2, & k = n^3 \ (n = 1, 2, 3...) \\ -p, & k + 1 = n^3 \\ 0, & \text{otherwise} \end{cases}..$$

and so

$$\lim_{n\to\infty}\frac{1}{n^\beta}\left|\left\{k \le n : \left|\Delta_p^m x_k - 0\right| \ge \varepsilon\right\}\right| \le \lim_n \frac{2\sqrt[3]{n}}{n^\beta} = 0$$

hence $stat(\beta) - \lim_{k\to\infty} \Delta_p^m x_k = 0$, i.e $x \in S^\beta\left(\Delta_p^m\right)$ for $\frac{1}{3} < \beta \le 1$, but $x \notin S^\alpha\left(\Delta_p^m\right)$ for $0 < \alpha \le \frac{1}{3}$ so that the inclusion $S^\alpha\left(\Delta_p^m\right) \subset S^\beta\left(\Delta_p^m\right)$ is strict. This holds for $\frac{1}{3} = \alpha < \beta < \frac{1}{2}$ for example, but there is no a number $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$. Therefore, the condition $\alpha < \frac{1}{k} < \beta$ is sufficient but not necessary for strictness of inclusion $S^\alpha\left(\Delta_p^m\right) \subset S^\beta\left(\Delta_p^m\right)$. □

Corollary 3.4. *If a sequence is Δ_p^m-statistically convergent of order α to L, for some $0 < \alpha \le 1$, then it is Δ_p^m-statistically convergent to L, that is $S^\alpha\left(\Delta_p^m\right) \subseteq S\left(\Delta_p^m\right)$ and inclusion is strict at least for $0 < \alpha < \frac{1}{2}$.*

We state the following theorems without proof, since these can be established using standard techniques.

Theorem 3.5. *Let $\alpha \in (0,1]$ and $x = (x_k)$, $y = (y_k)$ be sequences of real numbers. Then*
i) If $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L_1$ and $c \in \mathbb{C}$, then $stat(\alpha) - \lim_{k\to\infty} c\Delta_p^m x_k = cL_1$,
ii) If $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L_1$ and $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m y_k = L_2$, then $stat(\alpha) - \lim_{k\to\infty} \left(\Delta_p^m x_k + \Delta_p^m y_k\right) = L_1 + L_2$.

Theorem 3.6. *Let $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ be real sequences such that $\Delta_p^m x_k \le \Delta_p^m y_k \le \Delta_p^m z_k$. If $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m x_k = L = stat(\alpha) - \lim_{k\to\infty} \Delta_p^m z_k$, then $stat(\alpha) - \lim_{k\to\infty} \Delta_p^m y_k = L$.*

Theorem 3.7. *Let $\alpha \in (0,1]$ be arbitrary real number, then $S^\alpha\left(\Delta_p^m\right) \cap \ell_\infty\left(\Delta_p^m\right)$ is a closed subset of $\ell_\infty\left(\Delta_p^m\right)$.*

Theorem 3.8. *The set $S^\alpha\left(\Delta_p^m\right) \cap \ell_\infty\left(\Delta_p^m\right)$ is nowhere dense in $\ell_\infty\left(\Delta_p^m\right)$.*

Proof. Since every closed linear subspace of an arbitrary linear normed space E different from E is a nowhere dense set in E, by Theorem 3.7 we only need to show that $S^\alpha\left(\Delta_p^m\right) \cap \ell_\infty\left(\Delta_p^m\right) \neq \ell_\infty\left(\Delta_p^m\right)$. For this, choose $p = 1$ and consider a sequence $x = (x_k)$ defined by

$$\Delta^m x_k = \begin{cases} \sqrt{k}, & k = n^2 \\ & \qquad n = 1,2,3,\ldots \quad , \\ 0, & k \neq n^2 \end{cases} \tag{2}$$

then $x \in S^\alpha\left(\Delta_p^m\right)$, but $x \notin \ell_\infty\left(\Delta_p^m\right)$ by (1). □

Definition 3.9. Let $\alpha \in (0,1]$ be arbitrary real number and q be a positive real number. A sequence $x \in w$ is said to be $w_q\left(\Delta_p^m\right)$−summable of order α (or $w_q^\alpha\left(\Delta_p^m\right)$−summable) if there exists a real number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|\Delta_p^m x_k - L\right|^q = 0, \text{ where } p, m \in \mathbb{N}.$$

In this case we write $x_k \to L(w_q\left(\Delta_p^m\right))$. The set of all $w_q\left(\Delta_p^m\right)$−summable sequences of order α to L will be denoted by $w_q^\alpha\left(\Delta_p^m\right)$.

Theorem 3.10. *Let $\alpha_o \in (0,1]$ and q_o be a positive real number. The sequence space $w_{q_0}^{\alpha_0}\left(\Delta_p^m\right)$ is a Banach space for $1 \le q_o < \infty$ normed by*

$$\|x\|_2 = \sum_{i=1}^{m} \left|x_i\right| + \sup_n \left(\frac{1}{n^{\alpha_0}} \sum_{k=1}^{n} \left|\Delta_p^m x_k\right|^{q_0} \right)^{\frac{1}{q_0}}$$

and a complete q−normed space for $0 < q_o < 1$ by

$$\|x\|_3 = \sum_{i=1}^{m} \left|x_i\right|^q + \sup_n \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|\Delta_p^m x_k\right|^{q_0}$$

Proof. The proof has been omitted. □

In the next theorem, we give the relationship between Δ_p^m−statistically convergent of order α and $w_q\left(\Delta_p^m\right)$−summable sequences of order α.

Theorem 3.11. *Let α, β be fixed real numbers such that $0 < \alpha \le \beta \le 1$, $p, m \in \mathbb{N}$ and let q be a positive real number, then $w_q^\alpha\left(\Delta_p^m\right) \subset S^\beta\left(\Delta_p^m\right)$ and the inclusion is strict.*

Proof. The inclusion part of proof is easy. Taking $p = 1$ we show the strictness of the inclusion $w_q^\alpha\left(\Delta_p^m\right) \subset S^\beta\left(\Delta_p^m\right)$ for a special case. For this, choose $p = 1$ and consider the sequence $x = (x_k)$ defined by

$$\Delta^m x_k = \begin{cases} 1, & \text{if } k = n^2 \\ 0, & \text{if } k \neq n^2 \end{cases} \quad n = 1, 2, \dots. \tag{3}$$

For every $\varepsilon > 0$ and $\alpha \in \left(\frac{1}{2}, 1\right]$ we have

$$\frac{1}{n^\alpha}\left|\{k \leq n : |\Delta^m x_k - 0| \geq \varepsilon\}\right| \leq \frac{\sqrt{n}}{n^\alpha} = \frac{1}{n^{\alpha - \frac{1}{2}}}$$

so $x_k \to 0\,(S^\alpha(\Delta^m))$ for $\alpha \in \left(\frac{1}{2}, 1\right]$ by (1). On the other hand for $\alpha \in \left(0, \frac{1}{2}\right]$ we have

$$\frac{\sqrt{n} - 1}{n^\alpha} \leq \frac{1}{n^\alpha}\sum_{k \in I_n} |\Delta^m x_k|^p = \frac{1}{\lambda_n^\alpha}\sum_{k \in I_n} |\Delta^m x_k - 0|^p,$$

and so $x_k \nrightarrow 0\left(w_q^\alpha(\Delta^m)\right)$ by (1). □

Corollary 3.12. *If a sequence $x = (x_k)$ is $w_q\left(\Delta_p^m\right)$−summable of order α to L, then it is Δ_p^m−statistically convergent of order α to L.*

Even if $x = (x_k)$ is a Δ_p^m−bounded sequence, the converse of Theorem 3.11 and Corollary 3.12 do not hold, in general. To show this we must find a sequence that is Δ_p^m−bounded (that is $x \in \ell_\infty\left(\Delta_p^m\right)$) and Δ_p^m−statistically convergent of order β, but need not to be $w_q\left(\Delta_p^m\right)$−summable of order α, for some real numbers α and β such that $0 < \alpha \leq \beta \leq 1$. For this, choose $p = 1$ and consider a sequence $x = (x_k)$ defined by

$$\Delta^m x_k = \begin{cases} \frac{1}{\sqrt{k}}, & k \neq n^3 \\ 0, & k = n^3 \end{cases} \quad n = 1, 2, \dots \quad . \tag{4}$$

Then $x \in \ell_\infty\left(\Delta_p^m\right)$ and $x \in S^\alpha\left(\Delta_p^m\right)$ for $\alpha \in (\frac{1}{3}, 1]$, but $x \notin w_q^\alpha\left(\Delta_p^m\right)$ for $\alpha \in (0, \frac{1}{2})$ by (1).

Definition 3.13. Let $\alpha \in (0, 1]$. A sequence $x = (x_k)$ is said to be Δ_p^m−statistically Cauchy of order α if for every $\varepsilon \geq 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|\Delta_p^m x_k - \Delta_p^m x_N\right| \geq \varepsilon\right\}\right| = 0$$

that is; the set $\left|\left\{k \leq n : \left|\Delta_p^m x_k - \Delta_p^m x_N\right| \geq \varepsilon\right\}\right|$ has α−density zero.

We establish the following theorem with help of the method used by Fridy [13] and Tabib [24].

Theorem 3.14. *A real sequence $x = (x_k)$ is Δ_p^m−statistically convergent of order α if and only if $x = (x_k)$ is Δ_p^m−statistically Cauchy of order α.*

Proof. Let $\alpha \in (0, 1]$ be given. Suppose that the sequence $x = (x_k)$ is Δ_p^m−statistically convergent of order α to L. Then for every $\varepsilon > 0$ the set

$$A(\varepsilon) = \left\{k \leq n, \left|\Delta_p^m x_k - L\right| \geq \frac{\varepsilon}{2}\right\}$$

has $\alpha-$density zero. Choose positive integer number N such that $\left|\Delta_p^m x_N - L\right| \geq \varepsilon$. Now let us take the sets

$$B_\varepsilon = \left\{k \leq n, |\Delta_p^m x_k - \Delta_p^m x_N| \geq \frac{\varepsilon}{2}\right\},$$
$$C_\varepsilon = \left\{k \leq n, \left|\Delta_p^m x_k - L\right| \geq \frac{\varepsilon}{2}\right\},$$
$$D_\varepsilon = \left\{N \leq n, |\Delta_p^m x_N - L| \geq \frac{\varepsilon}{2}\right\}.$$

Then $B_\varepsilon \subseteq C_\varepsilon \cup D_\varepsilon$ and therefore $\delta_\alpha(B_\varepsilon) \leq \delta_\alpha(C_\varepsilon) + \delta_\alpha(D_\varepsilon) = 0$. Hence $x = (x_k)$ is Δ_p^m-statistically Cauchy of order α.

Conversely let $x = (x_k)$ be a Δ_p^m-statistically Cauchy sequence of order α, then for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\delta_\alpha\left(\left\{k \leq n : \left|\Delta_p^m x_k - L\right| < \varepsilon\right\}\right) = 1.$$

Hence, we obtain

$$\delta_\alpha\left(\left\{k \leq n : \Delta_p^m x_k < \Delta_p^m x_{N_0} + \varepsilon\right\}\right) = 1$$

and

$$\delta_\alpha\left(\left\{k \leq n : \Delta_p^m x_{N_0} - \varepsilon < \Delta_p^m x_k\right\}\right) = 1.$$

We define the following sets:

$$A = \left\{a \in \mathbb{R} : \delta^\alpha\left(\left\{k \leq n : \Delta_p^m x_k < a\right\}\right) = 1\right\},$$

and

$$B = \left\{b \in \mathbb{R} : \delta^\alpha\left(\left\{k \leq n : \Delta_p^m x_k > b\right\}\right) = 1\right\},$$

then $\left(\Delta_p^m x_{N_0} + \varepsilon\right) \in A$ and $\left(\Delta_p^m x_{N_0} - \varepsilon\right) \in B$. Let $a \in A$ and $b \in B$, then we have

$$\delta_\alpha\left(\left\{k \leq n : \Delta_p^m x_k < a\right\}\right) = 1 \text{ and } \delta_\alpha\left(\left\{k \leq n : \Delta_p^m x_k > b\right\}\right) = 1.$$

Therefore, we get

$$\delta_\alpha\left(\left\{k \leq n : b < \Delta_p^m x_k < a\right\}\right) = 1.$$

This implies $b < a$. We have

$$\Delta_p^m x_{N_0} - \varepsilon \leq \sup B \leq \inf A \leq \Delta_p^m x_{N_0} + \varepsilon.$$

Since ε was arbitrary positive number, we get $\sup B = \inf A$ and $\sup B = \inf A = L$. Let $\varepsilon > 0$ be given and there exists $a \in A$ and $b \in B$ such that $L - \varepsilon < b < a < L + \varepsilon$. The definitions of A and B imply

$$\delta_\alpha\left(\left\{k \leq n : L - \varepsilon < \Delta_p^m x_k < L + \varepsilon\right\}\right) = 1,$$

we obtain

$$\delta_\alpha\left(\left\{k \leq n : \left|\Delta_p^m x_k - L\right| < \varepsilon\right\}\right) = 1 \text{ or } \delta_\alpha\left(\left\{k \leq n : \left|\Delta_p^m x_k - L\right| \geq \varepsilon\right\}\right) = 0.$$

Therefore, $x = (x_k)$ is Δ_p^m-statistically convergent of order α. □

Theorem 3.15. *If $x = (x_k)$ is a sequence for which there exists a Δ_p^m-statistically convergent of order α sequence y such that $\Delta_p^m x_k = \Delta_p^m y_k$ for almost all $k(\alpha)$. Then, x is Δ_p^m-statistically convergent sequence of order α.*

Proof. The proof has been omitted. □

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