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# Asymptotic Farthest Points and Extreme Points

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**Abstract.** Let $X$ be a normed space, $G$ a nonempty bounded subset of $X$ and $\{x_n\}$ a bounded sequence in $X$. In this article, we introduce and discuss the concept of asymptotic farthest points of $\{x_n\}$ in $G$, which is a new definition in abstract approximation theory. Then, by applying the topics of functional analysis, we investigate the relation between this new concept and the concepts of extreme points and convexity. In particular, one of the main purposes of this paper is to study conditions under which the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $G$ is equivalent to the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $ext(G)$ or $\overline{co}(G)$.

## 1. Introduction

All over this section, assume that $X$ is a normed space. First, we recall the definition of farthest point in normed spaces from [9]. Let $G$ be a nonempty bounded subset of $X$ and $x \in X$. A point $z \in G$ is said to be a farthest point from $x$ in $G$ if $\|x - z\| = \delta(x, G)$, where $\delta(x, G) = \sup_{g \in G} \|x - g\|$. The set of all farthest points from $x$ in $G$ is denoted by $F_G(x)$.

The concept of farthest points is an important topics in approximation theory which has relation whit the concepts of extreme points and convexity (see [7], [9], [11], [12] and [13]). Similarly, in this section and Section 2, we define and discuss the new concept of asymptotic farthest points and in Sections 3 and 4, we study the relation between this new concept and the concepts of extreme points and convexity. Now we define the concept of asymptotic farthest point as follows:

Let $G$ be a nonempty bounded subset of $X$ and $\{x_n\}$ a bounded sequence in $X$. Consider the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ defined by

$$\delta_a(x, \{x_n\}) = \limsup_{n \to \infty} \|x - x_n\|, \quad x \in X.$$

The number $\sup_{x \in G} \delta_a(x, \{x_n\})$ is said to be the asymptotic farthest distance of $\{x_n\}$ from $G$ and is denoted by $\delta_a(G, \{x_n\})$. A point $z \in G$ is said to be an asymptotic farthest point of the sequence $\{x_n\}$ in the set $G$ if $\delta_a(z, \{x_n\}) = \delta_a(G, \{x_n\})$. The set of all asymptotic farthest points of $\{x_n\}$ in $G$ is denoted by $F_a(G, \{x_n\})$.

We note that the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ in the above definition is nonexpansive (and hence continuous) and convex. Thus, from Theorem 2.5.16 of [8] (The Mazur Theorem) and Proposition 2.5.2 of [1], we infer this function is weakly lower semicontinuous.

Infact, with providing the above new concept, we can expand the topics of abstract approximation theory. In Section 2, we give some results for this new concept.

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## 2. Some Results for Asymptotic Farthest Points

In this section, we obtain some results for the concept of asymptotic farthest points. This new concept is different from the concept of simultaneous farthest points (see [5]). First, we give the following example:

**Example 2.1.** *Let $X = \mathbb{C}$ with three real norms $\|x+iy\|_1 = |x|+|y|$, $\|x+iy\|_2 = \sqrt{x^2+y^2}$ and $\|x+iy\|_3 = \max\{|x|,|y|\}$, $z_n = (1+2^{1-n})i^{n-1}$ for all $n \in \mathbb{N}$ and $G_j = \{z \in \mathbb{C} : \|z\|_j \leq 1\}$ for $j = 1,2,3$. Then $F_a(G_j,\{z_n\}) = \{z \in \mathbb{C} : \|z\|_j = 1\}$ for $j = 1,3$ and $F_a(G_2,\{z_n\}) = \{\pm 1, \pm i\}$. Also, we have $F_a(\{z \in \mathbb{C} : \|z\|_j < 1\},\{z_n\}) = \emptyset$ for $j = 1,2,3$.*

In the following of this section, let $X$ be a normed space, $G$ a nonempty bounded subset of $X$, $\{x_n\}$ a bounded sequence in $X$ and $K$ a nonempty convex subset of $X$.

If $G$ is closed, then it follows from the continuity of the function $\delta_a(\cdot,\{x_n\}) : X \to \mathbb{R}$ that $F_a(G,\{x_n\})$ is closed. Also, if $G$ is compact, then by the continuity of this function, we have $F_a(G,\{x_n\})$ is compact and nonempty. But Example 2.1 shows that if $G$ is convex, then $F_a(G,\{x_n\})$ need not be convex. Now we give the following remarks and lemmas:

**Remark 2.2.** *Let $x \in X$. If $x_n \to x$, then $F_a(G,\{x_n\}) = F_G(x)$.*

**Remark 2.3.** *Let $x \in X$. Then, $x_n \to x$ if and only if $\delta_a(x,\{x_n\}) = 0$.*

**Remark 2.4.** *If $\delta_a(G,\{x_n\}) = 0$, then $\delta_a(x,\{x_n\}) = 0$ for all $x \in G$ and so from the above remark, $x_n \to x$ for all $x \in G$. This implies that $G$ is singleton and hence we have $F_a(G,\{x_n\}) = G$.*

**Lemma 2.5.** *We have $\delta_a(G,\{x_n\}) = \delta_a(\overline{G},\{x_n\})$.*

*Proof.* Because $G \subseteq \overline{G}$, it follows that $\delta_a(G,\{x_n\}) \leq \delta_a(\overline{G},\{x_n\})$. To prove the reversed inequality, let $z \in \overline{G}$. Then there exists a sequence $\{z_i\}$ in $G$ such that $z_i \to z$. Since the function $\delta_a(\cdot,\{x_n\}) : X \to \mathbb{R}$ is continuous, we have $\delta_a(z,\{x_n\}) = \lim_{i\to\infty} \delta_a(z_i,\{x_n\}) \leq \delta_a(G,\{x_n\})$. Now by taking the supremum over $z \in \overline{G}$, we obtain $\delta_a(\overline{G},\{x_n\}) \leq \delta_a(G,\{x_n\})$ and so we are done. □

**Lemma 2.6.** *We have $\delta_a(G,\{x_n\}) = \delta_a(\overline{G^w},\{x_n\})$.*

*Proof.* Because $G \subseteq \overline{G^w}$, it follows that $\delta_a(G,\{x_n\}) \leq \delta_a(\overline{G^w},\{x_n\})$. To prove the reversed inequality, let $z \in \overline{G^w}$. Then there exists a net $\{z_\alpha\}$ in $G$ such that $z_\alpha \xrightarrow{w} z$. Since the function $\delta_a(\cdot,\{x_n\}) : X \to \mathbb{R}$ is weakly lower semicontinuous, we have $\delta_a(z,\{x_n\}) \leq \liminf_\alpha \delta_a(z_\alpha,\{x_n\}) \leq \delta_a(G,\{x_n\})$ ([8], Page 217). Now by taking the supremum over $z \in \overline{G^w}$, we obtain $\delta_a(\overline{G^w},\{x_n\}) \leq \delta_a(G,\{x_n\})$ and so we are done. □

In the following, we want to establish a characterization of the set $F_a(G,\{x_n\})$ by balls. For this purpose, we need to the following definition:

**Definition 2.7.** *For any $\lambda \geq 0$, define a far level set of $\{x_n\}$ in $G$, denoted by $F_a^\lambda(G,\{x_n\})$, as follows:*

$$F_a^\lambda(G,\{x_n\}) = \{x \in G : \delta_a(G,\{x_n\}) - \lambda \leq \delta_a(x,\{x_n\})\}.$$

Obviously, from the above definition, we have the following remarks:

**Remark 2.8.** *For each $\lambda > 0$, the property of limit superior implies $F_a^\lambda(G,\{x_n\}) \neq \emptyset$.*

**Remark 2.9.** *If $0 \leq \lambda_1 \leq \lambda_2$, then $F_a^{\lambda_1}(G,\{x_n\}) \subseteq F_a^{\lambda_2}(G,\{x_n\})$.*

**Remark 2.10.** *If $\lambda \geq \delta_a(G,\{x_n\})$, then $F_a^\lambda(G,\{x_n\}) = G$.*

**Remark 2.11.** *For each $\varepsilon > 0$, we have*

$$F_a(G,\{x_n\}) = F_a^0(G,\{x_n\}) = \bigcap_{\lambda>0} F_a^\lambda(G,\{x_n\}) = \bigcap_{0<\lambda<\varepsilon} F_a^\lambda(G,\{x_n\}).$$

The following theorem give us a characterization of the set of all asymptotic farthest points:

**Theorem 2.12.** *Let $\delta_a(G,\{x_n\}) > 0$. Then*

$$F_a(G,\{x_n\}) = \left(\bigcap_{0<\lambda<\delta_a(G,\{x_n\})} \overline{\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty}\Big(B(x_i,\delta_a(G,\{x_n\})-\lambda)\Big)^c\right)\right)}\right)\bigcap G.$$

*Proof.* First, we show that for all $0<\lambda<\delta_a(G,\{x_n\})$, we have

$$\overline{\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty}\Big(B(x_i,\delta_a(G,\{x_n\})-\lambda)\Big)^c\right)\right)}\bigcap G \subseteq F_a^{\lambda}(G,\{x_n\}). \tag{1}$$

Let $0<\lambda<\delta_a(G,\{x_n\})$ and $z\in\overline{(\bigcap_{n=1}^{\infty}(\bigcup_{i=n}^{\infty}(B(x_i,\delta_a(G,\{x_n\})-\lambda))^c))}\bigcap G$. Then $z\in G$ and there exists a sequence $\{z_n\}$ in $\bigcap_{n=1}^{\infty}(\bigcup_{i=n}^{\infty}(B(x_i,\delta_a(G,\{x_n\})-\lambda))^c)$ such that $z_n\to z$. Let $\varepsilon>0$ be given. Thus, there exists $N\in\mathbb{N}$ such that

$$\|z_N - z\| < \varepsilon. \tag{2}$$

Because $z_N\in\bigcap_{n=1}^{\infty}(\bigcup_{i=n}^{\infty}(B(x_i,\delta_a(G,\{x_n\})-\lambda))^c)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\|z_N - x_{n_i}\| \geq \delta_a(G,\{x_n\}) - \lambda \quad \text{for all } i\in\mathbb{N}. \tag{3}$$

On the other hand, by the triangle inequality, we have $\|z_N - x_{n_i}\|\leq\|z_N - z\|+\|z-x_{n_i}\|$ for all $i\in\mathbb{N}$. Thus, from (2) and (3), we obtain $\|z-x_{n_i}\|>\delta_a(G,\{x_n\})-\lambda-\varepsilon$ for all $i\in\mathbb{N}$. Therefore,

$$\limsup_{i\to\infty}\|z-x_{n_i}\| \geq \delta_a(G,\{x_n\})-\lambda-\varepsilon. \tag{4}$$

Because $\{\|z-x_{n_i}\|\}$ is a subsequence of $\{\|z-x_n\|\}$, this implies that $\delta_a(z,\{x_n\})=\limsup_{n\to\infty}\|z-x_n\|\geq\limsup_{i\to\infty}\|z-x_{n_i}\|$ and hence it follows from (4) that $\delta_a(z,\{x_n\})+\varepsilon\geq\delta_a(G,\{x_n\})-\lambda$. Since $\varepsilon>0$ is arbitrary, we conclude $\delta_a(z,\{x_n\})\geq\delta_a(G,\{x_n\})-\lambda$, i.e., $z\in F_a^{\lambda}(G,\{x_n\})$ and so (1) is valid. Now from (1) and Remark 2.11, we have

$$\left(\bigcap_{0<\lambda<\delta_a(G,\{x_n\})} \overline{\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty}\Big(B(x_i,\delta_a(G,\{x_n\})-\lambda)\Big)^c\right)\right)}\right)\bigcap G \subseteq F_a(G,\{x_n\}).$$

Conversely, assume that $z\in F_a(G,\{x_n\})$. Then $z\in G$. Suppose, for contradiction,

$$z\notin\bigcap_{0<\lambda<\delta_a(G,\{x_n\})} \overline{\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty}\Big(B(x_i,\delta_a(G,\{x_n\})-\lambda)\Big)^c\right)\right)}.$$

Thus, there exists a $0<\lambda<\delta_a(G,\{x_n\})$ such that $z\notin\overline{(\bigcap_{n=1}^{\infty}(\bigcup_{i=n}^{\infty}(B(x_i,\delta_a(G,\{x_n\})-\lambda))^c))}$. Therefore, $z\notin\bigcap_{n=1}^{\infty}(\bigcup_{i=n}^{\infty}(B(x_i,\delta_a(G,\{x_n\})-\lambda))^c)$ and so by De Morgan law, $z\in\bigcup_{n=1}^{\infty}(\bigcap_{i=n}^{\infty}B(x_i,\delta_a(G,\{x_n\})-\lambda))$. Thus, there exists $n_0\in\mathbb{N}$ such that $\|z-x_i\|<\delta_a(G,\{x_n\})-\lambda$ for all $i\geq n_0$. Then $\delta_a(z,\{x_n\})=\limsup_{n\to\infty}\|z-x_n\|\leq\delta_a(G,\{x_n\})-\lambda$. Moreover, because $z\in F_a(G,\{x_n\})$, it follows that $\delta_a(z,\{x_n\})=\delta_a(G,\{x_n\})$ and hence $\delta_a(G,\{x_n\})\leq\delta_a(G,\{x_n\})-\lambda$, which is a contradiction. Therefore,

$$z\in\left(\bigcap_{0<\lambda<\delta_a(G,\{x_n\})} \overline{\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty}\Big(B(x_i,\delta_a(G,\{x_n\})-\lambda)\Big)^c\right)\right)}\right)\bigcap G$$

and so we are done. □

At the end of this section, we discuss conditions under which $F_a(G, \{x_n\}) \subseteq \partial G$ ($\partial G$ is the set of all frontier points of $G$). We remind that $K$ is a nonempty convex subset of $X$. According to Theorem 3.4.6 of [10] (The Maximum Principle), we have if $f : K \to \mathbb{R}$ is a convex function and attains a global maximum at an interior point of $K$, then $f$ is constant. Therefore, from the fact that $K \subseteq \overline{K} = K^\circ \cup \partial K$, we conclude the following proposition:

**Proposition 2.13.** *If $f : K \to \mathbb{R}$ is a nonconstant convex function, then the set of all points that $f$ attains a global maximum at them is a subset of $\partial K$.*

Now we infer the following theorem:

**Theorem 2.14.** *If $X$ is uniformly convex and $G$ convex, then $F_a(G, \{x_n\}) \subseteq \partial G$.*

*Proof.* Consider the convex function $\delta_a(\cdot, \{x_n\}) : G \to \mathbb{R}$. If this function is nonconstant, then from Proposition 2.13, we have $F_a(G, \{x_n\}) \subseteq \partial G$. But otherwise, if this function is constant, then $F_a(G, \{x_n\}) = G$ and so $F_a(G, \{x_n\})$ is nonempty and convex. Thus, from uniformly convexity of $X$ and Theorem 2.3.13 of [1], we conclude $F_a(G, \{x_n\})$ is singleton. Therefore, $G$ is singleton and hence $F_a(G, \{x_n\}) \subseteq \partial G$. $\square$

In the above theorem, if in addition, $G$ is open, then $F_a(G, \{x_n\}) = \emptyset$. Also, if $X$ is finite-dimensional, then the uniform convexity of $X$ and convexity of $G$ can be deleted in the above theorem. To prove this, we need to the following two propositions:

**Proposition 2.15.** *If $X$ is finite-dimensional, then $F_a(G, \{x_n\}) \subseteq \bigcup_{x \in A} F_G(x)$, where $A$ is the set of all limits of the convergent subsequences of $\{x_n\}$.*

*Proof.* Assume that $z \in F_a(G, \{x_n\})$. Then $z \in G$ and $\limsup_{n\to\infty} \|z - x_n\| = \sup_{x \in G} \limsup_{n\to\infty} \|x - x_n\|$. The property of limit superior implies the existence of a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty} \|z - x_n\| = \lim_{i\to\infty} \|z - x_{n_i}\|$. Thus, we obtain

$$\lim_{i\to\infty} \|z - x_{n_i}\| \geq \limsup_{n\to\infty} \|x - x_n\| \quad \text{for all } x \in G. \tag{5}$$

Because $X$ is finite-dimensional and $\{x_{n_i}\}$ is bounded, by Theorem 1.7.5 of [1] (The Heine-Borel Theorem), there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \to x_0 \in A$ as $j \to \infty$. Therefore, from (5), we conclude

$$\|z - x_0\| = \lim_{j\to\infty} \|z - x_{n_{i_j}}\| = \lim_{i\to\infty} \|z - x_{n_i}\| \geq \limsup_{n\to\infty} \|x - x_n\| \geq \lim_{j\to\infty} \|x - x_{n_{i_j}}\| = \|x - x_0\| \quad \text{for all } x \in G.$$

Then $\|x_0 - z\| \geq \delta(x_0, G)$. On the other hand, since $z \in G$, we have $\delta(x_0, G) \geq \|x_0 - z\|$ and so $\delta(x_0, G) = \|x_0 - z\|$, i.e., $z \in F_G(x_0)$. Hence $z \in \bigcup_{x \in A} F_G(x)$. $\square$

**Proposition 2.16.** *Let $x \in X$. Then $F_G(x) \subseteq \partial G$.*

*Proof.* Obviously, if $G$ is singleton, then $F_G(x) \subseteq \partial G$. Assume that $G$ is not singleton and $z \in F_G(x)$. Because $z \in G$, we have $z \in \overline{G}$. Suppose, for contradiction, $z \notin \overline{G^c}$. Then there exists $r > 0$ such that $B(z, r) \subseteq G$. For each $n \in \mathbb{N}$, let $z_n = \frac{-1}{n}x + (1 + \frac{1}{n})z$. Therefore, $\|z_n - z\| = \frac{1}{n}\|x - z\|$ for all $n \in \mathbb{N}$. Then for sufficiently large $N \in \mathbb{N}$, $\|z_N - z\| < r$, i.e., $z_N \in B(z, r) \subseteq G$. Since $G$ is not singleton, we have $x \neq z$ and so $\|x - z\| > 0$. Thus,

$$\|x - z_N\| = (1 + \frac{1}{N})\|x - z\| > \|x - z\| = \delta(x, G),$$

which is a contradiction. Therefore, $z \in \overline{G^c}$ and so $z \in \overline{G} \cap \overline{G^c} = \partial G$. $\square$

Now from Propositions 2.15 and 2.16, we obtain the following theorem:

**Theorem 2.17.** *If $X$ is finite-dimensional, then $F_a(G, \{x_n\}) \subseteq \partial G$.*

In the above theorem, if in addition, $G$ is open, then $F_a(G, \{x_n\}) = \emptyset$.

## 3. The Relation between Asymptotic Farthest Points and Extreme Points

In this section and next section, the space $X$ is described in any position, but always $G$ is a nonempty bounded subset of $X$ and $\{x_n\}$ a bounded sequence in $X$. Also, for a subset $A \subseteq X$, $ext(A)$ is the set of all extreme points of $A$ ([2], Definition 7.61) and $co(A)$ ($\overline{co}(A)$) is the convex (closed convex) hull of $A$.

One of the main purposes of this section is to discuss conditions under which $ext(F_a(G, \{x_n\})) \neq \emptyset$. Also, we investigate the relation between the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $G$ and the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $ext(G)$.

Let $X$ be a normed space. According to Lemma 7.64 of [2], if $K$ is a nonempty convex subset of $X$ and $f : K \to \mathbb{R}$ a convex function, then the set of all maximizers of $f$ is either an extreme subset of $K$ ([2], Definition 7.61) or is empty. In a similar way, because the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ is convex, we conclude that $F_a(G, \{x_n\})$ is either an extreme subset of $G$ ($G$ is not necessarily convex) or is empty. In particular, if $F_a(G, \{x_n\}) = \{z\}$, then $z \in ext(G)$. Now we infer the following proposition:

**Proposition 3.1.** *Let $X$ be a normed space. Then $ext(F_a(G, \{x_n\})) = F_a(G, \{x_n\}) \cap ext(G)$.*

*Proof.* If $F_a(G, \{x_n\}) = \emptyset$, we are done, so assume $F_a(G, \{x_n\}) \neq \emptyset$. Since $F_a(G, \{x_n\}) \subseteq G$, we have $F_a(G, \{x_n\}) \cap ext(G) \subseteq ext(F_a(G, \{x_n\}))$. Conversely, suppose $z \in ext(F_a(G, \{x_n\}))$. Then $z \in F_a(G, \{x_n\}) \subseteq G$. Assume $t \in (0, 1)$, $x, y \in G$ and $z = tx + (1 - t)y$. Because $F_a(G, \{x_n\})$ is an extreme subset of $G$, it follows that $x, y \in F_a(G, \{x_n\})$ and hence $x = y = z$. Therefore, $z \in F_a(G, \{x_n\}) \cap ext(G)$ and so we are done. □

In the following, we discuss conditions under which $ext(F_a(G, \{x_n\})) \neq \emptyset$, by the above proposition, this means conditions under which there exists at least one asymptotic farthest point of $\{x_n\}$ in $G$ such that it is an extreme point of $G$. For instance, from Lemma V.8.2 of [4], we have the following proposition:

**Proposition 3.2.** *Let $X$ be a normed space and $F_a(G, \{x_n\})$ nonempty and weakly compact. Then $ext(F_a(G, \{x_n\})) \neq \emptyset$.*

Obviously, it follows from the above proposition that if $X$ is a normed space and $F_a(G, \{x_n\})$ nonempty and finite, then $ext(F_a(G, \{x_n\})) \neq \emptyset$. Also, from this proposition, we conclude the following corollary:

**Corollary 3.3.** *Let $X$ be a normed space and $G$ compact. Then $ext(F_a(G, \{x_n\})) \neq \emptyset$.*

It follows from the above corollary that if $X$ is a normed space and $G$ finite, then $ext(F_a(G, \{x_n\})) \neq \emptyset$. Also, this corollary and Theorem 1.7.5 of [1] (The Heine-Borel Theorem) imply that if $X$ is a finite-dimensional normed space and $G$ closed, then $ext(F_a(G, \{x_n\})) \neq \emptyset$.

If we want to replace the compactness of $G$ in Corollary 3.3 by weakly compactness of $G$, then we obtain the following theorem:

**Theorem 3.4.** *Let $X$ be a real normed space, $G$ weakly compact and $F_a(G, \{x_n\}) \neq \emptyset$. Then $ext(F_a(G, \{x_n\})) \neq \emptyset$.*

*Proof.* Set $C := \{x \in X : \delta_a(x, \{x_n\}) \leq \delta_a(G, \{x_n\})\}$. Since the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ is convex, then $C$ is convex.

If $\delta_a(x, \{x_n\}) = \delta_a(G, \{x_n\})$ for all $x \in G$, then $F_a(G, \{x_n\}) = G$ and so from Lemma V.8.2 of [4], we are done. But otherwise, assume that there exists $x_0 \in G$ such that $\delta_a(x_0, \{x_n\}) < \delta_a(G, \{x_n\})$. Thus, by the continuity of the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$, we have $C^\circ \neq \emptyset$.

Consider the convex function $g : C \to \mathbb{R}$ defined by $g(x) = \delta_a(x, \{x_n\})$, $x \in C$, and suppose $z \in F_a(G, \{x_n\})$. This function is nonconstant and attains a global maximum at $z$. Thus, from Proposition 2.13, we obtain $z \in \partial C$. Therefore, from the fact that $\partial C \subseteq (C^\circ)^c$, we conclude $z \in (C^\circ)^c$.

Now from Theorem 1 in Section 5.12 of [6] (Mazur's Theorem, Geometric Hahn-Banach Theorem), there exists a nonzero element $f \in X^*$ and a constant $\alpha \in \mathbb{R}$ such that $f(z) = \alpha$ and $f(x) < \alpha$ for all $x \in C^\circ$, i.e., there exists a closed hyperplane $H = \{x \in X : f(x) = \alpha\}$ in $X$ such that $z \in H$ and

$$H \cap C^\circ = \emptyset. \tag{1}$$

It follows from Lemma 5.28 of [2] that

$$f(x) \leq \alpha \quad \text{for all } x \in C. \tag{2}$$

Set $K := H \cap F_a(G, \{x_n\})$. Then $z \in K$, i.e., $K \neq \emptyset$. We prove that $K$ is weakly compact and hence we infer from Lemma V.8.2 of [4] that $ext(K) \neq \emptyset$. To prove the weakly compactness of $K$, assume $\{z_n\}$ be a sequence in $K$. Therefore, $f(z_n) = \alpha$ and $z_n \in F_a(G, \{x_n\}) \subseteq G$ for all $n \in \mathbb{N}$. Because $G$ is weakly compact, it follows from Theorem 2.8.6 of [8] (The Eberlein-Smulian Theorem) that there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ and $z_0 \in G$ such that $z_{n_i} \xrightarrow{w} z_0$ as $i \to \infty$ and so we have $f(z_{n_i}) \to f(z_0)$ as $i \to \infty$. Thus, $f(z_0) = \alpha$, i.e., $z_0 \in H$. Suppose, for contradiction, $\delta_a(z_0, \{x_n\}) < \delta_a(G, \{x_n\})$. Then by the continuity of the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$, we have $z_0 \in C^\circ$ and hence from (1), we obtain $z_0 \notin H$, which is a contradiction. Therefore, $z_0 \in F_a(G, \{x_n\})$. Thus, $z_0 \in K$ and so from Theorem 2.8.6 of [8] (The Eberlein-Smulian Theorem), $K$ is weakly compact. This implies that $ext(K) \neq \emptyset$.
Finally, we prove $ext(K) \subseteq ext(F_a(G, \{x_n\}))$ and so we infer $ext(F_a(G, \{x_n\})) \neq \emptyset$. Let $z' \in ext(K)$. Then $f(z') = \alpha$ and $z' \in F_a(G, \{x_n\})$. Assume $t \in (0, 1)$, $x, y \in F_a(G, \{x_n\})$ and $z' = tx + (1-t)y$. Since $x, y \in F_a(G, \{x_n\})$, we have $x, y \in C$. Therefore, from (2), we obtain $f(x) \leq \alpha$ and $f(y) \leq \alpha$. If $f(x) < \alpha$ or $f(y) < \alpha$, then we have

$$\alpha = f(z') = f(tx + (1-t)y) = tf(x) + (1-t)f(y) < t\alpha + (1-t)\alpha = \alpha,$$

which is a contradiction. Thus, $f(x) = f(y) = \alpha$, i.e., $x, y \in H$. Then $x, y \in K$ and so because $z' \in ext(K)$, we conclude $x = y = z'$. Therefore, $z' \in ext(F_a(G, \{x_n\}))$. This implies that $ext(F_a(G, \{x_n\})) \neq \emptyset$. □

Now we obtain the following corollary:

**Corollary 3.5.** *Let $X$ be a reflexive real normed space, $G$ weakly closed and $F_a(G, \{x_n\}) \neq \emptyset$. Then $ext(F_a(G, \{x_n\})) \neq \emptyset$.*

*Proof.* Because $X$ is reflexive and $G$ bounded and weakly closed, it follows from Theorem 2.8.6 of [8] (The Eberlein-Smulian Theorem) and Corollary 2.8.9 of [8] that $G$ is weakly compact and so by Theorem 3.4, we are done. □

In the following we discuss conditions under which we have $F_a(G, \{x_n\})$ is nonempty (singleton) if and only if $F_a(ext(G), \{x_n\})$ is nonempty (singleton). Let $X$ be a normed space. According to Proposition 3.1 of [3], if $K$ is a nonempty weakly compact subset of $X$ and $f : X \to \mathbb{R}$ a continuous convex function, then $\sup_{x \in K} f(x) = \sup_{x \in ext(K)} f(x)$. In a similar way, because the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ is continuous and convex, we infer the following proposition:

**Proposition 3.6.** *Let $X$ be a normed space and $G$ weakly compact. Then $\delta_a(G, \{x_n\}) = \delta_a(ext(G), \{x_n\})$.*

Obviously, from the above proposition and Proposition 3.1, we have the following corollary:

**Corollary 3.7.** *Let $X$ be a normed space and $G$ weakly compact. Then*

$$F_a(ext(G), \{x_n\}) = F_a(G, \{x_n\}) \cap ext(G) = ext(F_a(G, \{x_n\})).$$

Now we give the following two theorems:

**Theorem 3.8.** *Let $X$ be a real normed space and $G$ weakly compact. Then $F_a(G, \{x_n\}) \neq \emptyset$ if and only if*

$$F_a(ext(G), \{x_n\}) \neq \emptyset.$$

*Proof.* From Theorem 3.4 and Corollary 3.7 is obvious. □

**Theorem 3.9.** *Let $X$ be a real normed space and $G$ weakly compact. Then $F_a(G, \{x_n\}) = \{z\}$ if and only if $F_a(ext(G), \{x_n\}) = \{z\}$.*

*Proof.* Suppose $F_a(G, \{x_n\}) = \{z\}$. Then by Theorem 3.8, $F_a(ext(G), \{x_n\}) \neq \emptyset$. On the other hand, from Corollary 3.7, we have $F_a(ext(G), \{x_n\}) \subseteq F_a(G, \{x_n\})$. Thus, we obtain $F_a(ext(G), \{x_n\}) = \{z\}$. Conversely, assume that $F_a(ext(G), \{x_n\}) = \{z\}$. Then by Theorem 3.8, $F_a(G, \{x_n\}) \neq \emptyset$. Set $C := \{x \in X : \delta_a(x, \{x_n\}) \leq \delta_a(G, \{x_n\})\}$. Since the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ is convex, then $C$ is convex.
If $\delta_a(x, \{x_n\}) = \delta_a(G, \{x_n\})$ for all $x \in ext(G)$, then $F_a(ext(G), \{x_n\}) = ext(G)$ and so $ext(G) = \{z\}$. Therefore,

$\overline{co}(ext(G)) = \{z\}$ and hence from Theorem V.8.4 of [4] (The Krein-Milman Theorem), we have $G = \{z\}$. This implies $F_a(G, \{x_n\}) = \{z\}$. But otherwise, assume that there exists $x_0 \in ext(G)$ such that $\delta_a(x_0, \{x_n\}) < \delta_a(G, \{x_n\})$. Thus, by the continuity of the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$, we have $C^\circ \neq \emptyset$.
Consider the convex function $g : C \to \mathbb{R}$ defined by $g(x) = \delta_a(x, \{x_n\})$, $x \in C$, and suppose $\overline{z} \in F_a(G, \{x_n\})$. This function is nonconstant and attains a global maximum at $\overline{z}$. Thus, from Proposition 2.13, we obtain $\overline{z} \in \partial C$. Therefore, from the fact that $\partial C \subseteq (C^\circ)^c$, we conclude $\overline{z} \in (C^\circ)^c$.
Now from Theorem 1 in Section 5.12 of [6] (Mazur's Theorem, Geometric Hahn-Banach Theorem), there exists a nonzero element $f \in X^*$ and a constant $\alpha \in \mathbb{R}$ such that $f(\overline{z}) = \alpha$ and $f(x) < \alpha$ for all $x \in C^\circ$, i.e., there exists a closed hyperplane $H = \{x \in X : f(x) = \alpha\}$ in $X$ such that $\overline{z} \in H$ and

$$H \cap C^\circ = \emptyset. \tag{3}$$

It follows from Lemma 5.28 of [2] that

$$f(x) \leq \alpha \quad \text{for all } x \in C. \tag{4}$$

Set $K := H \cap F_a(G, \{x_n\})$. Then $\overline{z} \in K$, i.e., $K \neq \emptyset$. Similar to the proof of Theorem 3.4, from weakly compactness of $G$ and (3), it follows that $K$ is weakly compact. Then by Lemma V.8.2 of [4], we conclude $ext(K) \neq \emptyset$ and by Theorem V.8.4 of [4] (The Krein-Milman Theorem), we conclude

$$K \subseteq \overline{co}(ext(K)). \tag{5}$$

Also, similar to the proof of Theorem 3.4, from (4), we have $ext(K) \subseteq ext(F_a(G, \{x_n\}))$. Now from Corollary 3.7, we obtain

$$ext(K) \subseteq ext(F_a(G, \{x_n\})) = F_a(ext(G), \{x_n\}). \tag{6}$$

If $ext(K)$ contain more than one element, then by (6), we infer $F_a(ext(G), \{x_n\})$ contain more than one element, which is a contradiction. Thus, from the fact that $ext(K) \neq \emptyset$, we infer $ext(K)$ is singleton. Therefore, from (5), we have $K$ is singleton. Infact, $K = \{\overline{z}\}$. Then $ext(K) = \{\overline{z}\}$ and hence from (6), we conclude $\overline{z} \in F_a(ext(G), \{x_n\})$. Thus, we have $\overline{z} = z$. This implies $F_a(G, \{x_n\}) = \{z\}$. □

At the end of this section, we infer some results that say us under which conditions we have

$$ext(F_a(\overline{co}(G), \{x_n\})) \neq \emptyset.$$

These results will be used in Section 4.

**Proposition 3.10.** *Let $X$ be a reflexive real normed space and $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Then $ext(F_a(\overline{co}(G), \{x_n\})) \neq \emptyset$.*

*Proof.* It follows from Theorem 2.5.16 of [8] (The Mazur Theorem) that $\overline{co}(G)$ is weakly closed. Thus, by Corollary 3.5, we are done. □

**Proposition 3.11.** *Let $X$ be a real Banach space, $G$ weakly compact and $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Then*

$$ext(F_a(\overline{co}(G), \{x_n\})) \neq \emptyset.$$

*Proof.* By Theorem 2.8.14 of [8] (The Krein-Smulian Weak Compactness Theorem), $\overline{co}(G)$ is weakly compact. Therefore, from Theorem 3.4, we are done. □

**Proposition 3.12.** *Let $X$ be a real normed space, $\overline{co}(G)$ weakly compact and $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Then*

$$ext(F_a(\overline{co}(G), \{x_n\})) \neq \emptyset.$$

*Proof.* From Theorem 3.4 is obvious. □

## 4. Convexity Results

In this section, we obtain some convexity results for the new concept of asymptotic farthest points. In particular, one of the main purposes of this section is to discuss conditions under which we have $F_a(G, \{x_n\})$ is nonempty (singleton) if and only if $F_a(\overline{co}(G), \{x_n\})$ is nonempty (singleton). At the beginning of this section, we give the following proposition:

**Proposition 4.1.** *Let $X$ be a normed space, $x, y \in G$ and $F_a(G, \{x_n\}) \cap L(x, y) \neq \emptyset$. Then $G \cap L[x, y] \subseteq F_a(G, \{x_n\})$ ($L(x, y)$ ($L[x, y]$) is the open (closed) line segment between $x$ and $y$).*

*Proof.* Suppose $z \in F_a(G, \{x_n\}) \cap L(x, y)$. Because $F_a(G, \{x_n\})$ is an extreme subset of $G$, we have $x, y \in F_a(G, \{x_n\})$. Assume that $z' \in G \cap L(x, y)$ and $z \neq z'$. Then there exist $t, t' \in (0, 1)$ such that $z = tx + (1 - t)y$ and $z' = t'x + (1 - t')y$. Therefore, we obtain

$$z = \frac{t}{t'}z' + (1 - \frac{t}{t'})y, \tag{1}$$

$$z = (1 - \frac{1-t}{1-t'})x + \frac{1-t}{1-t'}z'. \tag{2}$$

Because $z \neq z'$, we have either $t < t'$ or $t > t'$. If $t < t'$, then from (1) and the fact that $F_a(G, \{x_n\})$ is an extreme subset of $G$, we conclude $z' \in F_a(G, \{x_n\})$. Also, if $t > t'$, then from (2) and the fact that $F_a(G, \{x_n\})$ is an extreme subset of $G$, we conclude $z' \in F_a(G, \{x_n\})$. Thus, we have $G \cap L[x, y] \subseteq F_a(G, \{x_n\})$. □

In the above proposition, if in addition, $G$ is convex, then $L[x, y] \subseteq F_a(G, \{x_n\})$. Also, combining this proposition with Theorem 2.14 or Theorem 2.17, we obtain the following corollaries, respectively:

**Corollary 4.2.** *Let $X$ be a uniformly convex normed space, $G$ convex, $x, y \in G$ and $F_a(G, \{x_n\}) \cap L(x, y) \neq \emptyset$. Then $L[x, y] \subseteq \partial G$.*

**Corollary 4.3.** *Let $X$ be a finite-dimensional normed space, $x, y \in G$ and $F_a(G, \{x_n\}) \cap L(x, y) \neq \emptyset$. Then $G \cap L[x, y] \subseteq \partial G$.*

In the above corollary, if in addition, $G$ is convex, then $L[x, y] \subseteq \partial G$.

In Lemmas 2.5 and 2.6, we showed that if $X$ is a normed space, then $\delta_a(G, \{x_n\}) = \delta_a(\overline{G}, \{x_n\}) = \delta_a(\overline{G^w}, \{x_n\})$. Now we give the following lemma:

**Lemma 4.4.** *Let $X$ be a normed space. Then $\delta_a(G, \{x_n\}) = \delta_a(co(G), \{x_n\})$.*

*Proof.* Since $G \subseteq co(G)$, it follows that $\delta_a(G, \{x_n\}) \leq \delta_a(co(G), \{x_n\})$. To prove the reversed inequality, let $z \in co(G)$. Then there exist $n \in \mathbb{N}$, $t_1, t_2, ..., t_n \in [0, 1]$ and $z_1, z_2, ..., z_n \in G$ such that $\sum_{i=1}^n t_i = 1$ and $z = \sum_{i=1}^n t_i z_i$. Because the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$ is convex, we have

$$\delta_a(z, \{x_n\}) \leq \sum_{i=1}^n t_i \delta_a(z_i, \{x_n\}) \leq \sum_{i=1}^n t_i \delta_a(G, \{x_n\}) = \delta_a(G, \{x_n\}).$$

Now by taking the supremum over $z \in co(G)$, we obtain $\delta_a(co(G), \{x_n\}) \leq \delta_a(G, \{x_n\})$ and so we are done. □

Therefore, from the fact that $\overline{co}(G) = \overline{co(G)}$ ([8], Page 21), we infer the following corollary:

**Corollary 4.5.** *Let $X$ be a normed space. Then we have*

$$\delta_a(G, \{x_n\}) = \delta_a(\overline{G}, \{x_n\}) = \delta_a(\overline{G^w}, \{x_n\}) = \delta_a(co(G), \{x_n\}) = \delta_a(\overline{co}(G), \{x_n\}).$$

We now establish the relation between the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $G$ and the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $co(G)$.

**Theorem 4.6.** *Let $X$ be a normed space. Then we have $F_a(G, \{x_n\}) \neq \emptyset$ if and only if $F_a(co(G), \{x_n\}) \neq \emptyset$.*

*Proof.* Suppose $F_a(G, \{x_n\}) \neq \emptyset$. Then there exists $z \in G \subseteq co(G)$ such that $\delta_a(z, \{x_n\}) = \delta_a(G, \{x_n\})$. Thus, by Lemma 4.4, we have $\delta_a(z, \{x_n\}) = \delta_a(co(G), \{x_n\})$ and so $z \in F_a(co(G), \{x_n\})$, i.e., $F_a(co(G), \{x_n\}) \neq \emptyset$. Conversely, assume that $F_a(co(G), \{x_n\}) \neq \emptyset$. Then there exists $z \in co(G)$ such that $\delta_a(z, \{x_n\}) = \delta_a(co(G), \{x_n\})$. Thus, by Lemma 4.4, we have $\delta_a(z, \{x_n\}) = \delta_a(G, \{x_n\})$. If $z \in G$, then we are done. But if $z \notin G$, then there exist $n \geq 2$, $t_1, t_2, ..., t_n \in (0, 1)$ and $z_1, z_2, ..., z_n \in G$ such that $\sum_{i=1}^{n} t_i = 1$ and $z = \sum_{i=1}^{n} t_i z_i$. Suppose, for contradiction, $\delta_a(z_j, \{x_n\}) < \delta_a(G, \{x_n\})$ for some $j \in \{1, 2, ..., n\}$. Then by the convexity of the function $\delta_a(\cdot, \{x_n\}) : X \to \mathbb{R}$, we have

$$\delta_a(G, \{x_n\}) = \delta_a(z, \{x_n\}) \leq \sum_{i=1}^{n} t_i \delta_a(z_i, \{x_n\}) < \sum_{i=1}^{n} t_i \delta_a(G, \{x_n\}) = \delta_a(G, \{x_n\}),$$

which is a contradiction. Thus, $\delta_a(z_i, \{x_n\}) = \delta_a(G, \{x_n\})$ for all $i \in \{1, 2, ..., n\}$ and so $z_i \in F_a(G, \{x_n\})$ for all $i \in \{1, 2, ..., n\}$, i.e., $F_a(G, \{x_n\}) \neq \emptyset$. □

**Theorem 4.7.** *Let $X$ be a normed space. Then we have $F_a(G, \{x_n\}) = \{z\}$ if and only if $F_a(co(G), \{x_n\}) = \{z\}$.*

*Proof.* Suppose $F_a(co(G), \{x_n\}) = \{z\}$. Then by Theorem 4.6, $F_a(G, \{x_n\}) \neq \emptyset$. On the other hand, from Lemma 4.4, we have $F_a(G, \{x_n\}) \subseteq F_a(co(G), \{x_n\})$. Thus, we obtain $F_a(G, \{x_n\}) = \{z\}$. Conversely, assume that $F_a(G, \{x_n\}) = \{z\}$. Then by Theorem 4.6, $F_a(co(G), \{x_n\}) \neq \emptyset$. Let $z' \in F_a(co(G), \{x_n\})$. Therefore, $z' \in co(G)$ and $\delta_a(z', \{x_n\}) = \delta_a(co(G), \{x_n\})$. Thus, from Lemma 4.4, we have $\delta_a(z', \{x_n\}) = \delta_a(G, \{x_n\})$. Suppose, for contradiction, $z' \notin G$. Then there exist $n \geq 2$, $t_1, t_2, ..., t_n \in (0, 1)$ and $z_1, z_2, ..., z_n \in G$ such that $\sum_{i=1}^{n} t_i = 1$ and $z' = \sum_{i=1}^{n} t_i z_i$. Similar to the proof of Theorem 4.6, we have $z_i \in F_a(G, \{x_n\})$ for all $i \in \{1, 2, ..., n\}$. Thus, $z_i = z$ for all $i \in \{1, 2, ..., n\}$ and hence $z' = z$. Then $z \notin G$, which is a contradiction. Therefore, we have $z' \in G$ and so $z' \in F_a(G, \{x_n\})$. Thus, $z' = z$ and hence we conclude $F_a(co(G), \{x_n\}) = \{z\}$. □

The importance of the above results is that these two theorems say us for study conditions under which $F_a(G, \{x_n\})$ is nonempty or singleton, we can assume that $G$ is convex. If $co(G)$ is replaced by $\overline{G}$ or $\overline{G^w}$ or $\overline{co}(G)$, then Theorems 4.6 and 4.7 are not valid. To confirm this, we note to the following example:

**Example 4.8.** *Let $X = \mathbb{R}$ with usual metric and $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Then $F_a((-1, 1), \{x_n\}) = \emptyset$, $F_a([-1, 1), \{x_n\}) = \{-1\}$, $F_a((-1, 1], \{x_n\}) = \{1\}$ and $F_a([-1, 1], \{x_n\}) = \{-1, 1\}$.*

Now we discuss conditions under which the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $G$ is equivalent to the existence (uniqueness) of asymptotic farthest point of $\{x_n\}$ in $\overline{co}(G)$.

**Theorem 4.9.** *Let $X$ be a real Banach space and $G$ weakly compact. Then $F_a(G, \{x_n\}) \neq \emptyset$ if and only if $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$.*

*Proof.* Suppose $F_a(G, \{x_n\}) \neq \emptyset$. Then there exists $z \in G \subseteq \overline{co}(G)$ such that $\delta_a(z, \{x_n\}) = \delta_a(G, \{x_n\})$. Thus, by Corollary 4.5, we have $\delta_a(z, \{x_n\}) = \delta_a(\overline{co}(G), \{x_n\})$ and so $z \in F_a(\overline{co}(G), \{x_n\})$, i.e., $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Conversely, assume that $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Then from Proposition 3.11, we conclude $ext(F_a(\overline{co}(G), \{x_n\})) \neq \emptyset$. Thus, by Proposition 3.1, there exists $z \in F_a(\overline{co}(G), \{x_n\})$ such that $z \in ext(\overline{co}(G))$. Because $z \in F_a(\overline{co}(G), \{x_n\})$, we have $\delta_a(z, \{x_n\}) = \delta_a(\overline{co}(G), \{x_n\})$ and hence from Corollary 4.5, we obtain $\delta_a(z, \{x_n\}) = \delta_a(G, \{x_n\})$. On the other hand, since $X$ is a Banach space, $G$ weakly compact (and so weakly closed) and $z \in ext(\overline{co}(G))$, it follows from Theorem 2.8.14 of [8] (The Krein-Smulian Weak Compactness Theorem) and Corollary 2.10.16 of [8] that $z \in G$. This implies $z \in F_a(G, \{x_n\})$, i.e., $F_a(G, \{x_n\}) \neq \emptyset$. □

**Theorem 4.10.** *Let $X$ be a real normed space, $G$ weakly closed and $\overline{co}(G)$ weakly compact. Then $F_a(G, \{x_n\}) \neq \emptyset$ if and only if $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$.*

*Proof.* Similar to the beginning of the proof of Theorem 4.9, if $F_a(G, \{x_n\}) \neq \emptyset$, then $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Conversely, assume that $F_a(\overline{co}(G), \{x_n\}) \neq \emptyset$. Thus, from Proposition 3.12, we conclude $ext(F_a(\overline{co}(G), \{x_n\})) \neq \emptyset$. Therefore, by Proposition 3.1, there exists $z \in F_a(\overline{co}(G), \{x_n\})$ such that $z \in ext(\overline{co}(G))$. Since $z \in F_a(\overline{co}(G), \{x_n\})$, we have $\delta_a(z, \{x_n\}) = \delta_a(\overline{co}(G), \{x_n\})$ and so from Corollary 4.5, we obtain $\delta_a(z, \{x_n\}) = \delta_a(G, \{x_n\})$. On the other hand, because $G$ is weakly closed, $\overline{co}(G)$ weakly compact and $z \in ext(\overline{co}(G))$, it follows from Corollary 2.10.16 of [8] that $z \in G$. This implies $z \in F_a(G, \{x_n\})$, i.e., $F_a(G, \{x_n\}) \neq \emptyset$. □

**Theorem 4.11.** *Let $X$ be a real normed space, $G$ weakly closed and $\overline{co}(G)$ weakly compact. Then $F_a(G,\{x_n\}) = \{z\}$ if and only if $F_a(\overline{co}(G),\{x_n\}) = \{z\}$.*

*Proof.* Suppose $F_a(\overline{co}(G),\{x_n\}) = \{z\}$. Then by Theorem 4.10, $F_a(G,\{x_n\}) \neq \emptyset$. On the other hand, from Corollary 4.5, we have $F_a(G,\{x_n\}) \subseteq F_a(\overline{co}(G),\{x_n\})$. Thus, we obtain $F_a(G,\{x_n\}) = \{z\}$. Conversely, assume that $F_a(G,\{x_n\}) = \{z\}$. Then by Theorem 4.10, $F_a(\overline{co}(G),\{x_n\}) \neq \emptyset$. Set $C := \{x \in X : \delta_a(x,\{x_n\}) \leq \delta_a(G,\{x_n\})\}$. Since the function $\delta_a(\cdot,\{x_n\}) : X \to \mathbb{R}$ is convex, then $C$ is convex.
If $\delta_a(x,\{x_n\}) = \delta_a(G,\{x_n\})$ for all $x \in G$, then $F_a(G,\{x_n\}) = G$ and so $G = \{z\}$. Therefore, $\overline{co}(G) = \{z\}$ and hence $F_a(\overline{co}(G),\{x_n\}) = \{z\}$. But otherwise, assume that there exists $x_0 \in G$ such that $\delta_a(x_0,\{x_n\}) < \delta_a(G,\{x_n\})$. Thus, by the continuity of the function $\delta_a(\cdot,\{x_n\}) : X \to \mathbb{R}$, we have $C^\circ \neq \emptyset$.
Consider the convex function $g : C \to \mathbb{R}$ defined by $g(x) = \delta_a(x,\{x_n\})$, $x \in C$, and suppose $\bar{z} \in F_a(\overline{co}(G),\{x_n\})$. This function is nonconstant and attains a global maximum at $\bar{z}$. Thus, from Proposition 2.13, we obtain $\bar{z} \in \partial C$. Therefore, from the fact that $\partial C \subseteq (C^\circ)^c$, we conclude $\bar{z} \in (C^\circ)^c$.
Now from Theorem 1 in Section 5.12 of [6] (Mazur's Theorem, Geometric Hahn-Banach Theorem), there exists a nonzero element $f \in X^*$ and a constant $\alpha \in \mathbb{R}$ such that $f(\bar{z}) = \alpha$ and $f(x) < \alpha$ for all $x \in C^\circ$, i.e., there exists a closed hyperplane $H = \{x \in X : f(x) = \alpha\}$ in $X$ such that $\bar{z} \in H$ and

$$H \cap C^\circ = \emptyset. \tag{3}$$

It follows from Lemma 5.28 of [2] that

$$f(x) \leq \alpha \quad \text{for all } x \in C. \tag{4}$$

Set $K := H \cap F_a(\overline{co}(G),\{x_n\})$. Then $\bar{z} \in K$, i.e., $K \neq \emptyset$. Similar to the proof of Theorem 3.4, from weakly compactness of $\overline{co}(G)$ and (3), it follows that $K$ is weakly compact. Then by Lemma V.8.2 of [4], we conclude $ext(K) \neq \emptyset$ and by Theorem V.8.4 of [4] (The Krein-Milman Theorem), we conclude

$$K \subseteq \overline{co}(ext(K)). \tag{5}$$

Also, similar to the proof of Theorem 3.4, from (4), it follows that $ext(K) \subseteq ext(F_a(\overline{co}(G),\{x_n\}))$. Now from Proposition 3.1 and Corollary 2.10.16 of [8], we obtain

$$ext(K) \subseteq ext(F_a(\overline{co}(G),\{x_n\})) = F_a(\overline{co}(G),\{x_n\}) \cap ext(\overline{co}(G)) \subseteq F_a(\overline{co}(G),\{x_n\}) \cap G. \tag{6}$$

If $ext(K)$ contain more than one element, then from (6), we conclude $F_a(G,\{x_n\})$ contain more than one element, which is a contradiction. Thus, from the fact that $ext(K) \neq \emptyset$, we infer $ext(K)$ is singleton. Therefore, from (5), we have $K$ is singleton. Infact, $K = \{\bar{z}\}$. Then $ext(K) = \{\bar{z}\}$ and hence from (6), we conclude $\bar{z} \in F_a(G,\{x_n\})$. Thus, we have $\bar{z} = z$. This implies $F_a(\overline{co}(G),\{x_n\}) = \{z\}$. □

From the above theorem and Theorem 2.8.14 of [8] (The Krein-Smulian Weak Compactness Theorem), we obtain the following corollary:

**Corollary 4.12.** *Let $X$ be a real Banach space and $G$ weakly compact. Then $F_a(G,\{x_n\}) = \{z\}$ if and only if $F_a(\overline{co}(G),\{x_n\}) = \{z\}$.*

At the end of this section, we discuss conditions under which $F_a(\overline{G^w},\{x_n\}) \neq \emptyset$.

**Theorem 4.13.** *Let $X$ be a reflexive real normed space and $F_a(\overline{co}(G),\{x_n\}) \neq \emptyset$. Then $F_a(\overline{G^w},\{x_n\}) \neq \emptyset$.*

*Proof.* We know $\overline{co}(G)$ is weakly closed. Also, $\overline{co}(G)$ is bounded. Then from reflexivity of $X$, Theorem 2.8.6 of [8] (The Eberlein-Smulian Theorem) and Corollary 2.8.9 of [8], we obtain $\overline{co}(G)$ is weakly compact. Therefore, from Corollary 2.10.16 of [8], we have $ext(\overline{co}(G)) \subseteq \overline{G^w}$. On the other hand, from Proposition 3.10, $ext(F_a(\overline{co}(G),\{x_n\})) \neq \emptyset$. Thus, by Proposition 3.1, there exists $z \in F_a(\overline{co}(G),\{x_n\})$ such that $z \in ext(\overline{co}(G))$. Therefore, from Corollary 4.5, we have $z \in F_a(\overline{G^w},\{x_n\})$. This implies $F_a(\overline{G^w},\{x_n\}) \neq \emptyset$. □

**Theorem 4.14.** *Let $X$ be a real normed space, $\overline{co}(G)$ weakly compact and $F_a(\overline{co}(G),\{x_n\}) \neq \emptyset$. Then $F_a(\overline{G^w},\{x_n\}) \neq \emptyset$.*

*Proof.* It follows from Corollary 2.10.16 of [8] that $ext(\overline{co}(G)) \subseteq \overline{G^w}$. On the other hand, from Proposition 3.12, $ext(F_a(\overline{co}(G),\{x_n\})) \neq \emptyset$. Thus, by Proposition 3.1, there exists $z \in F_a(\overline{co}(G),\{x_n\})$ such that $z \in ext(\overline{co}(G))$. Therefore, from Corollary 4.5, we have $z \in F_a(\overline{G^w},\{x_n\})$. This implies $F_a(\overline{G^w},\{x_n\}) \neq \emptyset$. □

## References


[1] R.P. Agarwal, D. Regan and D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer-Verlag, New York, 2009.

[2] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, 3rd Edition, Springer-Verlag, Berlin, 2006.

[3] J.-M. Auge, *Perturbation of farthest points in weakly compact sets*, Math. Morav. **15** (2011), no. 1, 1-6.

[4] N. Dunford and J.T. Schwartz, *Linear Operators*, Part I (General Theory), Wiley-Interscience Pub., New York, 1958.

[5] P. Govindarajulu, *On simultaneous approximation in locally convex space*, Indian J. Pure Appl. Math. **16** (1985), no. 6, 617-626.

[6] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley-Interscience Pub., New York, 1969.

[7] M. Martin and T.S.S.R.K. Rao, *On remotality for convex sets in Banach spaces*, J. Approx. Theory **162** (2010), no. 2, 392-396.

[8] R.E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.

[9] T.D. Narang, *A study of farthest points*, Nieuw Arch. Wisk. (3) **25** (1977), no. 1, 54-79.

[10] C.P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, Springer-Verlag, Berlin, 2004.

[11] M. Sababheh and R. Khalil, *Remotality of closed bounded convex sets in reflexive spaces*, Numer. Funct. Anal. Optim. **29** (2008), no. 9-10, 1166-1170.

[12] Sangeeta and T.D. Narang, *On the farthest points in convex metric spaces and linear metric spaces*, Publ. Inst. Math. (Beograd) (N.S.) **95** (**109**) (2014), 229-238.

[13] Z.H. Zhang and C.Y. Liu, *Convexities and existence of the farthest point*, Abstr. Appl. Anal. **2011**, Art. ID 139597, 9pp.