Filomat 32:18 (2018), 6253–6266
https://doi.org/10.2298/FIL1818253A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Korovkin Type Approximation Theorems Proved via Weighted $\alpha\beta$-equistatistical Convergence for Bivariate Functions

Hüseyin Aktuğlu[a], Halil Gezer[b]

[a] *Eastern Mediterranean University*
[b] *Cyprus International University*

Abstract. Statistical convergence was extended to weighted statistical convergence in [24], by using a sequence of real numbers s_k, satisfying some conditions. Later, weighted statistical convergence was considered in [35] and [19] with modified conditions on s_k. Weighted statistical convergence is an extension of statistical convergence in the sense that, for $s_k = 1$, for all k, it reduces to statistical convergence. A definition of weighted $\alpha\beta$-statistical convergence of order γ, considered in [25] does not have this property. To remove this extension problem the definition given in [25] needs some modifications. In this paper, we introduced the modified version of weighted $\alpha\beta$-statistical convergence of order γ, which is an extension of $\alpha\beta$-statistical convergence of order γ. Our definition, with $s_k = 1$, for all k, reduces to $\alpha\beta$-statistical convergence of order γ.
Moreover, we use this definition of weighted $\alpha\beta$-statistical convergence of order γ, to prove Korovkin type approximation theorems via, weighted $\alpha\beta$-equistatistical convergence of order γ and weighted $\alpha\beta$-statistical uniform convergence of order γ, for bivariate functions on $[0,\infty)\times[0,\infty)$. Also we prove Korovkin type approximation theorems via $\alpha\beta$-equistatistical convergence of order γ and $\alpha\beta$-statistical uniform convergence of order γ, for bivariate functions on $[0,\infty)\times[0,\infty)$. Some examples of positive linear operators are constructed to show that, our approximation results works, but its classical and statistical cases do not work. Finally, rates of weighted $\alpha\beta$-equistatistical convergence of order γ is introduced and discussed.

1. Introduction

Recall that the natural density of a subset K of $\mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} n^{-1} \left|\{k \in [1,n] : \ k \in K\}\right|,$$

provided that limit exists and $|K|$ represents the cardinality of the set K. The concept of statistical convergence which was introduced by Steinhaus [40] and Fast [17] independently, is based on this density function. A sequence x_k is called statistically convergent to L and denoted by $st-\lim_{n\to\infty} x_n = L$, if, for each $\varepsilon > 0$, $\delta(\{k \in [1,n] : \ |x_k - L| \geq \varepsilon\}) = 0$. Later, by using different density functions, λ- statistical convergence [34] and lacunary statistical convergence [18] are defined and studied.

2010 *Mathematics Subject Classification*. Primary 41A25 ; Secondary 41A36, 40A35.
Keywords. Statistical convergence, $\alpha\beta-$statistical convergence, equistatistical convergence, weighted statistical convergence, Positive linear operators, Korovkin type approximation theorem, rates of convergence.
Received: 25 April 2018; Accepted: 27 August 2018
Communicated by Ivana Djolović
Email addresses: huseyin.aktuglu@emu.edu.tr (Hüseyin Aktuğlu), hgezer@ciu.edu.tr (Halil Gezer)

In [10], Çolak introduced statistical convergence of order γ by using the following generalization of δ.

$$\delta(K,\gamma) = \lim_{n\to\infty} n^{-\gamma} \left|\{k \in [1,n] : \ k \in K\}\right|,$$

where $0 < \gamma \leq 1$.

Let α and β be two non-decreasing sequences of positive numbers such that,

i) $\beta(n) - \alpha(n) \geq 0$, for all n,
ii) $\lim_{n\to\infty}(\beta(n) - \alpha(n)) = \infty$,

and let Λ be the set of all pairs (α,β) satisfying (i) and (ii). Then, for all $(\alpha,\beta) \in \Lambda$, $\delta^{\alpha,\beta}(K,\gamma)$ is introduced in [2] as follows

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_{n\to\infty} \frac{\left|\{k \in [\alpha(n),\beta(n)] : k \in K\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \tag{1}$$

where $0 < \gamma \leq 1$.

Remark 1.1. *i) If $\alpha(n) = 1$ and $\beta(n) = n$ then $\delta^{\alpha,\beta}(K,\gamma) = \delta(K,\gamma)$.*
ii) If $\alpha(n) = 1$, $\beta(n) = n$ and $\gamma = 1$ then $\delta^{\alpha,\beta}(K,\gamma) = \delta(K)$.

Lemma 1.2. *([2]) Let K and M be two subsets of $\mathbb{N}$ and $0 < \gamma \leq 1$, then for all $(\alpha,\beta) \in \Lambda$, we have the following properties.*
i) $\delta^{\alpha,\beta}(\phi,\gamma) = 0$.
ii) $\delta^{\alpha,\beta}(\mathbb{N},1) = 1$.
iii) If K is a finite set then $\delta^{\alpha,\beta}(K,\gamma) = 0$.
iv) If $K \subset M \Rightarrow \delta^{\alpha,\beta}(K,\gamma) \leq \delta^{\alpha,\beta}(M,\gamma)$.
v) $\delta^{\alpha,\beta}(K \cup M,\gamma) \leq \delta^{\alpha,\beta}(K,\gamma) + \delta^{\alpha,\beta}(M,\gamma)$
vi) If $0 < \gamma \leq \eta \leq 1$ then $\delta^{\alpha,\beta}(K,\eta) \leq \delta^{\alpha,\beta}(K,\gamma)$.

The $\alpha\beta$-statistical convergence of order $0 < \gamma \leq 1$ was introduced in [2] as follows.

Definition 1.3. *([2]) A sequence x is said to be $\alpha\beta-$statistically convergent to L of order γ, and denoted by $st^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} x_n = L$, if for every $\varepsilon > 0$,*

$$\delta^{\alpha,\beta}(\{k \in [\alpha(n),\beta(n)] : |x_k - L| \geq \varepsilon\},\gamma) = \lim_{n\to\infty} \frac{\left|\{k \in [\alpha(n),\beta(n)] : |x_k - L| \geq \varepsilon\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

If $\gamma = 1$, then $\alpha\beta-$statistical convergence of order γ is called $\alpha\beta-$statistical convergence.

2. Weighted $\alpha\beta$-statistical convergence of order γ

The concept of weighted statistical convergence was first introduced in [24]. Then Mursaleen et. al. [35] and Ghosal [19] considered modified forms of weighted statistical convergence. Recall that, a sequence x_k is said to be weighted statistically convergent of order γ to L (see [19],[20],[24],[35]), if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(S_n)^{\gamma}} \left|\{k \leq S_n : s_k|x_k - L| \geq \varepsilon\}\right| = 0,$$

where $\{s_n\}$ is a sequence of real numbers such that

$$s_n \geq 0, \ \ s_1 > 0, \ \liminf_{n\to\infty} s_n > 0 \ \text{ and } \ S_n = \sum_{k=1}^{n} s_k \to \infty \ \text{ as } \ n \to \infty. \tag{2}$$

Remark 2.1. *1) If $s_n = 1$, for all n, then weighted statistical convergence of order γ, reduces to statistical convergence of order γ.*
2) If $s_n = 1$, for all n and $\gamma = 1$ then weighted statistically convergence of order γ, reduces to statistical convergence.

On the other hand, the weighted $\alpha\beta$-statistical convergence for sequences of real numbers is introduced and discussed in [25] as follows.

Definition 2.2. *A sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-statistically convergent of order γ to l or $S^{\gamma}_{\alpha\beta}$-convergent, if for every $\varepsilon > 0$,*

$$\delta^{\alpha,\beta}(\{k : s_k|x_k - l| \geq \varepsilon\}, \gamma) = \lim_{n\to\infty} \frac{1}{S_n^{\gamma}} |\{k \leq S_n : s_k|x_k - l| \geq \varepsilon\}| = 0, \tag{3}$$

where s_k is a sequence of real numbers such that, $s_0 > 0$, and

$$S_n = \sum_{k\in[\alpha(n),\beta(n)]} s_k \to \infty \text{ as } n \to \infty.$$

It is natural to expect that, under the condition, $s_k = 1$ for all k (or $s_k = 1$ for all k and $\gamma = 1$), the weighted $\alpha\beta$-statistical convergence of order γ will be $\alpha\beta$-statistical convergence of order γ (or $\alpha\beta$-statistical convergence). The following example show that, Definition 2.2, does not have this property. In other words, Definition 2.2, and Definition 1.3, are not the same under the condition that $s_k = 1$ for all k. Therefore, Definition 2.2, is not an extension of $\alpha\beta$-statistical convergence of order γ. Moreover, it is well known that, for special choices of $\alpha(n)$ and $\beta(n)$, the $\alpha\beta$-statistical convergence reduces to λ-statistical convergence and lacunary statistical convergence (see [2]). If we use the same choices of $\alpha(n)$ and $\beta(n)$, Definition 2.2, does not have this property as well.

Example 2.3. *Consider the sequence,*

$$x_k = \begin{cases} 0, & k \in [2^{2n-1}, 2^{2n} - 1] \text{ for some } n = 1, 2, 3, ... \\ 1, & \text{otherwise} \end{cases}$$

and let $\alpha(n) = 2^{2n-1}$ and $\beta(n) = 2^{2n} - 1$. Then

$$\lim_{r\to\infty} \frac{\left|\left\{k \in [2^{2n-1}, 2^{2n} - 1] : |x_k| \geq \varepsilon\right\}\right|}{(2^{2n-1})^{\gamma}} = 0,$$

therefore $st^{\gamma}_{\alpha\beta} - lim x_k = 0$.
On the other hand, by Definition 2.2, with $0 < \varepsilon < 1$, and $s_k = 1$, we have $S_n = 2^{2n-1}$ and

$$\frac{1}{(2^{2n-1})^{\gamma}} |\{k \leq 2^{2n-1} : |x_k| \geq \varepsilon\}| \geq \frac{2^{2n-2}}{(2^{2n-1})^{\gamma}} \geq \frac{(2^{2n-2})^{\gamma}}{(2^{2n-1})^{\gamma}} = \left(\frac{1}{2}\right)^{\gamma} \nrightarrow 0,$$

where 2^{2n-2} is the number of 1's in the last block before the interval $[2^{2n-1}, 2^{2n} - 1]$.

The main motivation of the present section is to introduce the concept of weighted $\alpha\beta$-statistical convergence of order γ which is a natural extension of $\alpha\beta$-statistical convergence of order γ. In other words, weighted $\alpha\beta$-statistical convergence of order γ with $s_k = 1$ for all k will be $\alpha\beta$-statistical convergence of order γ.

Let s_n be any sequence satisfying (2), then for any pair $(\alpha, \beta) \in \Lambda$, define,

$$A_n = \frac{\alpha(n)}{[\alpha(n)]} \sum_{k=1}^{[\alpha(n)]} s_k \quad \text{and} \quad B_n = \frac{\beta(n)}{[\beta(n)]} \sum_{k=1}^{[\beta(n)]} s_k,$$

where $[r]$ is the integer part of r.
Now we introduce the following definition.

Definition 2.4. *A sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-statistically convergent of order γ to l, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{(B_n - A_n + 1)^\gamma} |\{k \in [A_n, B_n] : s_k|x_k - l| \geq \varepsilon\}| = 0, \tag{4}$$

where s_k is a sequence of real numbers satisfying (2).

Remark 2.5. *Taking $s_k = 1$ for all k, in (4), then $A_n = \alpha(n)$, $B_n = \beta(n)$, and Definition 2.4 reduces to Definition 1.3.*

Recall that, for special choices of $\alpha(n)$ and $\beta(n)$, the $\alpha\beta-$statistical convergence reduces to λ-statistical convergence and lacunary statistical convergence. If we use the same choices of $\alpha(n)$ and $\beta(n)$, in Definition 2.4, we get natural definitions of weighted λ-statistical convergence of order γ and weighted lacunary statistical convergence of order γ, satisfying the property that, taking $s_k = 1$ for all k, they gives λ-statistical convergence of order γ and lacunary statistical convergence of order γ.

3. $\alpha\beta$-Equistatistical convergence of order γ for bivariate functions

The main objective of this section is to introduce and discuss $\alpha\beta$-statistical pointwise, $\alpha\beta$-statistical uniform and $\alpha\beta$-equistatistical convergence for bivariate functions. We construct examples to show the differences among these definitions. Now, replacing $\delta(K)$ by $\delta^{\alpha,\beta}(K,\gamma)$, we can introduce following definitions for bivariate functions.

Definition 3.1. *(f_n) is said to be $\alpha\beta-$statistically pointwise convergent to f of order γ on $X^2 = X \times X \subset \mathbb{R}^2$ if for every $\varepsilon > 0$ and for each $(x,y) \in X^2$*

$$\lim_{n\to\infty} \frac{\left|\left\{k \in [\alpha(n), \beta(n)] : \left|f_k(x,y) - f(x,y)\right| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0,$$

then it is denoted by $st^\gamma_{\alpha\beta} - f_n \to f$.

Definition 3.2. *(f_n) is said to be $\alpha\beta-$equistatistically convergent to f of order γ on $X^2 \subset \mathbb{R}^2$ if for every $\varepsilon > 0$, the sequence of real valued functions*

$$p_{r,\varepsilon,\gamma}(x,y) := \frac{\left|\left\{k \in [\alpha(r), \beta(r)] : \left|f_k(x,y) - f(x,y)\right| \geq \varepsilon\right\}\right|}{(\beta(r) - \alpha(r) + 1)^\gamma}$$

converges uniformly to zero funtion on X^2 i.e $\left\|p_{r,\varepsilon,\gamma}(.)\right\|_{C(X^2)} \to 0$, where $\|f\|_{C(X^2)} = sup_{(x,y)\in X^2}|f(x,y)|$. Then it is denoted by $st^\gamma_{\alpha\beta} - f_n \twoheadrightarrow f$.

Definition 3.3. *(f_n) is said to be $\alpha\beta-$statistically uniform convergent to f of order γ on $X^2 \subset \mathbb{R}^2$ if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{\left|\left\{k \in [\alpha(n), \beta(n)] : \left\|f_k(x,y) - f(x,y)\right\|_{C(X^2)} \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0.$$

Then it is denoted by $st^\gamma_{\alpha\beta} - f_n \rightrightarrows f$.

Remark 3.4. *1) In the case $\gamma = 1$, $\alpha\beta-$statistical pointwise convergence of order γ, $\alpha\beta-$equistatistical convergence of order γ and $\alpha\beta-$statistical uniform convergence of order γ are called $\alpha\beta-$statistical pointwise convergence, $\alpha\beta-$equistatistical convergence and $\alpha\beta-$statistical uniform convergence.*
2) It is Obvious that, for any $0 < \gamma \leq 1$,

$$st^\gamma_{\alpha\beta} - f_n \rightrightarrows f \;\Rightarrow\; st^\gamma_{\alpha\beta} - f_n \twoheadrightarrow f \Rightarrow st^\gamma_{\alpha\beta} - f_n \to f.$$

Example 3.5. *Consider the sequence of continuous functions* $h_n : [0,\infty)\times[0,\infty) \longrightarrow [0,1]$, $n \in \mathbb{N}$, *defined by*

$$h_n(x,y) = \begin{cases} -4n^2(n+1)^2\left(x-\frac{1}{n}\right)\left(x-\frac{1}{n+1}\right) & , \text{if } \quad x \in \left(\frac{1}{n+1},\frac{1}{n}\right] \\ 0 & , \text{otherwise} \end{cases} \tag{5}$$

and let $h(x,y)=0$. *For a given* $\varepsilon>0$, $0<\gamma\le 1$ *and for all* $(\alpha,\beta)\in\Lambda$ *we have,*

$$p_{r,\varepsilon,\gamma}(x,y) = \frac{\left|\left\{k\in[\alpha(r),\beta(r)] : \left|h_k(x,y)-h(x,y)\right|\ge\varepsilon\right\}\right|}{(\beta(r)-\alpha(r)+1)^\gamma} \le \frac{1}{(\beta(r)-\alpha(r)+1)^\gamma} \to 0 \text{ as } r\to\infty$$

uniformly in (x,y) *which gives that* $st^\gamma_{\alpha\beta}-h_n \twoheadrightarrow h$. *But* $st^\gamma_{\alpha\beta}-h_n \rightrightarrows h$ *does not hold since* $\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\left|h_n(x,y)\right|=1$ *for all* n.

Example 3.6. *Consider the sequence of functions* $f_n : [0,\infty)\times[0,\infty)\to[0,1)$,

$$f_n(x,y) = \left(\frac{x}{x+1}\right)^n\left(\frac{y}{y+1}\right)^n. \tag{6}$$

Since $f(x,y)=0$ *is the pointwise limit of the sequence* $f_n(x,y)$ *in the ordinary sense it is obvious that* $f_n\to f(\alpha\beta-stat)$ *for all* $(\alpha,\beta)\in\Lambda$. *On the other hand choose* $\varepsilon=\frac{1}{4}$, *then for all* $k\in[\alpha(n),\beta(n)]$ *and* $(x,y)\in(\frac{1}{\sqrt[\beta(n)]{2}-1},\infty)\times(\frac{1}{\sqrt[\beta(n)]{2}-1},\infty)$ *we have,*

$$f_k(x,y) = \left(\frac{x}{1+x}\right)^k\left(\frac{y}{1+y}\right)^k \ge \left(\frac{1}{\sqrt[\beta(n)]{2}}\right)^k\left(\frac{1}{\sqrt[\beta(n)]{2}}\right)^k \ge \left(\frac{1}{\sqrt[\beta(n)]{2}}\right)^{\beta(n)}\left(\frac{1}{\sqrt[\beta(n)]{2}}\right)^{\beta(n)} = \frac{1}{4},$$

which implies that $st^\gamma_{\alpha\beta}-f_n\twoheadrightarrow f$ *does not hold for any* $0<\gamma\le 1$.

In the following example, we also show that $\alpha\beta$-statistical uniform convergence does not imply statistical uniform convergence or ordinary uniform convergence for functions of two variables.

Example 3.7. *Let* $g_k : D=[0,\infty)\times[0,\infty)\to\{0,1\}$, *be such that*

$$g_k(x,y) = \begin{cases} 0, & k\in[2^{2n-1},2^{2n}-1] \text{ for some } n=1,2,3,... \\ 1, & \text{otherwise} \end{cases}$$

for all (x,y) *and let* $\alpha(n)=2^{2n-1}$ *and* $\beta(n)=2^{2n}-1$. *Then*

$$\lim_{r\to\infty}\frac{\left|\left\{k\in[2^{2r-1},2^{2r}-1] : \left\|g_k(x,y)-g(x,y)\right\|_{C(D)}\ge\varepsilon\right\}\right|}{(2^{2r-1})}=0$$

where $g(x,y)=0$ *for all* (x,y). *Therefore* $st_{\alpha\beta}-g_n\rightrightarrows g$. *But since* $\delta(\{1\le k\le n : \|g_k(x,y)-g(x,y)\|_{C(D)}\ge\varepsilon\})$ *does not exist,* g_k *is not uniformly convergent to* g *in the statistical and ordinary sense.*

4. Weighted $\alpha\beta$-equistatistical convergence of order γ

Recently, weighted statistical pointwise, weighted statistical uniform and weighted equistatistical convergence are introduced and studied in [1] for functions of one variable, by using the modified form of weighted statistical convergence given in [19]. A Korovkin type approximation theorem, via weighted $\alpha\beta$-statistical uniform convergence of order γ on compact subset of $\mathbb{R}$, using Definition 2.2, is considered in [25].

In this section we extend Definition 2.4, to functions of two variables and we introduce and discuss, the weighted $\alpha\beta$-statistical pointwise convergence of order γ, the weighted $\alpha\beta$-statistical uniform convergence of order γ and the weighted $\alpha\beta$-equistatistical convergence of order γ, for sequences of real valued functions of two variables. Since Definition 2.4, is the natural extension of $\alpha\beta$-statistical convergence of order γ, following definitions includes λ-statistical and lacunary statistical versions.

Definition 4.1. *(f_n) is said to be weighted $\alpha\beta-$statistically pointwise convergent of order γ to f on $X \times X \subset \mathbb{R}^2$ if for every $\varepsilon > 0$ and for each $(x, y) \in X^2$*

$$\lim_{n\to\infty} \frac{\left|\left\{k \in [A_n, B_n] : s_k \left|f_k(x, y) - f(x, y)\right| \geq \varepsilon\right\}\right|}{(B_n - A_n + 1)^{\gamma}} = 0,$$

then it is denoted by $w - st^{\gamma}_{\alpha\beta} - f_n \to f$

Definition 4.2. *(f_n) is said to be weighted $\alpha\beta-$equistatistically convergent of order γ to f on $X^2 \subset \mathbb{R}^2$ if for every $\varepsilon > 0$, the sequence of real valued functions*

$$p_{r,\varepsilon,\gamma}(x, y) := \frac{\left|\left\{k \in [A_r, B_r] : s_k \left|f_k(x, y) - f(x, y)\right| \geq \varepsilon\right\}\right|}{(B_r - A_r + 1)^{\gamma}}$$

converges uniformly to zero funtion on X^2 i.e $\left\|p_{r,\varepsilon,\gamma}(.)\right\|_{C(X^2)} \to 0$, where $\|f\|_{C(X^2)} = sup_{(x,y)\in X^2}|f(x, y)|$ Then it is denoted by $w - st^{\gamma}_{\alpha\beta} - f_n \twoheadrightarrow f$.

Definition 4.3. *(f_n) is said to be weighted $\alpha\beta-$statistically uniform convergent of order γ to f on $X^2 \subset \mathbb{R}^2$ if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{\left|\left\{k \in [A_n, B_n] : s_k \left\|f_k(x, y) - f(x, y)\right\|_{C(X^2)} \geq \varepsilon\right\}\right|}{(B_n - A_n + 1)^{\gamma}} = 0.$$

Then it is denoted by $w - st^{\gamma}_{\alpha\beta} - f_n \rightrightarrows f$.

Lemma 4.4. *For any $0 < \gamma \leq 1$, $w - st^{\gamma}_{\alpha\beta} - f_n \rightrightarrows f \Rightarrow w - st^{\gamma}_{\alpha\beta} - f_n \twoheadrightarrow f \Rightarrow w - st^{\gamma}_{\alpha\beta} - f_n \to f$.*

Example 4.5. *Consider the sequence of continuous functions $h_n : [0, \infty) \times [0, \infty) \longrightarrow [0, 1]$, $n \in \mathbb{N}$, defined by*

$$h_n(x, y) = \begin{cases} -4n^2(n + 1)^2 \left(x - \frac{1}{n}\right)\left(x - \frac{1}{n+1}\right) & , \text{if} \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ 0 & , \text{otherwise} \end{cases} \tag{7}$$

and let $h(x, y) = 0$, $s_k = k$, $(\alpha, \beta) \in \Lambda$, such that $\alpha(n) = 1$ and $\beta(n) \in \mathbb{N}$ for all n. Then, $A_n = 1$ and $B_n = \frac{\beta(n)(\beta(n)+1)}{2}$. For a given $\varepsilon > 0$, and for any $0 < \gamma \leq 1$ we have,

$$p_{r,\varepsilon,\gamma}(x, y) = \frac{\left|\left\{k \in [1, B(r)] : s_k \left|h_k(x, y) - h(x, y)\right| \geq \varepsilon\right\}\right|}{B_r^{\gamma}} \leq \frac{1}{B_r^{\gamma}} \to 0 \text{ as } r \to \infty$$

uniformly in (x, y) which gives that $w - st^{\gamma}_{\alpha\beta} - h_n \twoheadrightarrow h$. But $w - st^{\gamma}_{\alpha\beta} - h_n \rightrightarrows h$ does not hold since $\sup_{(x,y)\in[0,\infty)\times[0,\infty)} \left|h_n(x, y)\right| = 1$ for all n.

Example 4.6. *Consider the sequence of functions $f_n : [0, \infty) \times [0, \infty) \to [0, 1)$,*

$$f_n(x, y) = \left(\frac{x}{x + 1}\right)^n \left(\frac{y}{y + 1}\right)^n, \tag{8}$$

and let $s_k = 2k$, $(\alpha, \beta) \in \Lambda$, such that $\alpha(n) = 1$ and $\beta(n) \in \mathbb{N}$ for all n. Then for all n, we have $A_n = 1$ and $B_n = \beta(n)(\beta(n) + 1)$. On the other hand since $f(x, y) = 0$ is the pointwise limit of the sequence $f_n(x, y)$ in the ordinary sense it is obvious that $w - f_n \to f(\alpha\beta - stat)$. Now choose $\varepsilon = \frac{1}{4}$, then for all $k \in [1, B_n]$ and $(x, y) \in (\frac{1}{\sqrt[B_n]{2}-1}, \infty) \times (\frac{1}{\sqrt[B_n]{2}-1}, \infty)$ we have,

$$f_k(x, y) = \left(\frac{x}{1 + x}\right)^k \left(\frac{y}{1 + y}\right)^k \geq \left(\frac{1}{\sqrt[B_n]{2}}\right)^k \left(\frac{1}{\sqrt[B_n]{2}}\right)^k \geq \left(\frac{1}{\sqrt[B_n]{2}}\right)^{B_n} \left(\frac{1}{\sqrt[B_n]{2}}\right)^{B_n} = \frac{1}{4},$$

which implies that $w - st^{\gamma}_{\alpha\beta} - f_n \twoheadrightarrow f$ does not hold for any $0 < \gamma \leq 1$.

5. Korovkin Type Approximation Theorems

Korovkin type approximation theory was initiated by P.P. Korovkin in [30] and used by many researchers. Later, Korovkin type approximation theorems by means of statistical convergence, A-statistical convergence, statistical C_1 summabilty, equistatistical convergence, $\alpha\beta$-statistical convergence etc. are considered in [2], [3], [4], [5], [6], [7], [12], [13], [14], [16], [20], [21], [22], [23], [25],[27],[27], [28], [29], [31], [32], [36], [37], [38] and [39]. Recently, a Korovkin type approximation theorem is considered via weighted equistatistical convergence in [1]. The main purpose of this section is to prove different Korovkin type approximation theorems in the sense of $\alpha\beta-$equistatistical convergence of order γ, weighted $\alpha\beta-$equistatistical convergence of order γ, $\alpha\beta-$statistical uniform convergence of order γ and weighted $\alpha\beta-$statistical uniform convergence of order γ, for bivariate functions on the set $D = [0, \infty) \times [0, \infty)$.

Let $C_B(D)$ be the space of all continuous and bounded functions on D, which is equipped with the usual norm

$$\|f\|_{C_B(D)} = sup_{(x,y)\in D}|f(x, y)|,$$

for $f \in C_B(D)$. Throughout the paper, we consider the space H_{ω_2} of real-valued functions, defined on D and satisfying

$$|f(u, v) - f(x, y)| \leq \omega_2(f; |\frac{u}{1+u} - \frac{x}{1+x}|, |\frac{v}{1+v} - \frac{y}{1+y}|).$$

where ω_2 is a non-negative function on $D = [0, \infty) \times [0, \infty)$, which is increasing for both variables and satisfying;
i) $\omega_2(f; \delta_1 + \delta_2, \delta) \leq \omega_2(f; \delta_1, \delta) + \omega_2(f; \delta_2, \delta)$.
ii) $\omega_2(f; \delta, \delta_1 + \delta_2) \leq \omega_2(f; \delta, \delta_1) + \omega_2(f; \delta, \delta_2)$.
iii) $\lim_{\delta_1\to 0,\delta_2\to 0} \omega_2(f; \delta_1, \delta_2) = 0$.

Theorem 5.1. *Let $L_n : H_{w_2} \to C_B(D)$ be a sequence of positive linear operators, $0 < \gamma \leq 1$ and let $(\alpha, \beta) \in \Lambda$. Then for all $f \in H_{w_2}$*

$$st^{\gamma}_{\alpha\beta} - L_n(f; x, y) \twoheadrightarrow f(x, y) \tag{9}$$

if and only if

$$st^{\gamma}_{\alpha\beta} - L_n(\varphi_i; x, y) \twoheadrightarrow \varphi_i(x, y) \tag{10}$$

for $i = 0, 1, 2, 3$ where $\varphi_0(u, v) = 1$, $\varphi_1(u, v) = \frac{u}{1+u}$, $\varphi_2(u, v) = \frac{v}{1+v}$, $\varphi_3(u, v) = \varphi_1^2(u, v) + \varphi_2^2(u, v)$.

Proof. Suppose that (10) holds, $f \in H_{w_2}$ is an arbitrary element and $(x, y) \in D$ is arbitrary but a fixed point. By the assumption, for every $\varepsilon > 0$, there exits δ_1, δ_2 such that $|f(u, v) - f(x, y)| < \varepsilon$ holds for all $(u, v) \in D$ satisfying $|\frac{u}{1+u} - \frac{x}{1+x}| < \delta_1$ and $|\frac{v}{1+v} - \frac{y}{1+y}| < \delta_2$.
Let

$$D_{\delta_1,\delta_2} = \{(u, v) \in D : |\frac{u}{1+u} - \frac{x}{1+x}| < \delta_1 \text{ and } |\frac{v}{1+v} - \frac{y}{1+y}| < \delta_2\}.$$

Then,

$$\begin{aligned} |f(u, v) - f(x, y)| &= |f(u, v) - f(x, y)|\chi_{D_{\delta_1,\delta_2}}(u, v) + |f(u, v) - f(x, y)|\chi_{D\setminus D_{\delta_1,\delta_2}}(u, v) \\ &< \varepsilon + 2M\chi_{D\setminus D_{\delta_1,\delta_2}}(u, v), \end{aligned}$$

where χ_D denotes the characteristic function of the set D and $M = \|f\|_{C_B(K)}$. On the other hand,

$$\chi_{D\setminus D_{\delta_1,\delta_2}}(u, v) \leq \frac{1}{\delta_1^2}(\frac{u}{1+u} - \frac{x}{1+x})^2 + \frac{1}{\delta_2^2}(\frac{v}{1+v} - \frac{y}{1+y})^2.$$

Now take $\delta = min\{\delta_1, \delta_2\}$ in the last two inequalities we have,

$$|f(u,v) - f(x,y)| \leq \varepsilon + \frac{2M}{\delta^2}\{(\frac{u}{1+u} - \frac{x}{1+x})^2 + (\frac{v}{1+v} - \frac{y}{1+y})^2\}. \tag{11}$$

By linearity and positivity of the operators L_n we have,

$$\begin{aligned}
|L_n(f;x,y) - f(x,y)| &\leq L_n(|f(u,v) - f(x,y)|;) \\
&+ |f(x,y)||L_n(\varphi_0;x,y) - \varphi_0(x,y)| \\
&\leq \varepsilon L_n(\varphi_0;x,y) + \frac{2M}{\delta^2}L_n((\frac{u}{1+u} - \frac{x}{1+x})^2;x,y) \\
&+ L_n((\frac{v}{1+v} - \frac{y}{1+y})^2;x,y) + M|L_n(\varphi_0;x,y) - \varphi_0(x,y)|.
\end{aligned}$$

Using the boundedness of f and (11) we get,

$$\begin{aligned}
|L_n(f;x,y) - f(x,y)| &\leq \varepsilon + (\varepsilon + M)|L_n(\varphi_0(x,y))| \\
&+ \frac{2M}{\delta^2}\{L_n(\varphi_3;x,y) - \frac{2x}{1+x}L_n(\varphi_1;x,y) \\
&- \frac{2y}{1+y}L_n(\varphi_2;x,y) \\
&+ ((\frac{x}{1+x})^2 + (\frac{y}{1+y})^2)L_n(\varphi_0;x,y)\} \\
&= \varepsilon + (\varepsilon + M)|L_n(\varphi_0;,x,y) - \varphi_0(x,y)| \\
&+ \frac{2M}{\delta^2}(L_n(\varphi_3;x,y) - \varphi_3(x,y)) \\
&- \frac{4M}{\delta^2}(\frac{x}{1+x})(L_n(\varphi_1;x,y) - \varphi_1(x,y)) \\
&- \frac{4M}{\delta^2}(\frac{y}{1+y})(L_n(\varphi_2;x,y) - \varphi_2(x,y)) \\
&+ \frac{2M}{\delta^2}((\frac{x}{1+x})^2 + (\frac{y}{1+y})^2)(L_n(\varphi_0;x,u) - \varphi_0(x,y)) \\
&\leq \varepsilon + (\varepsilon + M + \frac{4M}{\delta^2})|L_n(\varphi_0;x,y) - \varphi_0(x,y)| \\
&+ \frac{4M}{\delta^2}\{|L_n(\varphi_1;x,y) - \varphi_1(x,y)| + |L_n(\varphi_2;x,y) - \varphi_2(x,y)|\} \\
&+ \frac{2M}{\delta^2}|L_n(\varphi_3;x,y) - \varphi_3(x,y)|.
\end{aligned}$$

Let $B = \varepsilon + M + \frac{4M}{\delta^2}$, then we have

$$|L_n(f;x,y) - f(x,y)| \leq \varepsilon + B\sum_{i=0}^{3}|L_n(\varphi_i;x,y) - \varphi_i(x,y)|. \tag{12}$$

Now for a given s, choose $0 < \varepsilon < s$ and define the following sets :

$$U_s(x,y) = \{k \in [\alpha(n), \beta(n)] : |L_k(f;x,y) - f(x,y)| \geq s\}$$

$$U_s^i(x,y) = \{k \in [\alpha(n), \beta(n)] : |L_k(\varphi_i;x,y) - \varphi_i(x,y)| \geq \frac{s-\varepsilon}{4B}\}$$

for $i = 0, 1, 2, 3$. It is obvious that

$$U_s(x,y) \subset \bigcup_{i=0}^{3} U_s^i(x,y). \tag{13}$$

Also define the following real valued functions:

$$p_{m,s,\gamma}(x,y) = \frac{1}{(\beta(m)-\alpha(m)+1)^{\gamma}}|\{k \in [\alpha(m),\beta(m)] : |L_k(f;x,y) - f(x,y)| \geq \frac{s-\varepsilon}{4B}\}|$$

and

$$p^{i}_{m,s,\gamma}(x,y) = \frac{1}{(\beta(m)-\alpha(m)+1)^{\gamma}}|\{k \in [\alpha(m),\beta(m)] : |L_k(\varphi_i;x,y) - \varphi_i(x,y)| \geq \frac{s-\varepsilon}{4B}\}|$$

for $i = 0,1,2,3$ and $0 < \gamma \leq 1$. Then as a consequence of (13) we have

$$p_{m,s,\gamma}(x,y) \leq \sum_{i=0}^{3} p^{i}_{m,s,\gamma}(x,y) \tag{14}$$

for all $(x,y) \in D$. Taking supremum on both sides of (14) we get,

$$\|p_{m,s,\gamma}(.)\|_{C_B(D)} \leq \sum_{i=0}^{3} \|p^{i}_{m,s,\gamma}(.)\|_{C_B(D)}. \tag{15}$$

Applying limit to both sides of (15) as $m \to \infty$ and using (10) we obtain (9) which completes the proof of (10) ⇒ (9). The implication (9) ⇒ (10) is obvious. □

Theorem 5.2. *Let $L_n : H_{w_2} \to C_B(D)$, $0 < \gamma \leq 1$ and let $(\alpha,\beta) \in \Lambda$. Then for all $f \in H_{w_2}$*

$$st^{\gamma}_{\alpha\beta} - L_n(f;x,y) \rightrightarrows f(x,y) \tag{16}$$

if and only if

$$st^{\gamma}_{\alpha\beta} - L_n(\varphi_i;x,y) \rightrightarrows \varphi_i(x,y) \tag{17}$$

for $i = 0,1,2,3$ where $\varphi_0(u,v) = 1$, $\varphi_1(u,v) = \frac{u}{1+u}$, $\varphi_2(u,v) = \frac{v}{1+v}$, $\varphi_3(u,v) = \varphi_1(u,v)^2 + \varphi_2(u,v)^2$.

Proof. Using the same steps of the proof of Theorem 5.1 and taking supremum over $(x,y) \in D$ from (12) we get the following inequality;

$$\|L_n f - f\| \leq B\{\|L_n\varphi_0 - f_0\| + \|L_n\varphi_1 - \varphi_1\| + \|L_n\varphi_2 - \varphi_2\| + \|L_n\varphi_3 - \varphi_3\|\} + \varepsilon,$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}$. Now for a given $t > 0$, choose $0 < \varepsilon < t$ and define the following sets:

$$\begin{aligned} V^{\alpha,\beta} \quad &: \quad = \left\{k \in [\alpha(n),\beta(n)] : \left\|L_k(f;x,y) - f(x,y)\right\|_{C_B(D)} \geq t\right\} \\ V^{\alpha,\beta}_i \quad &: \quad = \left\{k \in [\alpha(n),\beta(n)] : \left\|L_k(\varphi_i;x,y) - \varphi_i(x,y)\right\|_{C_B(D)} \geq \frac{t-\varepsilon}{4B}\right\} \quad i = 0,1,2,3 \end{aligned}$$

Then, we have

$$V^{\alpha,\beta} \subset \bigcup_{i=0}^{3} V^{\alpha,\beta}_i.$$

This implies that,

$$\delta^{\alpha,\beta}(V^{\alpha,\beta},\gamma) \leq \sum_{i=0}^{3} \delta^{\alpha,\beta}(V^{\alpha,\beta}_i,\gamma) \tag{18}$$

using (17), completes the proof. The implication (16) ⇒ (17) is obvious. □

Now, consider the following Bleiman, Butzer and Hahn operators [9]

$$B_n(f,x,y)=\frac{1}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\sum_{l=0}^{n}f(\frac{k}{n-k+1},\frac{l}{n-l+1})\binom{n}{k}\binom{n}{l}x^ky^l,$$

where $D=[0,\infty)\times[0,\infty)$, $f\in H_{w_2}$, $(x,y)\in D$ and $n\in\mathbb{N}$. By using $B_n(f,x,y)$ we can define following positive linear operators;

$$T_n(f;x,y)=(1+h_n(x,y))B_n(f;x,y), \tag{19}$$

where $h_n(x,y)$ is the sequence of functions given in Example 3.5.
It is easy to see that

$$\begin{aligned}
T_n(\varphi_0;x,y) &= 1+h_n(x,y)\\
T_n(\varphi_1;x,y) &= \Big(1+h_n(x,y)\Big)\Big(\frac{n}{1+n}\Big)\Big(\frac{x}{1+x}\Big)\\
T_n(\varphi_2;x,y) &= \Big(1+h_n(x,y)\Big)\Big(\frac{n}{1+n}\Big)\Big(\frac{y}{1+y}\Big)\\
T_n(\varphi_3;x,y) &= \Big(1+h_n(x,y)\Big)\Big\{\frac{n(n-1)}{(n+1)^2}\frac{x^2}{(1+x)^2}+\frac{n}{(n+1)^2}\frac{x}{1+x}\\
&+ \frac{n(n-1)}{(n+1)^2}\frac{y^2}{(1+y)^2}+\frac{n}{(n+1)^2}\frac{y}{1+y}\Big\}.
\end{aligned}$$

and since, $st^{\gamma}_{\alpha\beta}-h_n\twoheadrightarrow 0$, T_n satisfies the conditions (10), hence by Theorem 5.1, we have

$$st^{\gamma}_{\alpha\beta}-T_n(f;x,y)\twoheadrightarrow f(x,y). \tag{20}$$

Moreover, $st^{\gamma}_{\alpha\beta}-h_n\not\rightrightarrows 0$ and T_n does not satisfy conditions (17), therefore

$$st^{\gamma}_{\alpha\beta}-T_n(f;x,y)\rightrightarrows f(x,y),$$

does not hold. In other words, $\alpha\beta$-equistatistical convergence of order γ can not be replaced by $\alpha\beta$-statistical uniform convergence of order γ, in (20).

Theorem 5.3. *Let $L_n: H_{w_2}\to C_B(D)$, $0<\gamma\le 1$, $(\alpha,\beta)\in\Lambda$ and let s_n be a sequence satisfying (2). Then for all $f\in H_{w_2}$*

$$w-st^{\gamma}_{\alpha\beta}-L_n(f;x,y)\twoheadrightarrow f(x,y) \tag{21}$$

if and only if

$$w-st^{\gamma}_{\alpha\beta}-L_n(\varphi_i;x,y)\twoheadrightarrow \varphi_i(x,y) \tag{22}$$

for $i=0,1,2,3$ where $\varphi_0(u,v)=1$, $\varphi_1(u,v)=\frac{u}{1+u}$, $\varphi_2(u,v)=\frac{v}{1+v}$, $\varphi_3(u,v)=\varphi_1(u,v)^2+\varphi_2(u,v)^2$.

Proof. By equation (12) we have,

$$|L_n(f;x,y)-f(x,y)|\le\varepsilon+B\sum_{i=0}^{3}|L_n(\varphi_i;x,y)-\varphi_i(x,y)|.$$

Now for a given s, choose $0<\varepsilon<s$ and define the following sets :

$$H_s(x,y)=\{k\in[A_n,B_n]: s_k|L_k(f;x,y)-f(x,y)|\ge s\}$$

$$H_s^i(x,y) = \{k \in [A_n, B_n] : s_k|L_k(\varphi_i; x, y) - \varphi_i(x,y)| \geq \frac{s-\varepsilon}{4B}\}$$

for $i = 0, 1, 2, 3$. It is obvious that

$$H_s(x,y) \subset \bigcup_{i=0}^{3} H_s^i(x,y). \tag{23}$$

Now define the following real valued functions:

$$h_{m,s,\gamma}(x,y) = \frac{1}{(B_n - A_n + 1)^\gamma}|\{k \in [A_n, B_n] : s_k|L_k(f; x, y) - f(x,y)| \geq \frac{s-\varepsilon}{4B}\}|$$

and

$$h_{m,s,\gamma}^i(x,y) = \frac{1}{(B_n - A_n + 1)^\gamma}|\{k \in [A_n, B_n] : s_k|L_k(\varphi_i; x, y) - \varphi_i(x,y)| \geq \frac{s-\varepsilon}{4B}\}|$$

for $i = 0, 1, 2, 3$ and $0 < \gamma \leq 1$. Then as a consequence of (23) we have

$$h_{m,s,\gamma}(x,y) \leq \sum_{i=0}^{3} h_{m,s,\gamma}^i(x,y) \tag{24}$$

for all $(x,y) \in D$. Taking supremum on both sides of (24) we get,

$$||h_{m,s,\gamma}(.)||_{C_B(D)} \leq \sum_{i=0}^{3} ||h_{m,s,\gamma}^i(.)||_{C_B(D)}. \tag{25}$$

Applying limit to both sides of (25) as $m \to \infty$ and using (22) we obtain (21) which completes the proof of (22) $\Rightarrow$ (21). The inverse implication (21) $\Rightarrow$ (22) is obvious. □

Theorem 5.4. *Let $L_n : H_{w_2} \to C_B(D)$, $0 < \gamma \leq 1$, $(\alpha, \beta) \in \Lambda$ and let s_n be a sequence satisfying (2). Then for all $f \in H_{w_2}$*

$$w - st_{\alpha\beta}^{\gamma} - L_n(f; x, y) \rightrightarrows f(x,y) \tag{26}$$

if and only if

$$w - st_{\alpha\beta}^{\gamma} - L_n(\varphi_i; x, y) \rightrightarrows \varphi_i(x,y) \tag{27}$$

for $i = 0, 1, 2, 3$ where $\varphi_0(u,v) = 1$, $\varphi_1(u,v) = \frac{u}{1+u}$, $\varphi_2(u,v) = \frac{v}{1+v}$, $\varphi_3(u,v) = \varphi_1(u,v)^2 + \varphi_2(u,v)^2$.

Proof. Taking supremum over $(x,y) \in D$ from (12) we get the following inequality;

$$||L_n f - f|| \leq B\{||L_n\varphi_0 - f_0|| + ||L_n\varphi_1 - \varphi_1|| + ||L_n\varphi_2 - \varphi_2|| + ||L_n\varphi_3 - \varphi_3||\} + \varepsilon,$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}$. Now for a given $t > 0$, choose $0 < \varepsilon < t$ and define the following sets:

$$G^{\alpha,\beta} \; : \; = \left\{k \in [A_n, B_n] : s_k \left\|L_k(f; x, y) - f(x,y)\right\|_{C_B(D)} \geq t\right\}$$

$$G_i^{\alpha,\beta} \; : \; = \left\{k \in [A_n, B_n] : s_k \left\|L_k(\varphi_i; x, y) - \varphi_i(x,y)\right\|_{C_B(D)} \geq \frac{t-\varepsilon}{4B}\right\} \quad i = 0, 1, 2, 3$$

Then, we have

$$G^{\alpha,\beta} \subset \bigcup_{i=0}^{3} G_i^{\alpha,\beta}.$$

This implies that,

$$\delta^{\alpha,\beta}(G^{\alpha,\beta},\gamma) \le \sum_{i=0}^{3} \delta^{\alpha,\beta}(G_i^{\alpha,\beta},\gamma) \tag{28}$$

which completes the proof of (27) ⇒ (26). The inverse implication is obvious. □

Let $T_n^*(f;x,y)$ be the positive linear operator,

$$T_n^*(f;x,y) = (1 + h_n(x,y))B_n(f;x,y), \tag{29}$$

where $h_n(x,y)$ is the sequence of functions considered in Example 4.5. If $s_n = n$, $\alpha(n) = 1$, $\beta(n) \in \mathbb{N}$ for all n, then for any $0 < \gamma \le 1$, the sequence of positive linear operators T_n^* satisfies the conditions (22) (see Example 4.5) hence by Theorem 5.3, we have

$$w - st_{\alpha\beta}^{\gamma} - T_n^*(f;x,y) \twoheadrightarrow f(x,y)$$

But since T_n^* does not satisfy conditions (27)

$$st_{\alpha\beta}^{\gamma} - T_n^*(f;x,y) \not\twoheadrightarrow f(x,y).$$

6. Rates of weighted $\alpha\beta$-equi statistical convergence of order γ

In this section we study the rates of weighted $\alpha\beta$-equistatistical convergence of order γ of a sequence L_n of positive linear operators defined on H_{w_2} by using the modulus of continuity.

Definition 6.1. *Let a_n be a non-increasing sequence then f_n is called weighted $\alpha\beta$-equistatistical convergent of order γ to function f with the rate of a_n and denoted by $w - (f_n - f) = o(a_n,\gamma)$ $(\alpha\beta - equistat)$ if for every $\varepsilon > 0$ we have,*

$$\frac{|\{k \in [A_r, B_r] : s_k|f_k(x,y) - f(x,y)| \ge \varepsilon|\}}{(B_r - A_r + 1)^{\gamma}\, a_r} \to 0$$

uniformly, with respect to $(x,y) \in D$, where s_k is a sequence satisfying (2).

Lemma 6.2. *Assume that f_n and g_n are two sequences in H_{w_2} such that $w - (f_n - f) = o(a_n,\gamma)$ $(\alpha\beta - equistat)$ and $w - (g_n - g) = o(b_n,\gamma)$ $(\alpha\beta - equistat)$ then,*
i) $w - ((f_n + g_n) - (f + g)) = o(c_n,\gamma)$ $(\alpha\beta - equistat)$.
ii)$w - ((f_n - f)(g_n - g)) = o(a_nb_n,\gamma)$ $(\alpha\beta - equistat)$.
iii) $w - (M(f_n - f)) = o(a_n,\gamma)$ $(\alpha\beta - equistat)$, for any scalar M.
iv) $w - (\sqrt{f_n - f}) = o(a_n,\gamma)$ $(\alpha\beta - equistat)$.
where, $c_n = \max\{a_n, b_n\}$.

Proof. i) Assume that $w - (f_n - f) = o(a_n,\gamma)$ $(\alpha\beta - equistat)$ and $w - (g_n - g) = o(b_n,\gamma)$ $(\alpha\beta - equistat)$ on D. For any $\varepsilon > 0$ and $(\alpha,\beta) \in \Lambda$ consider the following sets,

$$\begin{aligned} K_{n,\alpha,\beta}(x,y,\varepsilon) &= |\{k \in [A_n, B_n] : s_k|(f_k + g_k)(x,y) - (f+g)(x,y)| \ge \varepsilon|\}. \\ K^1_{n,\alpha,\beta}(x,y,\varepsilon) &= |\{k \in [A_n, B_n] : s_k|f_k(x,y) - f(x,y)| \ge \frac{\varepsilon}{2}|\}. \\ K^2_{n,\alpha,\beta}(x,y,\varepsilon) &= |\{k \in [A_n, B_n] : s_k|g_k(x,y) - g(x,y)| \ge \frac{\varepsilon}{2}|\}. \end{aligned}$$

It is obvious that,

$$\frac{K_{n,\alpha,\beta}(x,y,\varepsilon)}{(B_n - A_n + 1)^{\gamma}\, c_n} \le \frac{K^1_{n,\alpha,\beta}(x,y,\varepsilon)}{(B_n - A_n + 1)^{\gamma}\, a_n} + \frac{K^2_{n,\alpha,\beta}(x,y,\varepsilon)}{(B_n - A_n + 1)^{\gamma}\, b_n}. \tag{30}$$

If we apply limit to both sides of (30) as $n \to \infty$ and using the hypotheses of Lemma 6.2, we complete the proof for section i). Since (ii)-(iv) can be proved in a same way we omit them. □

Theorem 6.3. *Let $L_n : H_{w_2} \to C(D)$ be a sequence of positive linear operators. Assume that the following conditions hold:*
i) $L_n(f_0; x, y) - f_0 = o(a_n, \gamma)$ $(w - st_{\alpha\beta}\alpha\beta - equistat)$
ii) $\omega(f, \delta_n) = o(b_n, \gamma)$ $(\alpha\beta - equistat)$ where $\delta_n = \sqrt{L_n(\phi^2; x, y)}$ with $(\phi^2; x, y) = (u - x)^2 + (v - y)^2$.
Then, for all $f \in H_{w_2}$ we have,

$$L_n(f; x, y) - f = o(c_n, \gamma)\ (\alpha\beta - equistat))$$

where $c_n = \max\{a_n, b_n\}$.

Proof. Let f be any element of H_{w_2} and let (x, y) be a fixed point of D then it is well known that

$$\begin{aligned} |L_n(f; x, y) - f(x, y)| &\leq \|f(x, y)\|_{H_{w_2}} |L_n(f_0; x, y) - f_0(x, y)| + 2\omega(f, \delta_n)|L_n(f_0; x, y) - f_0(x, y)| \\ &\quad + \omega(f, \delta_n) \sqrt{|L_n(f_0; x, y) - f_0(x, y)|}. \end{aligned}$$

Using the hypothesis, and Lemma 6.2, in the above inequality, completes the proof. □

7. Concluding Remarks

It should be mentioned that, Definition 2.4 is an extension of $\alpha\beta$-statistical convergence of order γ. Therefore, for special choices of $\alpha(n)$ and $\beta(n)$ (see [2]), results obtained in this paper can be restated in the sense of weighted λ-statistical convergence of order γ and weighted lacunary statistical convergence of order γ.

References

[1] S. Akdağ, Weighted Equi-Statistical Convergence of the Korovkin type approximation theorems, Results in Math., 72 (2017) 1073–1085.
[2] H. Aktuğlu, Korovkin type approximation theorems proved via $\alpha\beta$-statistical convergence, Journal of Computational and Applied Mathematics 259 (2014) 174–181
[3] H. Aktuğlu, M. A. Özarslan, H. Gezer, A-Equistatistical Convergence of Positive Linear Operators, Journal of Computational analysis and Applications 12 (2010) 24–36.
[4] H. Aktuğlu, H. Gezer, Lacunary equistatistical convergence of positive linear operators ,Cent. Eur. J. Math. 7(3) (2009) 558–567.
[5] F. Altomere, M. Campiti, Korovkin type approximation theory and its applictions, de Gruyter Stud. Math. (Berlin: de Gruyter) (1994) vol. 17.
[6] G. A. Anastassiou and M.A. Khan, Korovkin type statistical approximation theorem for a function of two variables, Journal of Computational Analysis and Applications (2017) 21176–1184.
[7] G. A. Anastassiou, M. Mursaleen, S. A. Mohiuddine, Some approximation theorems for functions of two variables through almost convergence of double sequences, J. of Computational Analaysis and Applications, 13, n,1 (2011) 37–46 .
[8] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequence of fuctions, J. Math. Anal. Appl. 328 (2007) 715–729.
[9] G. Bleiman, P. L. Butzer, L. Hahn, A Bernstein type operator approximating continuous functions on semiaxis, Indag. Mayh., 42 (1980) 255–262.
[10] R. Çolak, Statistical Convergence of Order α, Modern Methods in Analysis and Its Applications. Anamaya Pub.,NewDelhi(2010) 121–129.
[11] M. Çinar, M. Karakas, M. Et, On pointwise and uniform statistical convergence of order α for sequences of functions, Fixed Point Theory Appl, (2013), 2013: 33 https://doi.org/10.1186/1687-1812-2013-33
[12] K. Demirci, S. Karakus, Korovkin-type approximation theorem for double sequences of positive linear operators via statistical A-summabilty, Results in Mathematics, 63, n.1-2 (2013) 1–13.
[13] F. Dirik, K. Demirci, Korovkin type approximation theorem for functions of two variables in statistical sense ,Turk J Math 34 (2010) 73–83.
[14] F. Dirik, K. Demirci, A korovkin type approximation theorem for double sequneces of positive linear operators of two variables in A-statistical sense, Bull. of the Korean Math, Soc., 47 n.4 (2010) 825–837.
[15] O. Duman, C. Orhan, μ-statistically convergent function sequences, Czechislovak Math. J. 54 (2004) 413–422.
[16] E. Erkuş, O. Duman, A-statistical extension of the Korovkin type approximation theorem, Proc. Indian Acad Sci. (Math. Sci.) Vol. 115, no. 4 (2005) 499–508.
[17] H. Fast, Sur la convegence statistique, Colloq. Math. 2 (1951) 241–244.
[18] J. A. Fridy, and C. Orhan, Lacunary Statistical Convergence, Pacific Journal of Mathematics 160 (1993) 45–51.

[19] S. Ghosal, Weighted statistical convergence of order α and its applications. Journal of the Egyptian Mathematical Society, 24 (2016) 60–67.
[20] S. Ghosal, M Banerjee, and A.Ghosh, Weighted modulus S_θ-convergence of order α in probability, Arab. J. Math. Sci. 23 (2017) 242–257.
[21] U. Kadak, Weighted statistical convergence based on generalized difference operator involving (p, q)-gamma function and its applications to approximation theorems, Jornal of Mathematical Analysis and Applications 48(2) (2017) 1633–1650.
[22] U. Kadak, On relative weighted summability in modular function spaces and associated approximation theorems, Positivity, 21(4) (2017) 1593–1614.
[23] U. Kadak, Braha, N. L. and Srivastava, H. M, Statistical weighted B-summability and its applications to approximation theorems, Applied Mathematics and Computation, 302 (2017) 80–96.
[24] V. Karakaya, and T.A. Chiristi, Weighted Statistical Convergence, Iran. Journal Sci. Technol. Trans. A Sci. 33 (2009) 219–223.
[25] V. Karakaya, and A. Karaisa, Korovkin type approximation theorems for weighted $\alpha\beta$-statistical convergence, Bull. Math. Sci. 5 (2015) 159–169.
[26] S. Karakuş, K. Demirci, O. Duman, Equi-statistical convergence of positive linear operators ,Journal of Mathematical Analysis and Applications. 339 (2008) 1065–1072.
[27] S. Karakuş, K. Demirci, K., Equi-statistical convergence of the Korovkin Type Approximation Theorem, Turk J Math 33 (2009) 159–168.
[28] S. Karakuş, K. Demirci, Equi-statistical $\sigma-$convergence of positive linear operators,Computers and Mathematics with Appl. 60 (2010) 2212–2218.
[29] Y. Kaya and N. Gönül, A Generalization of Lacunary Equistatistical Convergence of Positive Linear Operators, Abstract and Applied Analysis, 2013
[30] P.P. Korovkin, Linear operators and the theory of approximation, India, Delhi, 1960.
[31] S. A. Mohiuddine, A. Alotaibi, Statistical convergence and approximation theorems for functions of two variables, J. of Computational Analaysis and Applications, 15 n.2 (2013) 218–223.
[32] S. A. Mohiuddine, Statistical weighted A-summability with application to Korovkins type approximation theorem Journal of Inequalities and Applications, 1, (2016).
[33] S. A. Mohiuddine, A. Alotaibi and B. Hazarika: Weighted A-statistical convergence for sequences of positive linear operators, The scientific World J. 437863, 8, (2014).
[34] M. Mursaleen, $\lambda-$statistical convergence, Math. Slovaca, 50 (2000) 111–115.
[35] M. Mursaleen, V. Karakaya, M. Ertrk, and F. Grsoy.: Weighted statistical convergence and its applications to Korovkin type approximationtheorem, Appl. Math. Comput. 218 (2012) 9132–9137.
[36] M. Mursaleen, S. A. Mohiuddine, Korovkin type approximation theorem for functions of two variables via statistical summability $(C, 1)$ Acta Scientiarum. Technology, 37 n.2 (2015) 237–243.
[37] M. Mursaleen, A. Alotaibi, "Korovkin type approximation theorem for functions of two variables through statistical A-summability", Advances in Difference Equations (2012).
[38] M. Mursaleen, A. Alotaibi, Korovkin type approximation theorem for statistical A-summability of double sequences, J. of Computational Analaysis and Applications, 15 n.6 (2013) 1036–1045.
[39] H. Srivastava, M. Mursaleen, A. Khan, Generalized equi-statistical convergence of positive linear operators and associated approximation theorems, Mathematical and Computer Modelling 55 (2012) 2040–2051.
[40] H. Steinhaus, Surla convergence ordinaire et la convergence asymptotique, Colloq.Math., 2 (1951) 73–74.