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Generalised CR-submanifolds of a LP-Sasakian Manifolds

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Abstract. The study of CR-submanifolds of a Kaehler manifold was initiated by Bejancu [1]. Since then many papers have appeared on CR-submanifolds. The aim of the present paper is to study generalised CR-submanifolds of a LP-Sasakian manifolds.

1. Introduction

Many authors have studied the geometry of CR-submanifolds of Kaehler, Sasakian and trans-Sasakian manifolds. The main ones can be found in [12]. Prasad [4], Prasad and Ojha [15] studied submanifolds of a LP-Sasakian manifolds. The geometry of CR-submanifold of a LP-Sasakian manifolds have been studied by several authors such as De and Sengupta [16], Ozgur et al. [5], Ahmad [13] and Ahmad et al. [14]. Mihai [7] introduced a new class of submanifolds called "Generalised CR-submanifoldss" of a Kaehler manifold. Mihai [8] also studied generalised CR-submanifolds of a Sasakian manifold. In 2001, Sengupta and De [3] studied generalised CR-submanifol of a trans-Sasakian manifolds. Motivated by the above studies in the present paper we study the leaves and integrability conditions of the distributions on generalised CR-submanifolds of a LP-Sasakian manifolds.

2. LP-Sasakian manifolds

Matsumoto [9] introduced the notion of LP-Sasakian manifolds or in short *LP*-Sasakian manifolds. An example of a five dimensional LP-Sasakian manifold was given by Matsumoto, Mihai and Rosaca in [10]. Let M be an n-dimensional differential manifold endowed with a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor g_p: $T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,...,+)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space which satisfies the following relations

$$\phi^2(X) = X + \eta(X)\xi,\ \eta(\xi) = -1, \tag{1}$$

$$g(X,\xi) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2}$$

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for all vector fields X, Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold with the structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact manifold [9]. In the Lorentzian almost paracontact manifold M, the following relations hold [9]:

$$\phi\xi = 0, \ \eta(\phi X) = 0, \tag{3}$$

$$\phi(X, Y) = \phi(Y, X), \tag{4}$$

where $\phi(X, Y) = g(X, \phi Y)$.

A Lorentzian almost paracontact manifold M endowed with the structure (φ, ξ, η, g) is called an LP-Sasakian manifold [9] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \tag{5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In an LP-Sasakian manifold M with the structure (ϕ, ξ, η, g), it is easily seen that [9]

$$\nabla_X \xi = \phi X, \tag{6}$$

$$(\nabla_X \eta)(Y) = g(\phi X, Y) = (\nabla_Y \eta)(X). \tag{7}$$

for all vector fields X, Y on M. LP-Sasakian manifolds have been studied by several authors such as ([11], [6], [2]) and many others.

3. Generalised CR-submanifolds of LP-Sasakian manifolds

Let M be an m-dimensional submanifold isometrically immersed in a LP-Sasakian manifold $\bar{M}$ such that the structure vector field ξ of $\bar{M}$ is tangent to the submanifold M. We denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and by $\{\xi\}^\perp$ the complementary orthogonal distribution to $\{\xi\}$ in $T(M)$.

For any $X \in T(M)$ we have $g(\phi X, \xi) = 0$. Then we put

$$\phi X = bX + cX, \tag{8}$$

where $bX \in \{\xi\}^\perp$ and $cX \in T^\perp(M)$. Thus $X \to bX$ is an endomorphism of the tangent bundle $T(M)$ and $X \to cX$ is a normal bundle valued 1-form on M.

Definition 3.1. *[7] A submanifold M of an almost contact metric manifold $\bar{M}$ with almost contact metric structure (ϕ, ξ, η, g) is said to be a generalised CR-submanifold if*

$$D_x^\perp = T_x(M) \cap \phi T_x^\perp(M); x \in M$$

defines a differentiable subbundle of $T_x(M)$. Thus for $X \in D^\perp$ one has $bX = 0$.

We denote by D the complementary orthogonal subbundle to $D^\perp \oplus \{\xi\}$ in $T(M)$. For any $X \in D$, $bX \neq 0$. Also we have $bD = D$.

Thus for a generalised CR-submanifold M we have the orthogonal decomposition

$$T(M) = D \oplus D^\perp \oplus \{\xi\}.$$

Any vector field X tangent to M can be decomposed as

$$X = PX + QX + \eta(X)\xi, \tag{9}$$

where PX and QX belong to the distribution D and $D^\perp$, respectively. For any vector field N normal to M, we put

$$\phi N = tN + fN, \tag{10}$$

Where tN and fN denotes the tangential and normal component of ϕN. Now, we denote by $\bar{\nabla}$ the Riemannian connection on $\bar{M}$ with respect to the Riemannian metric g. The linear connection induced by $\bar{\nabla}$ on the normal bundle $T^{\perp}(M)$ is denoted by $\nabla^{\perp}$. Then the equations of Gauss and Weingarten are given by [12]

$$\bar{\nabla}_X Y = \bar{\nabla}_X Y + h(X,Y) \tag{11}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)X, \tag{12}$$

for $X, Y \in T(M)$, $N \in TM^{\perp}$, h (respectively A_N) is the second fundamental form (respectively tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreoverted by these tensor fields are related by

$$g(h(X,Y),N) = g(A_N X, Y), \tag{13}$$

for $X, Y \in T(M)$.

We denote

$$u(X,Y) = \nabla_X bPY - A_{cPY}X - A_{\phi QY}X.$$

Theorem 3.2. *Let M be a generalised CR-submanifold of LP-Sasakian manifold $\bar{M}$. Then we have*

$$P(u(X,Y)) - bP\nabla_X Y - P\, th(X,Y) = \eta(Y)PX, \tag{14}$$

$$Q(u(X,Y)) - Q\, th(X,Y) = \eta(Y)QX, \tag{15}$$

$$\eta(u(X,Y)) = g(\phi X, \phi Y) \tag{16}$$

$$h(X,bPY) + \nabla^{\perp}_X cPY + \nabla^{\perp}_X \phi QY - cP\nabla_X Y - \phi Q\nabla_X Y - fh(X,Y) = 0, \tag{17}$$

for $X, Y \in T(M)$.

Proof. Making use of (8), (9), (10), (11) and (12) in (5)

$$\begin{aligned} &\nabla_X bPY - bP\nabla_X Y + \nabla^{\perp}_X cPY - cP\nabla_X Y - A_{cPY}X \\ &-A_{\phi QY}X + \nabla^{\perp}_X \phi QY - \phi Q\nabla_X Y + h(X,bPY) - Pth(X,Y) \\ &-Qth(X,Y) - fh(X,Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \end{aligned} \tag{18}$$

Then equation (14)-(17) follows by taking components on each of the vector bundles $D, D^{\perp}, \{\xi\}$ and $T^{\perp}(M)$ respectively. □

Theorem 3.3. *Let M be a generalised CR-submanifold of LP-Sasakian manifold $\bar{M}$. Then we have*

$$P(t\nabla^{\perp}_X N + A_{fN}X - \nabla_X tN) = bPA_N X, \tag{19}$$

$$Q(t\nabla^{\perp}_X N + A_{fN}X - \nabla_X tN) = 0, \tag{20}$$

$$\eta(A_{fN} - \nabla_X tN) = 0, \tag{21}$$

$$h(X,tN) + \phi QA_N X + \nabla^{\perp}_X fN + cPA_N X = f\nabla^{\perp}_X N, \tag{22}$$

for $X \in T(M)$ and $N \in T^{\perp}(M)$.

Proof. For $X \in T(M)$ and $N \in T^{\perp}(M)$,

$$\begin{aligned} &P\nabla_X tN + Q\nabla_X tN + \eta(\nabla_X tN)\xi - PA_{fN}X - QA_{fN}X \\ &-\eta(A_{fN}X)\xi + \nabla^{\perp}_X fN + bPA_N X + cPA_N X + \phi QA_N X \\ &-Pt\nabla^{\perp}_X N - Qt\nabla^{b}_X otN - f\nabla^{\perp}_X N = 0. \end{aligned} \tag{23}$$

Hence the theorem follows by taking components on each of the vector bundles $D, D^{\perp}, \{\xi\}$ and $T^{\perp}(M)$ respectively. □

Theorem 3.4. *Let M be a generalised CR-submanifold of LP-Sasakian manifold $\bar{M}$. Then we have*

$$\nabla_X \xi = bX; \; h(X, \xi) = cX, \tag{24}$$

for $X \in D$

$$\nabla_Y \xi = 0; \; h(Y, \xi) = \phi Y, \tag{25}$$

for $Y \in D^{\perp}$

$$\nabla_\xi \xi = 0, \; h(\xi, \xi) = 0. \tag{26}$$

Proof. Hence the theorem follows from (6) by using (8), (9) and (11). □

Theorem 3.5. *Let M be a generalised CR-submanifold of LP-Sasakian manifold $\bar{M}$. Then we have*

$$A_{\phi X} Y = A_{\phi Y} X,$$

for $X, Y \in D^{\perp}$.

Proof. Using (2), (5), (11) and (13) we get

$$\begin{aligned} g(A_{\phi X} Y, Z) &= g(h(Y, Z), \phi X) = g(\bar{\nabla}_Z Y, \phi X) = -g(\phi \bar{\nabla}_Z Y, X) \\ &= -g(\bar{\nabla}_{Z\phi} Y, X) = g(\phi Y, \bar{\nabla}_Z X) = g(A_{\phi Y} X, Z), \end{aligned} \tag{27}$$

for $X, Y \in D^{\perp}$ and $Z \in T(M)$. □

Theorem 3.6. *Let M be a generalised CR-submanifold of a LP-Sasakian manifold $\bar{M}$. Then we have*

$$\nabla_\xi V \in D^{\perp}, \tag{28}$$

for $V \in D^{\perp}$ and

$$\nabla_\xi W \in D, \tag{29}$$

for $W \in D$.

Proof. Let us take $X = \xi$ and $V = \phi N$ in (19) where $N \in \phi D$. Taking account that $tN = \phi N$, $fN = 0$ we get

$$P \nabla_\xi V = P t \nabla^{\perp}_\xi N - bPA_N \xi. \tag{30}$$

The second relation of (24) gives

$$g(PN\xi, W) = g(A_N \xi, W) = g(h(W, \xi), N) = -g(cW, N) = 0, \tag{31}$$

for $W \in D$. Hence (30) becomes

$$P \nabla_\xi V = P t \nabla^{\perp}_\xi N. \tag{32}$$

On the other hand (22) implies

$$h(\xi, V) = f \nabla^{\perp}_\xi N - \phi Q A_N \xi. \tag{33}$$

For $V \in D^{\perp}$,

$$h(\xi, V) = h(V, \xi) = -\phi V \in \phi D^{\perp}. \tag{34}$$

Now for $X \in D^{bot}$ by virtue of Lemma and of (13) we have

$$g(h(\xi, V), \phi X) = g(h(V, \xi), \phi X) = g(A_{\phi X}V, \xi) = g(A_{\phi V}X, \xi)$$
$$= g(h(X, \xi), \phi V) = g(h(X, \xi), -N) = -g(A_N\xi, X) = -g(\phi A_N\xi, \phi X)$$
$$= -g(\phi PA_N\xi, \phi X) - g(\phi QA_N\xi, \phi X) = -g(\phi QA_N\xi, \phi X). \tag{35}$$

Therefore,

$$h(\xi, V) = -\phi QA_N\xi,$$

which together with (33) implies $f\nabla^{\perp}_{\xi}N = 0$. Hence $\nabla^{\perp}_{\xi}N \in \phi D^{\perp}$, since f is an automorphism of $cD \oplus v$. Thus $t\nabla^{\perp}_{\xi}N \in D^{\perp}$ and from (32) it follows that

$$P\nabla_{\xi}V = 0, \tag{36}$$

for all $V \in D^{\perp}$. Now from (21) we have

$$\eta(\nabla_{\xi}V) = 0, \tag{37}$$

for all $V = \phi N \in D^{\perp}$, where $N \in \phi D^{\perp}$.

Hence (28) follows from (21) and (22).

Finally by using (9), (26) and (28), we have

$$g(\nabla_{\xi}W, X) = g(\nabla_{\xi}W, PX),$$

for $X \in T(M)$ and $W \in D$. Thus we have $\nabla_{\xi}W$, for $W \in D$ and this completes the proof. □

Corollary 3.7. *Let M be a generalised CR-submanifold of the LP-Sasakian manifold $\bar{M}$. Then we have*

$$[Y, \xi] \in D^{\perp}, \tag{38}$$

for $Y \in D^{\perp}$ and

$$[X, \xi] \in D, \tag{39}$$

for $X \in D$.

The above corollary follows immediately from the Theorem 3.4 and THeorem 3.6.

Theorem 3.8. *Let M be a generalised CR-submanifold of the LP-Sasakian manifold $\bar{M}$. Then the distribution $D^{\perp}$ is always involutive.*

Proof. For $X, Y \in D^{\perp}$ by using (30) we get

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X) = g(X, \nabla_Y\xi) - g(Y, \nabla_X\xi) = 0. \tag{40}$$

□

On the other hand, from (14) we have

$$bP\nabla_X Y = -PA_{\phi Y}X - Pth(X, Y), \tag{41}$$

for $X, Y \in D^{\perp}$ and then by using Theorem 3.4 we get from (41)

$$bP[X, Y] = 0, \tag{42}$$

for $X, Y \in D^{\perp}$. As b is an automorphism of D, the theorem follows from (40) and (42).

Theorem 3.9. *Let M be a generalised CR-submanifold of the LP-Sasakian manifold $\bar{M}$. Then the distribution D is never involutive.*

Proof. For $X, Y \in D$ by using (24), we have

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y,\xi) - g(\nabla_Y X) = g(X,\nabla_Y \xi) - g(Y,\nabla_X \xi) \\ &= g(X,bY) - g(Y,bX) = 2g(Y,bX). \end{aligned} \tag{43}$$

Taking $X \neq 0$ and $Y = bX$ in (43), it follows that D is not involutive. □

Theorem 3.10. *Let M be a generalised CR-submanifold of the LP-Sasakian manifold $\bar{M}$. Then the distribution $D\oplus\{\xi\}$ is involutive if and only if*

$$h(bX,Y) - h(X,bY) + \nabla^{\perp}_{Y}cX - \nabla^{\perp}_{X}cY \in cD \oplus v. \tag{44}$$

Proof. Operating ϕ on both sides of (17) and then taking component in $D^{\perp}$ we have

$$Q\nabla_X Y = -Qt(h(X,bY)) + \nabla^{\perp}_{X}cPY - fh(X,Y),$$

$X, Y \in D$ and thus

$$Q[X,Y] = Qt(h(Y,bX) - h(X,bY) + \nabla^{\perp}_{Y}cX - \nabla^{\perp}_{X}cY), \tag{45}$$

for $X, Y \in D$. Hence the theorem follows from (45) and (39). □

Theorem 3.11. *Let M be a generalised CR-submanifold of the LP-Sasakian manifold $\bar{M}$. Then the leaves of distribution $D^{\perp}$ are totally geodesic in M if and only if*

$$h(X,bZ) + \nabla^{\perp}_{X}cZ \in cD \oplus v, \tag{46}$$

for $X \in D^{\perp}$ and $Z \in D \oplus \{\xi\}$.

Proof. For $X, Y \in D^{\perp}$ and $Z \in D \oplus \{\xi\}$ by using (2), (4), (11) and (12) we get

$$\begin{aligned} g(\nabla_X Y, Z) &= -g(Y,\bar{\nabla}_X Z) = -g(\phi\bar{\nabla}_X Z,\phi Y) \\ &= g((\bar{\nabla}_X\phi)Z,\phi Y) - g(\bar{\nabla}_X\phi Z,\phi Y) \\ &= -g(\nabla_X bZ + h(X,bZ) - A_{cZ}X + \nabla^{\perp}_{X}cZ,\phi Y) \\ &= -g(h(X,bZ) + \nabla^{\perp}_{X}cZ,\phi Y). \end{aligned} \tag{47}$$

Hence the theorem follows from (47). □

Theorem 3.12. *Let M be a generalised CR-submanifold of the LP-Sasakian manifold $\bar{M}$. Then the distribution $D\oplus\{\xi\}$ is involutive and its leaves are totally geodesic in M if and only if*

$$h(X,by) + \nabla^{\perp}_{X}cY \in cD \oplus v, \tag{48}$$

for $X, Y \in D \oplus \{\xi\}$.

Proof. For $X, Y \in D \oplus \{\xi\}$ and $Z \in D^{\perp}$ by using (2), (4), (8), (11) and (12) we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) = g(\phi\bar{\nabla}_X Y,\phi Z) = g(\bar{\nabla}_X\phi Y,\phi Z) \\ &= g(\nabla_X bY + h(X,bY) - A_{cY}X + \nabla^{\perp}_{X}cY,\phi Z). \end{aligned} \tag{49}$$

Hence the theorem follows from (49). □

References

[1] A. Bejancu., CR-submanifold of a Kaehler manifold I, Proc. Amer. Math. Soc. 69(1978), 135-142.
[2] A. Taleshian and N. Asghari., On LP-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor, Differ. Geom. Dyn. Syst. 12 (2010), 228-232.
[3] A. K. Sengupta and U. C. De., Generalised CR-submanifolds of a trans-sasakian manifold, Indian J. pure appl. Math., 32(2001), 573-580.
[4] B. Prasad., Semi-invariant submanifolds of a Lorentzian para-Sasakian manifold, Bull. Malaysian Math. Soc., 21(1998), 21-26.
[5] C. Ozgur., M. Ahmad and A. Haseeb., CR-submanifolds of a Lorentzian pra-Sasakian manifold with a semi-symmetric metric connection., Hacettepe J. of Math. and Stat., 39(2010), 489-496.
[6] C. Özgür., On ϕ-conformally flat Lorentzian para-Sasakian manifolds, Radovi Matematicki 12 (2003), 99-106.
[7] I. Mihai., Geometry and Topology of submanifolds, Vol. VII, 186-88, World Scientific, Singapore, 1995.
[8] I. Mihai., Geometry and Topology of submanifolds, Vol. VIII, 265-68, World Scientific, Singapore, 1996.
[9] K. Matsumoto., On Lorentzian paracontact manifolds, Bull. Yamagata Univ. Natur. Sci. 12(1989), $151-156$.
[10] K. Matsumoto, I. Mihai and R. Rosca., ξ-null geodesic gradient vector fields on a Lorentzian para-Sasakian manifolds, J. Korean Math. Soc., 32(1995), $17-31$.
[11] K. De and U. C. De., LP-Sasakian manifolds with quasi-conformal curvature tensor, SUT J. of Math., 49(2013), 33-46.
[12] M. Kobayashi., CR-submanifold of a Sasakian manifold, Tensor N. S. 35 (1981), 297-307.
[13] M. Ahmad., CR-submanifolds of a Lorentzian pra-Sasakian manifold endowed with a quarter-symmetric metric connection., Bull. Korean Math. Soc., 49(2012), 25-32.
[14] M. Ahmad, A. Haseeb, J. B. Jun and M. H. Shahid., CR-submanifolds and CR-product of a Lorentzian pra-Sasakian manifold endowed with a quarter symmetric semi-metric connection., Afr. Mat., 25(2014), 1113-1124.
[15] S. Prasad and R. H. Ojha., Lorentzian para-contact submanifolds, Publ. Math. Debrecen., 44(1994), 215-223.
[16] U. C. De and A. K. Sengupta., CR-submanifolds of a Lorentzian para-Sasakian manifold, Bull. Malaysian Math. Soc., 23(2000), 99-106.