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A Note on Preservation of Generalized Fredholm Spectra in Berkani's Sense

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Abstract. In this paper, we study the relationships between the spectra derived from B-Fredholm theory corresponding to two given bounded linear operators. The main goal of this paper is to obtain sufficient conditions for which the spectra derived from B-Fredholm theory corresponding to two given operators are respectively the same. Among other results, we prove that B-Fredholm type spectral properties for an operator and its restriction are equivalent, as well as obtain conditions for which B-Fredholm type spectral properties corresponding to two given operators are the same. As application of our results, we obtain conditions for which the above mentioned spectra and the spectra derived from the classical Fredholm theory are the same.

1. Introduction

In [3], B. Barnes studied the relationship between the Fredholm properties of an operator and the Fredholm properties of its extensions to certain superspaces, assuming some special conditions on the ranges. In [4], the same author studied the transmission of some properties from a bounded linear operator, as closed range and generalized inverses, to its restriction on certain subspaces and vice-versa. Motivated by these researches and by the generalized Fredholm theory in Berkani's sense (briefly, B-Fredholm theory) introduced recently by Berkani [6], in this paper we adopt the notation of [4] and investigate the behavior of several spectra derived from the B-Fredholm theory for an operator T and its restriction T_W on a proper closed and T-invariant subspace $W \subseteq X$ such that $T^n(X) \subseteq W$, for some $n \geq 1$, where $T \in L(X)$ and X is an infinite-dimensional complex Banach space. The main goal of this paper is to study the relationships between B-Fredholm type spectral properties of T and T_W in order to obtain sufficient conditions for which B-Fredholm type spectral properties for two given operators are equivalent. As application of our results, we give conditions for which the spectra derived from the B-Fredholm and the spectra derived from the classical Fredholm theory are the same.

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2. Preliminaries

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. The classes of operators studied in the classical Fredholm theory generate several spectra associated with an operator $T \in L(X)$. The *Fredholm spectrum* is defined by

$$\sigma_{\mathrm{f}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},$$

the *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{uf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Fredholm}\},$$

and the *lower semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{lf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Fredholm}\}.$$

The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{b}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},$$

and

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

Analogously, the *upper semi-Browder spectrum*, the *lower semi-Browder spectrum*, the *upper semi-Weyl spectrum* and the *lower semi-Weyl spectrum* are defined respectively by

$$\sigma_{\mathrm{ub}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

$$\sigma_{\mathrm{lb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Browder}\},$$

$$\sigma_{\mathrm{uw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\},$$

and

$$\sigma_{\mathrm{lw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Weyl}\}.$$

By duality, $\sigma_{\mathrm{lf}}(T) = \sigma_{\mathrm{uf}}(T^*)$, $\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*)$ and $\sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{uw}}(T^*)$, where T^* denotes the dual of T. For further information on Fredholm operators theory, we refer to [1] and [10].

According to [5] and [6], T_n denotes the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. Also, $T \in L(X)$ is said to be *B-Fredholm* (resp. *upper semi B-Fredholm*, *lower semi B-Fredholm*, *semi B-Fredholm*, *B-Browder*, *upper semi B-Browder*, *lower semi B-Browder*), if for some integer $n \geq 0$, the range $R(T^n)$ is closed and T_n, viewed as an operator from the space $R(T^n)$ into itself, is a Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm, Browder, upper semi-Browder, lower semi-Browder). If T_n is a semi-Fredholm operator, it follows from [6, Proposition 2.1] that also T_m is semi-Fredholm for every $m \geq n$, and $\operatorname{ind} T_m = \operatorname{ind} T_n$. This enables us to define the *index* of a semi B-Fredholm operator T as the index of the semi-Fredholm operator T_n. Thus, $T \in L(X)$ is said to be a *B-Weyl operator* if T is a B-Fredholm operator having index 0. $T \in L(X)$ is said to be *upper semi B-Weyl* (resp. *lower semi B-Weyl*) if T is upper semi B-Fredholm (resp. lower semi B-Fredholm) with index $\operatorname{ind} T \leq 0$ (resp. $\operatorname{ind} T \geq 0$). Note that if T is semi B-Fredholm and T^* denotes the dual of T, then also T^* is semi B-Fredholm with $\operatorname{ind} T^* = -\operatorname{ind} T$.

The spectra related with the B-Fredholm theory are defined as follows. The *B-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{bf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Fredholm}\},$$

the *upper semi B-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{ubf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Fredholm}\},$$

the *lower semi B-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{lbf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Fredholm}\},$$

while the *B-Browder spectrum* and *B-Weyl spectrum* are defined by

$$\sigma_{\mathrm{bb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\},$$

and

$$\sigma_{\mathrm{bw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}.$$

Similarly, the *upper semi B-Browder spectrum*, the *lower semi B-Browder spectrum*, the *upper semi B-Weyl spectrum* and the *lower semi B-Weyl spectrum* are defined respectively by

$$\sigma_{\mathrm{ubb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder}\},$$

$$\sigma_{\mathrm{lbb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder}\},$$

$$\sigma_{\mathrm{ubw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\},$$

and

$$\sigma_{\mathrm{lbw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\}.$$

Another class of operators related with semi B-Fredholm operators is the quasi-Fredholm operators defined in the sequel. Previously, we consider the following set.

$$\Delta(T) = \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap N(T) \subseteq T^m(X) \cap N(T)\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) = \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.1. *$T \in L(X)$ is said to be* quasi-Fredholm of degree *d, if there exists $d \in \mathbb{N}$ such that:*
(a) *$dis(T) = d$,*
(b) *$T^n(X)$ is a closed subspace of X for each $n \geq d$,*
(c) *$T(X) + N(T^d)$ is a closed subspace of X.*

Lemma 2.2. *([6]) . $T \in L(X)$ is an upper semi B-Fredholm (resp. a lower semi B-Fredholm) operator if and only if there exists an integer $d \in \mathbb{N}$ such that T is quasi-Fredholm of degree d and $N(T) \cap R(T^d)$ is of finite dimension (resp. $R(T) + N(T^d)$ is of finite codimension)*

.

Definition 2.3. *([9]). $T \in L(X)$ is said to have* the single valued extension property *at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at λ_0), if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0 \quad \textit{for all } \lambda \in \mathbb{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Also, the single valued extension property is inherited by restrictions on invariant closed subspaces. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{2}$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

Note that if $\sigma_{\mathrm{su}}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\mathrm{su}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\},$$

then $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{su}}(T^*)$, $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T^*)$ and $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{su}}(T)$.

It is easily seen from definition of localized SVEP, that

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{ap}}(T) \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{su}}(T) \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{4}$$

where $\operatorname{acc} K$ means the set of all accumulation points of a subset $K \subseteq \mathbb{C}$.

Remark 2.4. The implications (1), (2), (3) and (4) are actually equivalences, if $T \in L(X)$ is semi-Fredholm (see [1, Chapter 3]). More generally, if $T \in L(X)$ is quasi-Fredholm (see [2]).

Lemma 2.5. *([8]) . Let $T \in L(X)$. Then the following properties are equivalent:*

(i) $\lambda I - T$ is upper (resp. lower) semi B-Browder;
(ii) $\lambda I - T$ is quasi-Fredholm operator having finite ascent (resp. descent);
(iii) $\lambda I - T$ is quasi-Fredholm and T (resp. T^*) has the SVEP at λ.

.

According to the notations of Barnes [4], in the sequel of this paper we always assume that W is a proper closed subspace of a Banach space X. Also, we denote

$$\mathcal{P}(X, W) = \{T \in L(X) : T(W) \subseteq W \text{ and for some integer } n \geq 1, T^n(X) \subseteq W\}.$$

For each $T \in \mathcal{P}(X, W)$, T_W denote the restriction of T on the subspace T-invariant W of X. Observe that $0 \in \sigma_{\mathrm{su}}(T)$ for all $T \in \mathcal{P}(X, W)$, but $\sigma_{\mathrm{su}}(T)$ and $\sigma_{\mathrm{su}}(T_W)$ may differ only in 0.

3. Relations between the spectra of T and T_W

In this section we give some fundamental facts, by citing several previous results which will be used in the proof of the main results of this paper.

Lemma 3.1. *(see [7]) Let $T \in L(X)$. Then*

$$(T^n)^{-1}(R(T^{n+m})) = R(T^m) + N(T^n),$$

for any non-negative integers n, m.

Theorem 3.2. *(see [7]) Let $T \in \mathcal{P}(X, W)$. Then for all $\lambda \neq 0$, we have*

$$R((\lambda I - T)^m) \text{ is closed in } X \text{ if and only if } R((\lambda I - T_W)^m) \text{ is closed in } W$$

for any integer $m \geq 1$.

Lemma 3.3. *(see [7]) If $T \in \mathcal{P}(X, W)$, then for all $\lambda \neq 0$:*

(i) $N((\lambda I - T_W)^m) = N((\lambda I - T)^m)$, *for any* m,
(ii) $R((\lambda I - T_W)^m) = R((\lambda I - T)^m) \cap W$, *for any* m,
(iii) $\alpha(\lambda I - T_W) = \alpha(\lambda I - T)$,
(iv) $p(\lambda I - T_W) = p(\lambda I - T)$,
(v) $\beta(\lambda I - T_W) = \beta(\lambda I - T)$.

Moreover, we have the following equivalences.

Lemma 3.4. *(see [7]) If* $T \in \mathcal{P}(X, W)$, *then:*

(i) $p(T) < \infty$ *if and only if* $p(T_W) < \infty$,
(ii) $q(T) < \infty$ *if and only if* $q(T_W) < \infty$.

The following is a generalization of Lemma 3.3, part (v).

Theorem 3.5. *Let* $T \in \mathcal{P}(X, W)$. *Then for all* $\lambda \neq 0$,

$$\beta((\lambda I - T_W)^m) = \beta((\lambda I - T)^m)$$

for any integer $m \geq 1$.

Proof. Observe that if $T \in \mathcal{P}(X, W)$, for any integer $m \geq 1$, we have

$$\begin{aligned}(\lambda I - T)^m &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^{m-k} T^k \\ &= \lambda^m I - \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \lambda^{m-k} T^k. \\ &= \mu I - S,\end{aligned}$$

where $S = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \lambda^{m-k} T^k \in \mathcal{P}(X, W)$ and $\mu = \lambda^m \neq 0$.

Similarly,

$$\begin{aligned}(\lambda I - T_W)^m &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^{m-k} (T_W)^k \\ &= \lambda^m I - \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \lambda^{m-k} (T_W)^k. \\ &= \mu I - S_W.\end{aligned}$$

From the above equalities and by Lemma 3.3, part (v), we obtain the equality $\beta(\lambda I - S_W) = \beta(\lambda I - S)$ or equivalently $\beta((\lambda I - T_W)^m) = \beta((\lambda I - T)^m)$.

∎

Theorem 3.6. *(see [7]) If* $T \in \mathcal{P}(X, W)$ *and* $p(T) = \infty$, *or* $q(T) = \infty$, *then the following equalities are true:*

(i) $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{su}}(T_W)$;
(ii) $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T_W)$;
(iii) $\sigma(T) = \sigma(T_W)$;
(iv) $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T_W)$;
(v) $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T_W)$;

(vi) $\sigma_{\rm b}(T) = \sigma_{\rm b}(T_W)$;
(vii) $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(T_W)$;
(viii) $\sigma_{\rm f}(T) = \sigma_{\rm f}(T_W)$;
(ix) $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(T_W)$.

Remark 3.7. Recall that for $T \in L(X)$, $0 < p(\lambda I-T) = q(\lambda I-T) < \infty$ precisely when λ is a pole of the resolvent of T (see [10, Prop. 50.2]). Also, is well known that if λ is a pole of the resolvent of T, then $\lambda \in \operatorname{iso}\sigma(T)$. Evidently, if $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \in \partial\sigma(T)$. Thus, for $T \in \mathcal{P}(X, W)$, if $0 \notin \operatorname{iso}\sigma(T)$ (resp. $0 \notin \partial\sigma(T)$) then $p(T) = \infty$ or $q(T) = \infty$. Therefore, the conclusions of Theorem 3.6 remain true if the hypothesis $p(T) = \infty$ or $q(T) = \infty$ is replaced by one of the following hypothesis: $0 \notin \operatorname{iso}\sigma(T)$ or $0 \notin \partial\sigma(T)$. On the other hand, according to Lemma 3.4 we can change the hipothesis $p(T) = \infty$ or $q(T) = \infty$ by $p(T_W) = \infty$ or $q(T_W) = \infty$ in the Theorem 3.6. Hence, the conclusions of Theorem 3.6 remain true if the hypothesis $p(T) = \infty$ or $q(T) = \infty$ is replaced by one of the following hypothesis: $0 \notin \operatorname{iso}\sigma(T_W)$ or $0 \notin \partial\sigma(T_W)$.

4. B-Fredholm properties for T and T_W

In this section, we present the main results of this paper. We give sufficient conditions for which B-Fredholm type spectral properties for an operator $T \in \mathcal{P}(X, W)$ and its restriction T_W are equivalent, as well as we obtain conditions for which B-Fredholm type spectral properties corresponding to two given operators $T, S \in \mathcal{P}(X, W)$ are equivalent. Also, we give conditions for which the spectra derived from the B-Fredholm theory and the spectra derived from the classical Fredholm theory are the same.

The following proposition will play an important role in this paper.

Lemma 4.1. *Let $T \in \mathcal{P}(X, W)$. Then for all $\lambda \neq 0$, we have*

$$\lambda I - T \text{ is quasi-Fredholm if and only if } \lambda I - T_W \text{ is quasi-Fredholm.}$$

Proof. (Sufficiency) Suppose that $\lambda I - T$ is quasi-Fredholm. If $d = \operatorname{dis}(\lambda I - T)$, then for all $m \geq d$

$$\begin{aligned} N(\lambda I - T) \cap R((\lambda I - T)^m) &= N(\lambda I - T) \cap R((\lambda I - T)^d) \\ N(\lambda I - T) \cap R((\lambda I - T)^m) \cap W &= N(\lambda I - T) \cap R((\lambda I - T)^d) \cap W \end{aligned}$$

By Lemma 3.3,

$$N(\lambda I - T_W) \cap R((\lambda I - T_W)^m) = N(\lambda I - T_W) \cap R((\lambda I - T_W)^d),$$

for all $m \geq d$. On the other hand, since $R(\lambda I - T)^m)$ is closed in X for $m \geq d$ and $\lambda \neq 0$, then from Lemma 3.2, $R(\lambda I - T_W)^m)$ is closed in W for any $m \geq d$. But by Lemma 3.1, $R(\lambda I - T_W) + N(\lambda I - T_W)^m = ((T_W)^d)^{-1}(R(\lambda I - T_W)^{1+m}))$. So $R(\lambda I - T_W) + N(\lambda I - T_W)^m$ is closed in W for $m \geq d$. Therefore, we conclude that $\lambda I - T_W$ is quasi-Fredholm.

(Necessity) Suppose that $\lambda I - T_W$ is quasi-Fredholm. If $d' = \operatorname{dis}(\lambda I - T_W)$, then $N(\lambda I - T_W) \cap R((\lambda I - T_W)^m) = N(\lambda I - T_W) \cap R((\lambda I - T_W)^{d'})$ for $m \geq d'$. Using Lemma 3.3, we obtain that

$$\begin{aligned} N(\lambda I - T_W) \cap R((\lambda I - T)^m) \cap W &= N(\lambda I - T_W) \cap R((\lambda I - T)^{d'}) \cap W \\ (N(\lambda I - T_W) \cap W) \cap R((\lambda I - T)^m) &= (N(\lambda I - T_W) \cap W) \cap R((\lambda I - T)^{d'}) \\ N(\lambda I - T_W) \cap R((\lambda I - T)^m) &= N(\lambda I - T_W) \cap R((\lambda I - T)^{d'}) \end{aligned}$$

Again, by Lemma 3.3, $N(\lambda I - T) \cap R((\lambda I - T)^m) = N(\lambda I - T) \cap R((\lambda I - T)^{d'})$ for $m \geq d'$. As above, being $R(\lambda I - T_W)^m)$ closed in W for $m \geq d'$ and $\lambda \neq 0$, from Lemma 3.2, $R(\lambda I - T)^m)$ is closed in X for $m \geq d'$. By Lemma 3.1,

$$R(\lambda I - T) + N((\lambda I - T)^m) = (T^d)^{-1}(R(\lambda I - T)^{1+m})).$$

Hence $R(\lambda I - T) + N(\lambda I - T)^m$ is closed in X for $m \geq d'$. Then $\lambda I - T$ is quasi-Fredholm.

∎

The following result summarize interesting spectral relationships between an operator $T \in \mathcal{P}(X, W)$ and its restriction T_W for several spectra derived from the B-Fredholm theory.

Theorem 4.2. *If $T \in \mathcal{P}(X, W)$ and $0 \notin \operatorname{iso}\sigma(T)$, then the following equalities are true:*

(i) $\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{ubf}}(T_W)$;
(ii) $\sigma_{\mathrm{bf}}(T) = \sigma_{\mathrm{bf}}(T_W)$;
(iii) $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ubb}}(T_W)$;
(iv) $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bb}}(T_W)$;
(v) $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ubw}}(T_W)$;
(vi) $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bw}}(T_W)$.

Proof. (i) Observe that for any $n \in \mathbb{N}$ and $\lambda \neq 0$, by Lemma 3.3,

$$\begin{aligned} N(\lambda I - T) \cap R((\lambda I - T)^n) &= N(\lambda I - T_W) \cap R((\lambda I - T)^n) \\ &= N(\lambda I - T_W) \cap W \cap R((\lambda I - T)^n) \\ &= N(\lambda I - T_W) \cap R((\lambda I - T_W)^n) \end{aligned}$$

Hence, $N(\lambda I - T) \cap R((\lambda I - T)^n)$ has finite dimension if and only if $N(\lambda I - T_W) \cap R((\lambda I - T_W)^n)$ has finite dimension. Also, by Lemma 4.1, $\lambda I - T$ is quasi-Fredholm if and only if $\lambda I - T_W$ is quasi-Fredholm. Consequently, by Lemma 2.2, $\lambda I - T$ is upper semi B-Fredholm if and only if $\lambda I - T_W$ is upper semi B-Fredholm. Therefore $\sigma_{\mathrm{ubf}}(T) \setminus \{0\} = \sigma_{\mathrm{ubf}}(T_W) \setminus \{0\}$. In the case that $\lambda = 0$, observe first that if T is upper semi B-Fredholm, then there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is upper semi Fredholm. From this, $0 \notin \sigma_{\mathrm{uf}}(T_n)$. Since $0 \notin \operatorname{iso}\sigma(T)$, according to Remark 3.7, $p(T) = \infty$ or $q(T) = \infty$. But this is equivalent, by Lemma 3.4, to $p(T_n) = \infty$ or $q(T_n) = \infty$ because $T \in \mathcal{P}(X, R(T^n))$. Hence $T \in \mathcal{P}(X, R(T^n))$ and $p(T_n) = \infty$ or $q(T_n) = \infty$, and by Theorem 3.6, $\sigma_{\mathrm{uf}}(T_n) = \sigma_{\mathrm{uf}}(T)$. This implies that $0 \notin \sigma_{\mathrm{uf}}(T)$. As by hypothesis $T \in \mathcal{P}(X, W)$, using Theorem 3.6, $\sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{uf}}(T_W)$. So $0 \notin \sigma_{\mathrm{uf}}(T_W)$ which implies that T_W is upper semi Fredholm. Therefore T_W is upper semi B-Fredholm. Conversely, as seen above, if T_W is upper semi B-Fredholm, then there exists $n \in \mathbb{N}$ such that $R((T_W)^n)$ is closed and $(T_W)_n$ is upper semi Fredholm. From this, $0 \notin \sigma_{\mathrm{uf}}((T_W)_n)$. Since $T_W \in \mathcal{P}(W, R((T_W)^n))$, from Remark 3.7, Lemma 3.4 and Theorem 3.6, we conclude that $\sigma_{\mathrm{uf}}((T_W)_n) = \sigma_{\mathrm{uf}}(T_W)$ and hence $0 \notin \sigma_{\mathrm{uf}}(T_W)$. Since $T \in \mathcal{P}(X, W)$, again by Theorem 3.6, $\sigma_{\mathrm{uf}}(T_W) = \sigma_{\mathrm{uf}}(T)$. So $0 \notin \sigma_{\mathrm{uf}}(T)$ which implies that T is upper semi Fredholm, consequently T is upper semi B-Fredholm. This shows that T is upper semi B-Fredholm if and only if T_W is upper semi B-Fredholm. From this, and the equality $\sigma_{\mathrm{ubf}}(T) \setminus \{0\} = \sigma_{\mathrm{ubf}}(T_W) \setminus \{0\}$, we conclude the equality $\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{ubf}}(T_W)$.

(ii) Similarly to the part (i), but using Lemma 3.5 and the isomorphism $\frac{R(T^k)}{R(T^{k+1})} \cong \frac{X}{N(T^k)+R(X)}$ (via $T^k x + R(T^{k+1}) \to x + N(T^k) + R(T)$), we can conclude that

$$\begin{aligned} \beta((\lambda I - T)^{d+1}) &= \beta((\lambda I - T_W)^{d+1}) \\ \beta((\lambda I - T)_d) &= \beta((\lambda I - T_W)_d) \\ \dim \frac{R((\lambda I - T)^d)}{R((\lambda I - T)^{d+1})} &= \dim \frac{R((\lambda I - T_W)^d)}{R((\lambda I - T_W)^{d+1})} \\ \dim \frac{X}{N((\lambda I - T)^d) + R(\lambda I - T)} &= \dim \frac{W}{N((\lambda I - T_W)^d) + R(\lambda I - T_W)} \end{aligned}$$

Then $N((\lambda I - T)^d) + R(\lambda I - T)$ has finite codimension if and only if $N((\lambda I - T_W)^d) + R(\lambda I - T_W)$ has finite codimension. Consequently, $\lambda I - T$ is B-Fredholm if and only if $\lambda I - T_W$ is B-Fredholm. For the case $\lambda = 0$, arguing as in the proof of part (i), we obtain that $\sigma_{\mathrm{bf}}(T) = \sigma_{\mathrm{bf}}(T_W)$.

(iii) First, notice that, by Lemma 2.5 and Lemma 4.1, $\sigma_{\mathrm{ub}}(T) \setminus \{0\} = \sigma_{\mathrm{ub}}(T_W) \setminus \{0\}$. Now, we show that also $0 \in \sigma_{\mathrm{ub}}(T) \cap \sigma_{\mathrm{ub}}(T_W)$ holds. To see this, suppose that $0 \notin \sigma_{\mathrm{ub}}(T)$. By duality $0 \notin \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{lb}}(T^*)$ and hence, by Lemma 2.5, T is quasi-Fredholm and T, T^* have the SVEP at 0. Thus, by Remark 2.4, $0 \in \text{iso}\,\sigma(T)$, contradicting our hypothesis (observe that $0 \in \sigma(T)$). On the other hand, if $0 \notin \sigma_{\text{ub}}(T_W)$. Arguing as above, $0 \notin \sigma_{\text{ub}}(T_W) = \sigma_{\text{lb}}((T_W)^*)$ and hence, by Lemma 2.5, T_W is quasi-Fredholm and T_W, $(T_W)^*$ have the SVEP at 0. Again, by Remark 2.4, $0 \in \text{iso}\,\sigma(T_W)$, a contradiction. Because $\sigma(T_W) = \sigma(T)$, by Theorem 3.6.

(iv) The case $\lambda \neq 0$, follows immediately from Lemma 2.5 and Lemma 2.2. The case $\lambda = 0$ is analogous to the part (iii).

For (v) and (vi). The case $\lambda \neq 0$, follow from (i) and (ii), by using the equality $\text{ind}(\lambda I - T) = \text{ind}(\lambda I - T_W)$ derived from Lemma 3.3. The case $\lambda = 0$, is analogous to the parts (i), (ii), (iii) and (iv).

∎

In the next result, we give alternative conditions for which the equalities of Theorem 4.2 remain true.

Corollary 4.3. *Suppose that $T \in \mathcal{P}(X, W)$. If ones of the following conditions is valid:*

(i) $0 \notin \partial\sigma(T)$,
(ii) $0 \notin \text{iso}\,\sigma(T_W)$,
(iii) $0 \notin \partial\sigma(T_W)$.

Then all equalities in Theorem 4.2 are true too.

Proof. The given corollary immediately follows from Theorem 4.2 and Remark 3.7.

∎

As consequence of the above corollary, we obtain sufficient conditions for which the spectra derived from the B-Fredholm theory corresponding to two given operators are respectively the same.

Corollary 4.4. *Suppose that T, $S \in \mathcal{P}(X, W)$ and T, S agree on W. If ones of the following conditions is valid:*

(i) $0 \notin \text{iso}\,\sigma(T_W)$ *(or* $0 \notin \text{iso}\,\sigma(S_W)$*)*,
(ii) $0 \notin \partial\sigma(T_W)$ *(or* $0 \notin \partial\sigma(S_W)$*)*.

Then

(i) $\sigma_{\text{ubf}}(T) = \sigma_{\text{ubf}}(S)$ *and* $\sigma_{\text{bf}}(T) = \sigma_{\text{bf}}(S)$,
(ii) $\sigma_{\text{ubb}}(T) = \sigma_{\text{ubb}}(S)$ *and* $\sigma_{\text{bb}}(T) = \sigma_{\text{bb}}(S)$,
(iii) $\sigma_{\text{ubw}}(T) = \sigma_{\text{ubw}}(S)$ *and* $\sigma_{\text{bw}}(T) = \sigma_{\text{bw}}(S)$.

∎

The following theorem ensures that bounded operators acting on complemented subspaces can always be extended on the entire space preserving its generalized spectra in Berkani's sense.

Theorem 4.5. *Let W be a complemented subspace of X and $T \in L(W)$. If ones of the following conditions is valid:*

(i) $0 \notin \text{iso}\,\sigma(T)$,
(ii) $0 \notin \partial\sigma(T)$.

Then T has an extension $\overline{T} \in \mathcal{P}(X, W)$ and the following equalities are true:

(i) $\sigma_{\text{ubf}}(T) = \sigma_{\text{ubf}}(\overline{T})$ *and* $\sigma_{\text{bf}}(T) = \sigma_{\text{bf}}(\overline{T})$,
(ii) $\sigma_{\text{ubb}}(T) = \sigma_{\text{ubb}}(\overline{T})$ *and* $\sigma_{\text{bb}}(T) = \sigma_{\text{bb}}(\overline{T})$,
(iii) $\sigma_{\text{ubw}}(T) = \sigma_{\text{ubw}}(\overline{T})$ *and* $\sigma_{\text{bw}}(T) = \sigma_{\text{bw}}(\overline{T})$.

Proof. Since W is a complemented subspace of X, then there exists a bounded projection $P \in L(X)$ such that $P(X) = W$. Thus $\overline{T} = TP$ defines an operator in $\mathcal{P}(X, W)$ and $T = \overline{T}_W$. From this, by Corollary 4.3, we obtain the equalities (i), (ii) and (iii).

■

As a particular consequence of the above theorem, we obtain the following corollary.

Corollary 4.6. *Let W be a closed proper subspace of a Hilbert space H and $T \in L(W)$. If ones of the following conditions is valid:*

(i) $0 \notin \mathrm{iso}\,\sigma(T)$,

(ii) $0 \notin \partial\sigma(T)$.

Then T has an extension $\overline{T} \in L(H)$ and the following equalities are true:

(i) $\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{ubf}}(\overline{T})$ *and* $\sigma_{\mathrm{bf}}(T) = \sigma_{\mathrm{bf}}(\overline{T})$,

(ii) $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ubb}}(\overline{T})$ *and* $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bb}}(\overline{T})$,

(iii) $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ubw}}(\overline{T})$ *and* $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bw}}(\overline{T})$.

Proof. Follows immediately from Theorem 4.5, because every closed subspaces of a Hilbert space is complemented.

■

The following theorem gives sufficient conditions for which B-Fredholm type properties and Fredholm type properties are essentially the same for an operator.

Theorem 4.7. *If $T \in \mathcal{P}(X, W)$ and T has connected spectrum, then the following equalities are true:*

(i) $\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{uf}}(T)$;

(ii) $\sigma_{\mathrm{bf}}(T) = \sigma_{\mathrm{f}}(T)$;

(iii) $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ub}}(T)$;

(iv) $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{b}}(T)$;

(v) $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{uw}}(T)$;

(vi) $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{w}}(T)$.

Proof. Observe first that, under the hypothesis T has connected spectrum, $\mathrm{iso}\,\sigma(T) = \emptyset$ and hence λ is not an isolated point of $\sigma(T)$ for all λ. This implies that $0 \notin \mathrm{iso}\,\sigma(\lambda I - T)$ for all λ. Also $\lambda I - T \in \mathcal{P}(X, R((\lambda I - T)^n))$ for all $n \in \mathbb{N}$, such that $R((\lambda I - T)^n)$ is closed. According this facts, we have the following.

(i) If $\lambda \notin \sigma_{\mathrm{ubf}}(T)$, then $\lambda I - T$ is upper semi B-Fredholm. Thus, there exists $n \in \mathbb{N}$ such that $R((\lambda I - T)^n)$ is closed and $(\lambda I - T)_n$ is upper semi-Fredholm. This implies that $0 \notin \sigma_{\mathrm{uf}}((\lambda I - T)_n))$. Since $\lambda I - T \in \mathcal{P}(X, R((\lambda I - T)^n))$ and $0 \notin \mathrm{iso}\,\sigma(\lambda I - T)$, by Theorem 3.6, $\sigma_{\mathrm{uf}}(\lambda I - T) = \sigma_{\mathrm{uf}}((\lambda I - T)_n))$. In consequence $0 \notin \sigma_{\mathrm{uf}}(\lambda I - T)$, so $\lambda I - T$ is upper semi Fredholm. Hence $\lambda \notin \sigma_{\mathrm{uf}}(T)$. Conversely, if $\lambda \notin \sigma_{\mathrm{uf}}(T)$, then $\lambda I - T$ is upper semi-Fredholm. It follows that $R(\lambda I - T)^n$ is closed for any $n \in \mathbb{N}$, and hence $0 \notin \sigma_{\mathrm{uf}}(\lambda I - T)$. Again, since $\lambda I - T \in \mathcal{P}(X, R((\lambda I - T)^n))$ and $0 \notin \mathrm{iso}\,\sigma(\lambda I - T)$, by Theorem 3.6, $\sigma_{\mathrm{uf}}(\lambda I - T) = \sigma_{\mathrm{uf}}((\lambda I - T)_n))$. Consequently $0 \notin \sigma_{\mathrm{uf}}((\lambda I - T)_n)$, so $(\lambda I - T)_n$ is upper semi- Fredholm. Then $\lambda I - T$ is upper semi B-Fredholm and hence $\lambda \notin \sigma_{\mathrm{ubf}}(T)$. This proves the equality $\sigma_{\mathrm{ubf}}(T) = \sigma_{\mathrm{uf}}(T)$.

The proofs of (ii), (iii), (iv), (v) and (vi) are similar to that of part (i) and hence omitted.

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