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Hermite-Hadamard Type Inequalities for $\mathbb{B}^{-1}$-Convex Functions Involving Generalized Fractional Integral Operators

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Abstract. $\mathbb{B}^{-1}$-convexity is an abstract convexity type. We obtained Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions. But now, there are new and more general integral operator types that are fractional integrals. Thus, we need to prove Hermite-Hadamard inequalities involving different fractional integral operator types with this article.

1. Introduction

Recently, abstract convexity which becomes different convexity types is studied more than classic convexity. Because, abstract convexity has an important area in convexity theory and also, abstract convexity has significant applications variety fields like mathematical economy, operation research, inequality theory and optimization theory. Additionally, one of these abstract convexity types is $\mathbb{B}^{-1}$-convexity ([5, 7]). It has applications to mathematical economy and inequality theory ([7, 18]). Many articles were written about $\mathbb{B}^{-1}$-convexity ([4, 6, 12, 16]).

As an application on inequality theory for abstract convexity types, we can give the Hermite-Hadamard inequalities that are shown for integral mean value of a convex function. ([1–3, 9–11, 14, 17]). For $\mathbb{B}^{-1}$-convex functions Hermite-Hadamard inequality via classic integral was proven in [18]. Hermite-Hadamard inequality is an integral inequality and it has be given with classic integral operator up to now. But, recently, fractional integral operators because of generallity have been used when proven the integral inequalities ([8, 15]). Also, fractional integral operators can be compared in themselves according to generality. Therefore, we study Hermite-Hadamard inequalities involving different fractional integral operator types for $\mathbb{B}^{-1}$-convexity and we compare generality of these inequalities.

In this article, three parts that are definitions of fractional integral types, required definitions and theorems of $\mathbb{B}^{-1}$-convexity and Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions are given in Section 2. In third section, we give Hermite-Hadamard type inequalities involving Riemann - Liouville fractional integral. Next, Hermite-Hadamard type inequalities involving Hadamard fractional integral is shown. In last section, we prove Hermite-Hadamard type inequalities involving fractional integrals of a function with respect to another function. Consequently, we show that the last inequalities are more general.

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2. Preliminaries

In this section, we give some required definition and theorems.

2.1. Fractional Integral Types

Lets recall the following definitions of fractional integral types. Along the paper, let $f : [a,b] \to \mathbb{R}$ be a given function, where $0 \leq a < b < +\infty$ and $f \in L_1[a,b]$. Also, $\Gamma(\alpha)$ is the Gamma function.

Definition 2.1. *[13] The left-sided Riemann-Liouville integral $J_{a^+}^{\alpha} f$ and the right-sided Riemann-Liouville integral $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a \tag{1}$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b \tag{2}$$

respectively.

Definition 2.2. *[13] The left-sided Hadamard fractional integral $\boldsymbol{J}_{a^+}^{\alpha}$ of order $\alpha > 0$ of f is defined by*

$$\boldsymbol{J}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\,dt, \quad x > a \tag{3}$$

provided that the integral exists. The right-sided Hadamard fractional integral $\boldsymbol{J}_{b^-}^{\alpha}$ of order $\alpha > 0$ of f is defined by

$$\boldsymbol{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t}\,dt, \quad x < b \tag{4}$$

provided that the integral exists.

Definition 2.3. *[13] Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The left-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ is defined by*

$$I_{a^+;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{[g(x) - g(t)]^{1-\alpha}}\,dt, \quad x > a \tag{5}$$

provided that the integral exists. The right-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ is defined by

$$I_{b^-;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)}{[g(t) - g(x)]^{1-\alpha}}\,dt, \quad x < b \tag{6}$$

provided that the integral exists.

2.2. $\mathbb{B}^{-1}$-convexity

For $r \in \mathbb{Z}^-$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$ to itself; $\boldsymbol{x} = (x_1, x_2, ..., x_n) \to \Phi_r(\boldsymbol{x}) = (\varphi_r(x_1), \varphi_r(x_2), ..., \varphi_r(x_n))$ is homeomorphism from $\mathbb{R}_*^n$ to itself.

For a finite nonempty set $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)}\right\} \subset \mathbb{R}_*^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(\boldsymbol{x}^{(i)})\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1\right\} .$$

We denote by $\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)} \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m)}\right\}, ..., \min\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $\boldsymbol{x}^{(i)}$.

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.4. [5] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r\in\mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of $\mathbb{R}_*^n$, is called $\mathbb{B}^{-1}$-polytope of A.

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}_{++}^n$.

Theorem 2.5. [5] For all nonempty finite subsets $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)}\right\} \subset \mathbb{R}_{++}^n$ we have

$$Co^{-\infty}(A) = \lim_{r\to-\infty} Co^r(A) = \left\{\overset{m}{\underset{i=1}{\wedge}} t_i \boldsymbol{x}^{(i)} : t_i \geq 1, \min_{1\leq i\leq m} t_i = 1\right\} .$$

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.6. [5] A subset U of $\mathbb{R}_*^n$ is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.

By Theorem 2.5, we can reformulate the above definition for subsets of $\mathbb{R}_{++}^n$:

Theorem 2.7. [5] A subset U of $\mathbb{R}_{++}^n$ is $\mathbb{B}^{-1}$-convex if and only if for all $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in U$ and all $\lambda \in [1, \infty)$ one has $\lambda \boldsymbol{x}^{(1)} \wedge \boldsymbol{x}^{(2)} \in U$.

Remark 2.8. As a result of Theorem 2.7, we can say that $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are positive intervals.

Definition 2.9. [12] For $U \subset \mathbb{R}_*^n$, a function $f : U \to \mathbb{R}_*$ is called a $\mathbb{B}^{-1}$-convex function if $epi^*(f) = \left\{(\boldsymbol{x}, \mu) \middle| \boldsymbol{x} \in U, \mu \in \mathbb{R}_*, \mu \geq f(\boldsymbol{x})\right\}$ is a $\mathbb{B}^{-1}$-convex set.

In $\mathbb{R}_{++}^n$, we can give the following fundamental theorem which provides a sufficient and necessary condition for $\mathbb{B}^{-1}$-convex functions [12].

Theorem 2.10. Let $U \subset \mathbb{R}_{++}^n$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-convex if and only if the set U is $\mathbb{B}^{-1}$-convex and one has the inequality

$$f(\lambda \boldsymbol{x} \wedge \boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) \wedge f(\boldsymbol{y}) \tag{7}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in U$ and all $\lambda \in [1, +\infty)$.

2.3. Hermite-Hadamard Inequality for $\mathbb{B}^{-1}$-convex Functions

We proved the following theorem that gives the Hermite-Hadamard inequality involving classic integral for $\mathbb{B}^{-1}$-convex functions in [18].

Theorem 2.11. Suppose $f : [a,b] \subset \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function. Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \begin{cases} \frac{f(a)(a+b)}{2a}, & \frac{b}{a} \le \frac{f(b)}{f(a)} \\ \frac{2bf(a)f(b)-a\left[\left(f(a)\right)^2+\left(f(b)\right)^2\right]}{2(b-a)f(a)}, & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{8}$$

3. Hermite-Hadamard Type Inequalities Involving Riemann - Liouville Fractional Integral

Lets recall the Riemann-Liouville fractional Hermite-Hadamard inequalities for $\mathbb{B}^{-1}$-convex functions which were given the following theorems for left-sided integral and right-sided integral, respectively ([19]).

Theorem 3.1. *Let $f : [a,b] \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $f \in L_1[a,b]$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then the following inequality holds:*

$$J_{a^+}^{\alpha} f(b) \le \begin{cases} \frac{f(a)(b-a)^{\alpha}(\alpha a+b)}{a\Gamma(\alpha+2)}, & \frac{b}{a} \le \frac{f(b)}{f(a)} \\ \frac{\left(f(a)\right)^{\alpha+1}(b-a)^{\alpha}(\alpha a+b)-\left(bf(a)-af(b)\right)^{\alpha+1}}{a\left(f(a)\right)^{\alpha}\Gamma(\alpha+2)}, & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases} \tag{9}$$

with $\alpha > 0$.

Theorem 3.2. *Let $f : [a,b] \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $f \in L_1[a,b]$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then the following inequality holds:*

$$J_{b^-}^{\alpha} f(a) \le \begin{cases} \frac{f(a)(b-a)^{\alpha}(\alpha b+a)}{a\Gamma(\alpha+2)}, & \frac{b}{a} \le \frac{f(b)}{f(a)} \\ \frac{a(\alpha+1)\left(f(a)\right)^{\alpha}f(b)(b-a)^{\alpha}-\left(af(b)-af(a)\right)^{\alpha+1}}{a\left(f(a)\right)^{\alpha}\Gamma(\alpha+2)}, & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases} \tag{10}$$

with $\alpha > 0$.

4. Hermite-Hadamard Type Inequalities Involving Hadamard Fractional Integral

Hadamard fractional integral is one of the important fractional integral types. So, the authors introduced Hermite-Hadamard type inequalities including Hadamard fractional integral in [20].

Theorem 4.1. *Let $\alpha > 0$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then*

$$\boldsymbol{J}_{a^+}^{\alpha} f(b) \le \begin{cases} \frac{f(a)}{\Gamma(\alpha)}\int_1^{\frac{b}{a}}\left(\ln\frac{b}{\lambda a}\right)^{\alpha-1}d\lambda, & \frac{b}{a} \le \frac{f(b)}{f(a)} \\ \frac{f(a)}{\Gamma(\alpha)}\int_1^{\frac{f(b)}{f(a)}}\left(\ln\frac{b}{\lambda a}\right)^{\alpha-1}d\lambda + \frac{f(b)\left(\ln\frac{bf(a)}{af(b)}\right)^{\alpha}}{\Gamma(\alpha+1)}, & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{11}$$

Theorem 4.2. *Let $\alpha > 0$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then*

$$\boldsymbol{J}_{b^-}^{\alpha} f(a) \le \begin{cases} \frac{f(a)}{\Gamma(\alpha)}\int_1^{\frac{b}{a}}(\ln\lambda)^{\alpha-1}d\lambda, & \frac{b}{a} \le \frac{f(b)}{f(a)} \\ \frac{f(a)}{\Gamma(\alpha)}\int_1^{\frac{f(b)}{f(a)}}(\ln\lambda)^{\alpha-1}d\lambda + \frac{f(b)}{\Gamma(\alpha+1)}\left[\left(\ln\frac{b}{a}\right)^{\alpha}-\left(\ln\frac{f(b)}{f(a)}\right)^{\alpha}\right], & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{12}$$

5. Hermite-Hadamard Type Inequalities Involving Fractional Integral with respect to The Function g

Theorem 5.1. *Let $\alpha > 0$ and $0 \le a < b < +\infty$, $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). If $f : [a,b] \to \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function and $f \in L_1[a,b]$, then*

$$I^{\alpha}_{a^+;g} f(b) \le \begin{cases} \frac{af(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\lambda}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} d\lambda, & \frac{b}{a} \le \frac{f(b)}{f(a)} \\ \frac{af(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a)\lambda}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} d\lambda + \frac{f(b)\left[g(b)-g\left(\frac{af(b)}{f(a)}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}, & 1 \le \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{13}$$

Proof. Since $\mathbb{B}^{-1}$-convexity of $f : [a,b] \to \mathbb{R}_{++}$, the inequality (7) is valid for all $\lambda \in [1,+\infty)$ and $0 < a < b < +\infty$. We have to multiply both sides of this inequality by $\frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}}$ and integrate the resulting inequality with respect to λ over $[1,+\infty)$ to obtain desired inequality. We have the following equation for left hand side of inequality

$$\begin{aligned} &\int_1^{+\infty} \frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda + \int_{\frac{b}{a}}^{+\infty} \frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\, a}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} f(\lambda a)\, d\lambda + \int_{\frac{b}{a}}^{+\infty} 0 f(b)\, d\lambda \\ &= \int_a^b \frac{g'(t)}{\left[g(b)-g(t)\right]^{1-\alpha}} f(t)\, dt \\ &= \Gamma(\alpha) I^{\alpha}_{a^+;g} f(b) . \end{aligned}$$

For the right hand side of inequality, we have to consider two cases:
Firstly, it can be $\frac{b}{a} \le \frac{f(b)}{f(a)}$. In this case, for the right hand side we obtain that

$$\begin{aligned} &\int_1^{+\infty} \frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\, a}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} \lambda f(a)\, d\lambda \\ &= af(a) \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\, \lambda}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} d\lambda . \end{aligned}$$

Hence,

$$\begin{aligned} \int_1^{+\infty} \frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda &\le \int_1^{+\infty} \frac{g'(\min\{\lambda a,b\})}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\ \Gamma(\alpha) I^{\alpha}_{a^+;g} f(b) &\le af(a) \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\, \lambda}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} d\lambda \\ I^{\alpha}_{a^+;g} f(b) &\le \frac{af(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\, \lambda}{\left[g(b)-g(\lambda a)\right]^{1-\alpha}} d\lambda . \end{aligned}$$

When we handle the second case $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$, we meet following;

$$\begin{aligned}
&\int_1^{+\infty} \frac{g'(\min\{\lambda a, b\})}{[g(b) - g(\lambda a)]^{1-\alpha}} \min\{\lambda f(a), f(b)\} d\lambda \\
&= \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a) a}{[g(b) - g(\lambda a)]^{1-\alpha}} \lambda f(a) d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{g'(\lambda a) a}{[g(b) - g(\lambda a)]^{1-\alpha}} f(b) d\lambda \\
&= a f(a) \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a) \lambda}{[g(b) - g(\lambda a)]^{1-\alpha}} d\lambda + \frac{f(b)\left[g(b) - g\left(\frac{a f(b)}{f(a)}\right)\right]^{\alpha}}{\alpha} .
\end{aligned}$$

Thus, we have that

$$I^{\alpha}_{a^+;g} f(b) \leq \frac{a f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a) \lambda}{[g(b) - g(\lambda a)]^{1-\alpha}} d\lambda + \frac{f(b)\left[g(b) - g\left(\frac{a f(b)}{f(a)}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} .$$

□

Theorem 5.2. *Let $\alpha > 0$ and $0 < a < b < +\infty$, $g : [a, b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b). If $f : [a, b] \to \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function and $f \in L_1[a, b]$, then the following inequality for fractional integrals holds:*

$$I^{\alpha}_{b^-;g} f(a) \leq \begin{cases} \frac{a f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{a f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda + \frac{f(b)\left[(g(b) - g(a))^{\alpha} - \left(g\left(\frac{a f(b)}{f(a)}\right) - g(a)\right)^{\alpha}\right]}{\Gamma(\alpha+1)}, & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} . \end{cases} \tag{14}$$

Proof. For the $\mathbb{B}^{-1}$-convex function $f : [a, b] \to \mathbb{R}_{++}$, we have the following inequality

$$f(\min\{\lambda a, b\}) \leq \min\{\lambda f(a), f(b)\} \tag{15}$$

for all $\lambda \in [1, +\infty)$. To obtain the inequality (14), we should multiply both sides of (15) by $\frac{g'(\min\{\lambda a, b\})}{[g(\min\{\lambda a, b\}) - g(a)]^{1-\alpha}}$ and integrate the resulting inequality with respect to λ over $[1, +\infty)$. Thus, for the left hand side of inequality we obtain that

$$\begin{aligned}
&\int_1^{+\infty} \frac{g'(\min\{\lambda a, b\})}{[g(\min\{\lambda a, b\}) - g(a)]^{1-\alpha}} f(\min\{\lambda a, b\}) d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{g'(\lambda a)(\lambda a)'}{[g(\lambda a) - g(a)]^{1-\alpha}} f(\lambda a) d\lambda + \int_{\frac{b}{a}}^{+\infty} \frac{g'(b) b'}{[g(b) - g(a)]^{1-\alpha}} f(b) d\lambda \\
&= \int_a^b \frac{g'(t)}{[g(t) - g(a)]^{1-\alpha}} f(t) dt = \Gamma(\alpha) I^{\alpha}_{b^-;g} f(a) .
\end{aligned}$$

Additionally, for the right hand side of inequality, we meet two possibilities. One of these is possibility of $\frac{b}{a} \leq \frac{f(b)}{f(a)}$. In this case, the equality is

$$\begin{aligned}
\int_1^{+\infty} \frac{g'(\min\{\lambda a, b\})}{[g(\min\{\lambda a, b\}) - g(a)]^{1-\alpha}} \min\{\lambda f(a), f(b)\} d\lambda &= \int_1^{\frac{b}{a}} \frac{g'(\lambda a)}{[g(\lambda a) - g(a)]^{1-\alpha}} \lambda f(a) d\lambda \\
&= a f(a) \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda .
\end{aligned}$$

Then we deduce that

$$\Gamma(\alpha) I^{\alpha}_{b^-;g} f(a) \leq a f(a) \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda$$

$$I^{\alpha}_{b^-;g} f(a) \leq \frac{a f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda \,. \tag{16}$$

In the possibility of $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$, we have to analyse the following:

$$\begin{aligned} &\int_1^{+\infty} \frac{g'(\min\{\lambda a, b\})}{[g(\min\{\lambda a, b\}) - g(a)]^{1-\alpha}} \min\{\lambda f(a), f(b)\} d\lambda \\ &= \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a) a}{[g(\lambda a) - g(a)]^{1-\alpha}} \lambda f(a) d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{g'(\lambda a) a}{[g(\lambda a) - g(a)]^{1-\alpha}} f(b) d\lambda \\ &= a f(a) \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda + \frac{f(b)\left[(g(b) - g(a))^{\alpha} - \left(g\left(\frac{a f(b)}{f(a)}\right) - g(a)\right)^{\alpha}\right]}{\alpha} \,. \end{aligned}$$

Therefore, we deduce that

$$\Gamma(\alpha) I^{\alpha}_{b^-;g} f(a) \leq a f(a) \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda + \frac{f(b)\left[(g(b) - g(a))^{\alpha} - \left(g\left(\frac{a f(b)}{f(a)}\right) - g(a)\right)^{\alpha}\right]}{\alpha}$$

$$I^{\alpha}_{b^-;g} f(a) \leq \frac{a f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \frac{g'(\lambda a)\lambda}{[g(\lambda a) - g(a)]^{1-\alpha}} d\lambda + \frac{f(b)\left[(g(b) - g(a))^{\alpha} - \left(g\left(\frac{a f(b)}{f(a)}\right) - g(a)\right)^{\alpha}\right]}{\Gamma(\alpha + 1)} \,. \tag{17}$$

As a result, from (16) and (17), we have the inequality (14).
□

Theorem 5.1 and Theorem 5.2 which are proven above are the most general Hermite-Hadamard inequalities for $\mathbb{B}^{-1}$-convex functions.

Corollary 5.3. *The inequalities (9) and (10) can be obtained the inequalities (13) and (14), respectively.*

Actually, if we observe that for $g(x) = x$, the fractional integral (5) reduces to the left-sided Riemann-Liouville fractional integral (1), and the fractional integral (6) reduces to the right-sided Riemann-Liouville fractional integral (2) in general.

Additionally, this hypothesis are valid for our results. Namely, if we get $g(x) = x$ in (13), the inequality return to (9). Similarly, getting $g(x) = x$ in (14), it gives (10).

Corollary 5.4. *Hermite-Hadamard inequality via generalized fractional integral operator for $\mathbb{B}^{-1}$-convex function is generalized form of the Hermite-Hadamard inequality involving Hadamard fractional integral.*

The conclusion can be proven by using the same method in Corollary 5.3. For this, observe that for $g(x) = \ln x$, the fractional integral (5) reduces to the left-sided Hadamard fractional integral (3), and the fractional integral (6) reduces to the right-sided Hadamard fractional integral (4).

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