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The Extremal Ranks and Inertias of Matrix Expressions with Respect to Generalized Reflexive and Anti-Reflexive Matrices with Applications

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Abstract. The extremal ranks of matrix expressions of $A - BXC$ and $D - EYE^*$, and the extremal inertias of $D - EYE^*$ are discussed, where X and Y are reflexive (or anti-reflexive) and Hermitian reflexive (or anti-reflexive) matrices respectively. For the applications, we derive the extremal ranks of the reflexive and anti-reflexive solutions to $AX = B$. In addition, we also establish some conditions for the existence of common reflexive and anti-reflexive solutions to $AX = B$ and $CXD = E$, and conditions for the solvability of some matrix equations and matrix inequalities.

1. Introduction

Let $\mathbb{C}^{m\times n}$ and $\mathbb{C}_H^{m\times m}$ denote the set of all $m \times n$ matrices and $m \times m$ Hermitian matrices respectively. For $A \in \mathbb{C}^{m\times n}$, its rank, conjugate transpose and Moore-Penrose inverse will be denoted by $r(A)$, A^* and $A^\dagger$ respectively. For Hermitian matrix A, its positive and negative index of inertia are symbolled by $i_+(A)$ and $i_-(A)$ respectively, and $A \geqslant 0$ (or $A \leqslant 0$) means that A is a nonnegative-definite (or non-positive) matrix. I_n represents the identity matrix of size n. For convenience, we denote $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ and $P^2 = I$. Chen [1] defined two subspaces of matrix:

$$\mathbb{C}_r^{m\times n}(P,Q) = \{A \in \mathbb{C}^{m\times n} : A = PAQ\}, \quad \mathbb{C}_a^{m\times n}(P,Q) = \{A \in \mathbb{C}^{m\times n} : A = -PAQ\},$$

where P, Q are generalize reflection matrices of size m and n, respectively. In addition, the following symbols are also needed,

$$\mathbb{HC}_r^{m\times m}(P) = \left\{A \in \mathbb{C}_H^{m\times m} : A = PAP\right\}, \quad \mathbb{HC}_a^{m\times m}(P) = \left\{A \in \mathbb{C}_H^{m\times m} : A = -PAP\right\},$$

The matrices $A \in \mathbb{C}_r^{m\times n}(P,Q)$, $B \in \mathbb{C}_a^{m\times n}(P,Q)$ are said to be (P,Q) generalized reflexive and (P,Q) generalized anti-reflexive matrices respectively with respect to the generalized reflection matrix dual (P,Q). The matrices

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$C \in \mathbb{HC}_r^{m\times m}(P)$, $D \in \mathbb{HC}_a^{m\times m}(P)$ are said to be Hermitian reflexive and Hermitian anti-reflexive matrices respectively with respect to the generalized reflection matrix P. The (P, Q) generalized reflexive and anti-reflexive matrices have applications in system and control theory, in engineering, in scientific computations and various other fields (see [1]).

In this article, we consider the following matrix expressions

$$A - BXC, \tag{1}$$
$$D - EYE^*, \tag{2}$$

where $A \in \mathbb{C}^{k\times l}$, $B \in \mathbb{C}^{k\times m}$, $C \in \mathbb{C}^{n\times l}$, $D \in \mathbb{C}_H^{n\times n}$, $E \in \mathbb{C}^{n\times m}$ are given, and $X \in \mathbb{C}_r^{m\times n}(P, Q)$ (or $X \in \mathbb{C}_a^{m\times n}(P, Q)$) and $Y \in \mathbb{HC}_r^{m\times m}(P)$ (or $Y \in \mathbb{HC}_a^{m\times m}(P)$) are variable.

The rank and inertia of a matrix are two basic concepts in matrix theory, and have important applications in matrix theory and its applications. For example, matrix equation $BXC = A$ is consistent if and only if the minimal rank of $A - BXC$ with respect X equals 0; matrix inequality $EYE^* \geqslant D$ has a Hermitian solution if and only if the minimal positive inertia index of $D - EYE^*$ with respect Hermitian matrix Y equals 0; in addition, they are also applied to discuss the properties of solutions for matrix equations. Some previous systematical researches on ranks and inertias of linear matrix functions with respect to variable matrix or Hermitian matrix and their applications can be found in [2-16].

In [18, 19], the authors derived some conditions for the existence of reflexive and anti-reflexive solutions to $AXB = C$ by using matrix decomposition. However, the conditions seem to be complicated. In order to investigate some new conditions, and considering the applications of the rank and inertia of matrix expressions in determining the consistency of matrix equations, so, firstly, we will study the extremal ranks and inertias of matrix expressions (1) and (2).

To the best of our knowledge, there is no article yet discussing the ranks and inertias of a linear matrix functions with respect to generalized reflexive and anti-reflexive matrices. This paper is organized as follows. In section 2, we give a group of closed-form formulas for the extremal ranks of $A - BXC$. In section 3, we give the formulas for the extremal ranks and inertias of $D - EYE^*$. For applications, in section 4, some properties on the extremal ranks of the reflexive and anti-reflexive solutions to $AX = B$ are established, and conditions for the existence of common reflexive and anti-reflexive solutions to $AX = B$ and $CXD = E$ are also provided.

Before proceeding to the next sections, we first introduce the following results which will come in handy in the proofs of our theorems.

Lemma 1.1. ([20]) *Let P, Q be generalize reflection matrices of size m and n respectively, and $A \in \mathbb{C}_r^{m\times n}(P, Q)$, $B \in \mathbb{C}_a^{m\times n}(P, Q)$. Then*

$$P = U\begin{pmatrix} I_l & 0 \\ 0 & -I_{m-l} \end{pmatrix}U^*, \quad Q = V\begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}V^*, \quad A = U\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}V^*, \quad B = U\begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}V^*,$$

where U, V are unitary matrices, and $A_1 \in \mathbb{C}^{l\times k}$, $A_2 \in \mathbb{C}^{(m-l)\times(n-k)}$, $B_1 \in \mathbb{C}^{l\times(n-k)}$, $B_2 \in \mathbb{C}^{(m-l)\times k}$.

Lemma 1.2. ([2]) *Let $A \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $B_1 \in \mathbb{C}^{m\times p_2}$, $C_1 \in \mathbb{C}^{q_1\times n}$ and $C_1 \in \mathbb{C}^{q_2\times n}$ be given, $X_1 \in \mathbb{C}^{p_1\times q_1}$ and $X_2 \in \mathbb{C}^{p_2\times q_2}$ be variable. Then*

$$\max_{X_1, X_2} r[A - B_1X_1C_1 - B_2X_2C_2] = \min\left\{ r\begin{pmatrix} A & B_1 & B_2 \end{pmatrix}, \ r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix}, \ r\begin{pmatrix} A & B_1 \\ C_2 & 0 \end{pmatrix}, \ r\begin{pmatrix} A & B_2 \\ C_1 & 0 \end{pmatrix} \right\},$$

$$\min_{X_1,X_2} r[A - B_1X_1C_1 - B_2X_2C_2] = r\begin{pmatrix} A & B_1 & B_2 \end{pmatrix} + r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix} + \max\left\{ r\begin{pmatrix} A & B_1 \\ C_2 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix}, r\begin{pmatrix} A & B_2 \\ C_1 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix} \right\}.$$

Lemma 1.3. ([3]) *Let $A \in \mathbb{C}_H^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given. Then*

$$\max_{X\in\mathbb{C}^{n\times p}} r[A - BXC - (BXC)^*] = \min\left\{ r\begin{pmatrix} A & B & C^* \end{pmatrix}, r\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}, r\begin{pmatrix} A & C^* \\ C & 0 \end{pmatrix} \right\},$$

$$\min_{X\in\mathbb{C}^{n\times p}} r[A - BXC - (BXC)^*] = 2r\begin{pmatrix} A & B & C^* \end{pmatrix} + \max\{r(M_1) - 2r(N_1),\ r(M_2) - 2r(N_2), s_+ + t_-, s_- + t_+\},$$

$$\max_{X\in\mathbb{C}^{n\times p}} i_\pm[A - BXC - (BXC)^*] = \min\{i_\pm(M_1),\ i_\pm(M_2)\},$$

$$\min_{X\in\mathbb{C}^{n\times p}} i_\pm[A - BXC - (BXC)^*] = r\begin{pmatrix} A & B & C^* \end{pmatrix} + \max\{i_\pm(M_1) - r(N_1),\ i_\pm(M_2) - r(N_2)\},$$

where $s_\pm = i_\pm(M_1) - r(N_1)$, $t_\pm = i_\pm(M_2) - r(N_2)$, and

$$M_1 = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} A & C^* \\ C & 0 \end{pmatrix},\ N_1 = \begin{pmatrix} A & B & C^* \\ B^* & 0 & 0 \end{pmatrix},\ N_2 = \begin{pmatrix} A & B & C^* \\ C & 0 & 0 \end{pmatrix}.$$

Lemma 1.4. ([4, 11]) *Let $A \in \mathbb{C}_H^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{m\times k}$ be given. Then*

$$\max_{X=X^*,Y=Y^*} r[A - BXB^* - CYC^*] = \min\left\{ r\begin{pmatrix} A & B & C \end{pmatrix}, r\begin{pmatrix} A & B \\ C^* & 0 \end{pmatrix} \right\},$$

$$\min_{X=X^*,Y=Y^*} r[A - BXB^* - CYC^*] = 2r\begin{pmatrix} A & B & C \end{pmatrix} + r\begin{pmatrix} A & B \\ C^* & 0 \end{pmatrix} - r\begin{pmatrix} A & B & C \\ B^* & 0 & 0 \end{pmatrix} - r\begin{pmatrix} A & B & C \\ C^* & 0 & 0 \end{pmatrix}.$$

Lemma 1.5. ([5]) *Let $A \in \mathbb{C}_H^{m\times m}$, $B_i \in \mathbb{C}^{m\times n_i}$ be given, and $X_i \in \mathbb{C}_H^{n_i\times n_i}$, $i = 1, \ldots \cdots, k$, be variable, denote $p(X_1, \cdots, X_k) = A - B_1X_1B_1^* - \cdots - B_kX_kB_k^*$. Then*

$$\max_{X_i\in\mathbb{C}_H^{n_i\times n_i}} i_\pm p(X_1, \cdots, X_k) = i_\pm(M),$$

$$\min_{X_i\in\mathbb{C}_H^{n_i\times n_i}} i_\pm p(X_1, \cdots, X_k) = r\begin{pmatrix} A & B \end{pmatrix} - i_\mp(M),$$

where $B = \begin{pmatrix} B_1 & \cdots & B_k \end{pmatrix}$ and $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$.

Lemma 1.6. ([5]) *Let $A \in \mathbb{C}_H^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and denote $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$. Then*

$$i_\pm(M) = r(B) + i_\pm(E_BAE_B).$$

Lemma 1.7. ([5]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then*

$$r\begin{pmatrix} A & B \end{pmatrix} = r(A) + r(E_AB),\ \ r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A),\ \ r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = r(B) + r(C) + r(E_BAF_C).$$

2. The extremal ranks of $A - BXC$

In this section, our purpose is to derive the formulae for the extremal ranks of $A - BXC$ with respect to reflexive and anti-reflexive matrices X. Moreover, we present some conditions for the existence of reflexive and anti-reflexive solutions to matrix equation $AXB = C$.

Theorem 2.1. *Let $A \in \mathbb{C}^{k\times l}$, $B \in \mathbb{C}^{k\times m}$, $C \in \mathbb{C}^{n\times l}$ be given, and $X \in \mathbb{C}_r^{m\times n}(P,Q)$. Then*

$$\begin{aligned}\max_{X\in\mathbb{C}_r^{m\times n}(P,Q)} r(A-BXC) &= \min\left\{ r\begin{pmatrix} A & B \end{pmatrix},\ r\begin{pmatrix} A \\ C \end{pmatrix},\ r\begin{pmatrix} A & B(I_m+P) \\ (I_n-Q)C & 0 \end{pmatrix},\right.\\ &\qquad \left. r\begin{pmatrix} A & B(I_m-P) \\ (I_n+Q)C & 0 \end{pmatrix}\right\},\end{aligned} \tag{3}$$

$$\begin{aligned}&\min_{X\in\mathbb{C}_r^{m\times n}(P,Q)} r(A-BXC) = r\begin{pmatrix} A & B \end{pmatrix} + r\begin{pmatrix} A \\ C \end{pmatrix}\\ &+\max\left\{ r\begin{pmatrix} A & B(I_m+P) \\ (I_n-Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B \\ (I_n-Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B(I_m+P) \\ C & 0 \end{pmatrix},\right.\\ &\left. r\begin{pmatrix} A & B(I_m-P) \\ (I_n+Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B \\ (I_n+Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B(I_m-P) \\ C & 0 \end{pmatrix}\right\}.\end{aligned} \tag{4}$$

Proof. It follows from Lemma 1.1 that X can be written as

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}V^*.$$

Denote $BU = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$ and $V^*C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$. Then

$$A - BXC = A - B_1X_1C_1 - B_2X_2C_2.$$

In view of Lemma 1.2, we have

$$\begin{aligned}&\max_{X\in\mathbb{C}_r^{m\times n}(P,Q)} r(A-BXC)\\ =\ &\min\left\{ r\begin{pmatrix} A & B_1 & B_2 \end{pmatrix},\ r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix},\ r\begin{pmatrix} A & B_1 \\ C_2 & 0 \end{pmatrix},\ r\begin{pmatrix} A & B_2 \\ C_1 & 0 \end{pmatrix}\right\},\end{aligned} \tag{5}$$

$$\begin{aligned}&\min_{X\in\mathbb{C}_r^{m\times n}(P,Q)} r(A-BXC) = r\begin{pmatrix} A & B_1 & B_2 \end{pmatrix} + r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix}\\ &+\max\left\{ r\begin{pmatrix} A & B_1 \\ C_2 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix},\right.\\ &\left. r\begin{pmatrix} A & B_2 \\ C_1 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{pmatrix} - r\begin{pmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix}\right\}.\end{aligned} \tag{6}$$

It follows from Lemma 1.1 that

$$\begin{pmatrix} B_1 & 0 \end{pmatrix} = BU \times \frac{I_m + U^*PU}{2} = \frac{1}{2}B(I_m+P)U,\ \begin{pmatrix} 0 & B_2 \end{pmatrix} = \frac{1}{2}B(I_m-P)U,$$

$$\begin{pmatrix} C_1 \\ 0 \end{pmatrix} = \frac{1}{2}V^*(I_n+Q)C, \ \begin{pmatrix} 0 \\ C_2 \end{pmatrix} = \frac{1}{2}V^*(I_n-Q)C,$$

And simple computations show that

$$\begin{aligned}
r\begin{pmatrix} A & B_1 & B_2 \end{pmatrix} &= r\begin{pmatrix} A & BU \end{pmatrix} = r\begin{pmatrix} A & B \end{pmatrix}, \ r\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix}, \\
r\begin{pmatrix} A & B_1 \\ C_2 & 0 \end{pmatrix} &= r\begin{pmatrix} A & B_1 & 0 \\ 0 & 0 & 0 \\ C_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A & \frac{1}{2}B(I_m+P)U \\ \frac{1}{2}V^*(I_n-Q)C & 0 \end{pmatrix} \\
&= r\begin{pmatrix} A & B(I_m+P) \\ (I_n-Q)C & 0 \end{pmatrix}, \\
r\begin{pmatrix} A & B_2 \\ C_1 & 0 \end{pmatrix} &= r\begin{pmatrix} A & B(I_m-P) \\ (I_n+Q)C & 0 \end{pmatrix}, \\
r\begin{pmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix} &= r\begin{pmatrix} A & B(I_m+P) \\ C & 0 \end{pmatrix}, \ r\begin{pmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix} = r\begin{pmatrix} A & B(I_m-P) \\ C & 0 \end{pmatrix}, \\
r\begin{pmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{pmatrix} &= r\begin{pmatrix} A & B \\ (I_n-Q)C & 0 \end{pmatrix}, \ r\begin{pmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A & B \\ (I_n+Q)C & 0 \end{pmatrix}.
\end{aligned}$$

Substituting the above equalities into (5) and (6) yields (3) and (4). □

Similarly, we have the following results.

Theorem 2.2. *Let $A \in \mathbb{C}^{k\times l}$, $B \in \mathbb{C}^{k\times m}$, $C \in \mathbb{C}^{n\times l}$ be given, and $X \in \mathbb{C}_a^{m\times n}(P,Q)$. Then*

$$\begin{aligned}
\max_{X\in\mathbb{C}_a^{m\times n}(P,Q)} r(A-BXC) &= \min\left\{ r\begin{pmatrix} A & B \end{pmatrix}, \ r\begin{pmatrix} A \\ C \end{pmatrix}, \ r\begin{pmatrix} A & B(I_m+P) \\ (I_n+Q)C & 0 \end{pmatrix}, \right. \\
&\left. r\begin{pmatrix} A & B(I_m-P) \\ (I_n-Q)C & 0 \end{pmatrix} \right\},
\end{aligned}$$

$$\begin{aligned}
&\min_{X\in\mathbb{C}_a^{m\times n}(P,Q)} r(A-BXC) = r\begin{pmatrix} A & B \end{pmatrix} + r\begin{pmatrix} A \\ C \end{pmatrix} \\
&+\max\left\{ r\begin{pmatrix} A & B(I_m+P) \\ (I_n+Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B \\ (I_n+Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B(I_m+P) \\ C & 0 \end{pmatrix}, \right. \\
&\left. r\begin{pmatrix} A & B(I_m-P) \\ (I_n-Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B \\ (I_n-Q)C & 0 \end{pmatrix} - r\begin{pmatrix} A & B(I_m-P) \\ C & 0 \end{pmatrix} \right\}.
\end{aligned}$$

In [18, 19], the authors derived conditions for the existence of reflexive and anti-reflexive solutions of the matrix equation $AXB = C$, here, we present some new conditions on this topic. The following results are obtained directly by Theorem 2.1 and Theorem 2.2.

Corollary 2.3. *Let $A \in \mathbb{C}^{k\times m}$, $B \in \mathbb{C}^{n\times l}$ and $C \in \mathbb{C}^{k\times l}$ be given. Then*
(i) Consistent matrix equation $AXB = C$ has a solution $X \in \mathbb{C}_r^{m\times n}(P,Q)$ if and only if

$$r[A(I_m+P)] + r[(I_n-Q)B] = r\begin{pmatrix} C & A(I_m+P) \\ (I_n-Q)B & 0 \end{pmatrix},$$

$$r[A(I_m-P)] + r[(I_n+Q)B] = r\begin{pmatrix} C & A(I_m-P) \\ (I_n+Q)B & 0 \end{pmatrix}.$$

(*ii*) *Consistent matrix equation* $AXB = C$ *has a solution* $X \in \mathbb{C}_a^{m\times n}(P,Q)$ *if and only if*

$$r[A(I_m + P)] + r[(I_n + Q)B] = r\begin{pmatrix} C & A(I_m + P) \\ (I_n + Q)B & 0 \end{pmatrix},$$

$$r[A(I_m - P)] + r[(I_n - Q)B] = r\begin{pmatrix} C & A(I_m - P) \\ (I_n - Q)B & 0 \end{pmatrix}.$$

Proof. It is well known that matrix equation $AXB = C$ is consistent if and only if $r\begin{pmatrix} A & C \end{pmatrix} = r(A)$ and $r\begin{pmatrix} B \\ C \end{pmatrix} = r(B)$. Hence, it follows from Theorem 2.1 that consistent matrix equation $AXB = C$ has a solution $X \in \mathbb{C}_r^{m\times n}(P,Q)$ if and only if

$$\begin{aligned} 0 \quad = \quad \min\Bigg\{ & r[(I_n - Q)B] + r[A(I_m + P)] - r\begin{pmatrix} C & A(I_m + P) \\ (I_n - Q)B & 0 \end{pmatrix}, \\ & r[(I_n + Q)B] + r[A(I_m - P)] - r\begin{pmatrix} C & A(I_m - P) \\ (I_n + Q)B & 0 \end{pmatrix}\Bigg\}. \end{aligned}$$

Actually,

$$r[A(I_m + P)] + r[(I_n - Q)B] \leqslant r\begin{pmatrix} C & A(I_m + P) \\ (I_n - Q)B & 0 \end{pmatrix},$$

and

$$r[A(I_m - P)] + r[(I_n + Q)B] \leqslant r\begin{pmatrix} C & A(I_m - P) \\ (I_n + Q)B & 0 \end{pmatrix}.$$

So, statement (*i*) is obvious. Similarly, we have statement (*ii*). □

Corollary 2.4. ([17]) *Let* $A_1 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{p\times k}$, $A_2 \in \mathbb{C}^{m\times l}$, $B_2 \in \mathbb{C}^{q\times k}$ *and* $C \in \mathbb{C}^{m\times k}$ *be given,* $X_1 \in \mathbb{C}^{n\times p}$, $X_2 \in \mathbb{C}^{l\times q}$ *unknown. Then matrix equation* $A_1X_1B_1 + A_2X_2B_2 = C$ *is solvable if and only if*

$$r\begin{pmatrix} A_1 & C \\ 0 & B_2 \end{pmatrix} = r\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad r\begin{pmatrix} A_2 & C \\ 0 & B_1 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ 0 & B_1 \end{pmatrix},$$

$$r\begin{pmatrix} C & A_1 & A_2 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix}, \quad r\begin{pmatrix} B_1 \\ B_2 \\ C \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Proof. Rewrite $A_1X_1B_1 + A_2X_2B_2 = C$ as $AXB = C$ where $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, and $X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$ is a (P,Q) generalized reflexive matrix, with $P = \begin{pmatrix} I_n & 0 \\ 0 & -I_l \end{pmatrix}$ and $Q = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Then it follows from Corollary 2.3 that this corollary is evident. □

3. The extremal ranks and inertias of $D - EYE^*$

In this section, we derive the formulae for the extremal ranks and inertias of $D - EYE^*$ with respect to Hermitian reflexive and Hermitian anti-reflexive matrices Y, and consider their applications in matrix equation and matrix inequality.

Theorem 3.1. *Let $D \in \mathbb{C}_H^{n\times n}$, $E \in \mathbb{C}^{n\times m}$ be given, and $Y \in \mathbb{HC}_r^{m\times m}(P)$. Then*

$$\begin{aligned}
\max_{Y\in\mathbb{HC}_r^{m\times m}(P)} r(D-EYE^*) &= \min\left\{ r\begin{pmatrix} D & E \end{pmatrix}, r\begin{pmatrix} D & E(I+P) \\ (I-P)E^* & 0 \end{pmatrix} \right\}, \\
\min_{Y\in\mathbb{HC}_r^{m\times m}(P)} r(D-EYE^*) &= 2r\begin{pmatrix} D & E \end{pmatrix} + r\begin{pmatrix} D & E(I+P) \\ (I-P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix}, \\
\max_{Y\in\mathbb{HC}_r^{m\times m}(P)} i_\pm(D-EYE^*) &= i_\pm\begin{pmatrix} D & E \\ E^* & 0 \end{pmatrix}, \\
\min_{Y\in\mathbb{HC}_r^{m\times m}(P)} i_\pm(D-EYE^*) &= r\begin{pmatrix} D & E \end{pmatrix} - i_\mp\begin{pmatrix} D & E \\ E^* & 0 \end{pmatrix}.
\end{aligned}$$

Proof. It follows from Lemma 1.1 that Y can be written as

$$Y = U\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}U^*,$$

where Y_1 and Y_2 are Hermitian matrices.
Denote $EU = \begin{pmatrix} E_1 & E_2 \end{pmatrix}$. Then

$$D - EYE^* = D - E_1Y_1E_1^* - E_2Y_2E_2^*.$$

And,

$$\begin{gathered}
r\begin{pmatrix} D & E_1 & E_2 \end{pmatrix} = r\begin{pmatrix} D & E \end{pmatrix}, \; r\begin{pmatrix} D & E_1 \\ E_2^* & 0 \end{pmatrix} = r\begin{pmatrix} D & E(I+P) \\ (I-P)E^* & 0 \end{pmatrix}, \\
r\begin{pmatrix} D & E_1 & E_2 \\ E_1^* & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix}, \; r\begin{pmatrix} D & E_1 & E_2 \\ E_2^* & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix}, \\
i_\pm\begin{pmatrix} D & E_1 & E_2 \\ E_1^* & 0 & 0 \\ E_2^* & 0 & 0 \end{pmatrix} = i_\pm\begin{pmatrix} D & E \\ E^* & 0 \end{pmatrix}.
\end{gathered}$$

Thus, the conclusions of this theorem are obtained by Lemma 1.4 and Lemma 1.5. □

Theorem 3.2. *Let $D \in \mathbb{C}_H^{n\times n}$, $E \in \mathbb{C}^{n\times m}$ be given, and $Y \in \mathbb{HC}_a^{m\times m}(P)$. Then*

$$\begin{aligned}
\max_{Y\in\mathbb{HC}_a^{m\times m}(P)} r(D-EYE^*) &= \min\left\{ r\begin{pmatrix} D & E \end{pmatrix}, \; r\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix}, \; r\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} \right\}, \\
\min_{Y\in\mathbb{HC}_a^{m\times m}(P)} r(D-EYE^*) &= 2r\begin{pmatrix} D & E \end{pmatrix} + \max\left\{ r\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix} - 2r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix}, \right. \\
&\qquad \left. r\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} - 2r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix}, \; s_+ + t_-, \; s_- + t_+ \right\}, \\
\max_{Y\in\mathbb{HC}_a^{m\times m}(P)} i_\pm(D-EYE^*) &= \min\left\{ i_\pm\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix}, \; i_\pm\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} \right\}, \\
\min_{Y\in\mathbb{HC}_a^{m\times m}(P)} i_\pm(D-EYE^*) &= r\begin{pmatrix} D & E \end{pmatrix} + \max\left\{ i_\pm\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix}, \right. \\
&\qquad \left. i_\pm\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix} \right\},
\end{aligned}$$

where

$$\begin{aligned}
s_\pm &= i_\pm\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix}, \\
t_\pm &= i_\pm\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix}.
\end{aligned}$$

Proof. It follows from Lemma 1.1 that Y can be written as

$$Y = U\begin{pmatrix} 0 & Y_1^* \\ Y_1 & 0 \end{pmatrix}U^*.$$

Denote $EU = \begin{pmatrix} E_1 & E_2 \end{pmatrix}$. Then,

$$D - EYE^* = D - E_2Y_1E_1^* - (E_2Y_1E_1^*)^*.$$

Moreover,

$$\begin{aligned}
r\begin{pmatrix} D & E_2 & E_1 \end{pmatrix} &= r\begin{pmatrix} D & E \end{pmatrix},\ r\begin{pmatrix} D & E_2 \\ E_2^* & 0 \end{pmatrix} = r\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix},\\
r\begin{pmatrix} D & E_1 \\ E_1^* & 0 \end{pmatrix} &= r\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix},\\
i_\pm\begin{pmatrix} D & E_1 \\ E_1^* & 0 \end{pmatrix} &= i_\pm\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix},\\
i_\pm\begin{pmatrix} D & E_2 \\ E_2^* & 0 \end{pmatrix} &= i_\pm\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix},\\
r\begin{pmatrix} D & E_2 & E_1 \\ E_2^* & 0 & 0 \end{pmatrix} &= r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix},\ r\begin{pmatrix} D & E_2 & E_1 \\ E_1^* & 0 & 0 \end{pmatrix} = r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix}.
\end{aligned}$$

In view of Lemma 1.3, these results of this theorem are obvious. □

Corollary 3.3. *Let $D \in \mathbb{C}_H^{n\times n}$, $E \in \mathbb{C}^{n\times m}$ be given. Then*
(*i*) *Matrix equation $EYE^* = D$ has a solution $Y \in \mathbb{HC}_r^{m\times m}(P)$ if and only if*

$$2r\begin{pmatrix} D & E \end{pmatrix} + r\begin{pmatrix} D & E(I+P) \\ (I-P)E^* & 0 \end{pmatrix} = r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix} + r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix}.$$

(*ii*) *Matrix equation $EYE^* = D$ has a solution $Y \in \mathbb{HC}_a^{m\times m}(P)$ if and only if*

$$\begin{aligned}
2r\begin{pmatrix} D & E \end{pmatrix} + \max\Bigg\{ & r\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix} - 2r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix},\\
& r\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} - 2r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix},\ s_+ + t_-,\ s_- + t_+\Bigg\} = 0,
\end{aligned}$$

where $s_\pm$ and $t_\pm$ are given by Theorem 3.2.
(*iii*) *Matrix inequality $EYE^* \geqslant D$ has a solution $Y \in \mathbb{HC}_r^{m\times m}(P)$ if and only if*

$$r\begin{pmatrix} D & E \end{pmatrix} = i_-\begin{pmatrix} D & E \\ E^* & 0 \end{pmatrix}.$$

(*iv*) *Matrix inequality $EYE^* \geqslant D$ has a solution $Y \in \mathbb{HC}_a^{m\times m}(P)$ if and only if*

$$\begin{aligned}
r\begin{pmatrix} D & E \end{pmatrix} + \max\Bigg\{ & i_+\begin{pmatrix} D & E(I-P) \\ (I-P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I-P)E^* & 0 \end{pmatrix},\\
& i_+\begin{pmatrix} D & E(I+P) \\ (I+P)E^* & 0 \end{pmatrix} - r\begin{pmatrix} D & E \\ (I+P)E^* & 0 \end{pmatrix}\Bigg\} = 0.
\end{aligned}$$

Corollary 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^{m\times m}$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. Then the following statements are equivalent:*
(i) Matrix inequality $AXB + (AXB)^* \geqslant C$ is solvable;
(ii)

$$r\begin{pmatrix} C & A & B^* \end{pmatrix} + \max\left\{ i_+\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix},\ i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix} \right\} = 0;$$

(iii) $E_ACE_A \leqslant 0$, $F_BCF_B \leqslant 0$, and

$$\begin{aligned} r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(A) &= r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, \\ r\begin{pmatrix} C & A & B^* \end{pmatrix} + r(B) &= r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix}; \end{aligned}$$

(iv) $r(E_GCE_A) = r(E_GCF_B) = r(E_GC)$, $E_ACE_A \leqslant 0$ and $F_BCF_B \leqslant 0$.

Proof. Rewrite $AXB + (AXB)^* \geqslant C$ as $GYG^* \geqslant C$, where $Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ is a Hermitian anti-reflexive matrix with respect to the generalized reflection matrix $P = \begin{pmatrix} I_n & 0 \\ 0 & -I_p \end{pmatrix}$. Then the equivalence of statements (*i*) and (*iii*) is followed by Corollary 3.3-(*iv*), and the equivalence of statements (*ii*) – (*iv*) can be proved by Lemmas 1.6, 1.7. □

The following results follow from Corollary 3.3.

Corollary 3.5. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}_H^{m\times m}$ be given, $X \in \mathbb{C}_H^{n\times n}$ and $Y \in \mathbb{C}_H^{k\times k}$ be variable matrices. Then*
(i) Matrix equation $AXA^* + BYB^* = C$ is solvable if and only if

$$2r\begin{pmatrix} C & A & B \end{pmatrix} + r\begin{pmatrix} C & A \\ B^* & 0 \end{pmatrix} = r\begin{pmatrix} C & A & B \\ A^* & 0 & 0 \end{pmatrix} + r\begin{pmatrix} C & A & B \\ B^* & 0 & 0 \end{pmatrix}.$$

(ii) Matrix inequality $AXA^* + BYB^* \geqslant C$ is solvable if and only if

$$r\begin{pmatrix} C & A & B \end{pmatrix} = i_-\begin{pmatrix} C & A & B \\ A^* & 0 & 0 \\ B^* & 0 & 0 \end{pmatrix}.$$

4. Properties of the reflexive and anti-reflexive solutions to $AX = B$

For convenience, the following notations will be used in this section. For generalized reflection matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$, matrices $A \in \mathbb{C}^{k\times m}$ and $B \in \mathbb{C}^{k\times n}$, we set

$$A_1(P) = \begin{pmatrix} A(I+P) \\ A(I-P) \end{pmatrix},\ A_2(P) = \begin{pmatrix} A(I-P) \\ A(I+P) \end{pmatrix},$$

and denote $[A_i(P)]^*$ and $[A_i(P)]^\dagger$ by $A_i^*(P)$ and $A_i^\dagger(P)$ $(i = 1, 2)$ for short respectively. Further define

$$S = \left\{ X \in \mathbb{C}_r^{m\times n}(P,Q) \middle| AX = B \right\},\quad T = \left\{ X \in \mathbb{C}_a^{m\times n}(P,Q) \middle| AX = B \right\}. \tag{7}$$

Lemma 4.1. ([21])*Let $A \in \mathbb{C}^{k\times m}$ and $B \in \mathbb{C}^{k\times n}$ be given. Then*
(i) $AX = B$ has a solution $X \in \mathbb{C}_r^{m\times n}(P,Q)$ if and only if $A_1(P)A_1^\dagger(P)B_1(Q) = B_1(Q)$. In this case, a general solution X can be written as

$$X = A_1^\dagger(P)B_1(Q) + F_{A_1(P)}V,$$

where $V \in \mathbb{C}_r^{m\times n}(P,Q)$ *is arbitrary.*
(ii) $AX = B$ *has a solution* $X \in \mathbb{C}_a^{m\times n}(P,Q)$ *if and only if* $A_1(P)A_1^{\dagger}(P)B_2(Q) = B_2(Q)$*. In this case, a general solution* X *can be written as*

$$X = A_1^{\dagger}(P)B_2(Q) + F_{A_1(P)}W,$$

where $W \in \mathbb{C}_a^{m\times n}(P,Q)$ *is arbitrary.*

Note that condition $A_1(P)A_1^{\dagger}(P)B_1(Q) = B_1(Q)$ is equivalent to $r\begin{pmatrix} A_1(P) & B_1(Q) \end{pmatrix} = r[A_1(P)]$, i.e. $r\begin{pmatrix} A & B \\ AP & BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}$. Similarly, $A_1(P)A_1^{\dagger}(P)B_2(Q) = B_2(Q)$ is equivalent to $r\begin{pmatrix} A & B \\ AP & -BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}$.

Theorem 4.2. *Let* $A \in \mathbb{C}^{k\times m}$*,* $B \in \mathbb{C}^{k\times n}$*, and* S, T *be defined by* (7)*. Then*

$$\begin{aligned}\max_{X\in S} r(X) &= \min\left\{m + r\begin{pmatrix} B \\ BQ \end{pmatrix},\ n + r\begin{pmatrix} A \\ AP \end{pmatrix},\ r\begin{pmatrix} B+BQ \\ I_n - Q \end{pmatrix} + r\begin{pmatrix} A-AP \\ I_m + P \end{pmatrix},\right. \\ &\qquad \left. r\begin{pmatrix} B-BQ \\ I_n + Q \end{pmatrix} + r\begin{pmatrix} A+AP \\ I_m - P \end{pmatrix}\right\} - r\begin{pmatrix} A \\ AP \end{pmatrix}; \end{aligned} \tag{8}$$

$$\min_{X\in S} r(X) = r\begin{pmatrix} B \\ BQ \end{pmatrix}; \tag{9}$$

$$\begin{aligned}\max_{X\in T} r(X) &= \min\left\{m + r\begin{pmatrix} B \\ BQ \end{pmatrix},\ n + r\begin{pmatrix} A \\ AP \end{pmatrix},\ r\begin{pmatrix} B-BQ \\ I_n + Q \end{pmatrix} + r\begin{pmatrix} A-AP \\ I_m + P \end{pmatrix},\right. \\ &\qquad \left. r\begin{pmatrix} B+BQ \\ I_n - Q \end{pmatrix} + r\begin{pmatrix} A+AP \\ I_m - P \end{pmatrix}\right\} - r\begin{pmatrix} A \\ AP \end{pmatrix}; \end{aligned} \tag{10}$$

$$\min_{X\in T} r(X) = r\begin{pmatrix} B \\ BQ \end{pmatrix}. \tag{11}$$

Proof. In view of Lemma 4.1 and Theorem 2.1, we have

$$\begin{aligned}\max_{X\in S} r(X) &= \max_{V\in\mathbb{C}_r^{m\times n}(P,Q)} r[-A_1^{\dagger}(P)B_1(Q) - F_{A_1(P)}V] \\ &= \min\left\{ r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) & F_{A_1(P)} \end{pmatrix},\ r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) \\ I_n \end{pmatrix},\right. \\ &\qquad \left. r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) & F_{A_1(P)}(I_m+P) \\ I_n - Q & 0 \end{pmatrix},\ r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) & F_{A_1(P)}(I_m-P) \\ I_n + Q & 0 \end{pmatrix}\right\}. \end{aligned}$$

Recall the fact that if $M^*N = 0$, then $r\begin{pmatrix} M & N \end{pmatrix} = r(M) + r(N)$, and together with Lemma 1.7, we obtain

$$\begin{aligned} r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) & F_{A_1(P)} \end{pmatrix} &= r[A_1^{\dagger}(P)B_1(Q)] + r[F_{A_1(P)}] \\ &= r[B_1(Q)] + m - r[A_1(P)] \\ &= m + r\begin{pmatrix} B \\ BQ \end{pmatrix} - r\begin{pmatrix} A \\ AP \end{pmatrix}, \\ r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) & F_{A_1(P)}(I_m+P) \\ I_n - Q & 0 \end{pmatrix} &= r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) \\ I_n - Q \end{pmatrix} + r\begin{pmatrix} F_{A_1(P)}(I_m+P) \\ 0 \end{pmatrix} \\ &= r\begin{pmatrix} B_1(Q) \\ I_n - Q \end{pmatrix} + r\begin{pmatrix} A_1^*(P) & I_m + P \end{pmatrix} - r[A_1(P)] \\ &= r\begin{pmatrix} B+BQ \\ I_n - Q \end{pmatrix} + r\begin{pmatrix} A-AP \\ I_m + P \end{pmatrix} - r\begin{pmatrix} A \\ AP \end{pmatrix}, \\ r\begin{pmatrix} -A_1^{\dagger}(P)B_1(Q) & F_{A_1(P)}(I_m-P) \\ I_n + Q & 0 \end{pmatrix} &= r\begin{pmatrix} B-BQ \\ I_n + Q \end{pmatrix} + r\begin{pmatrix} A+AP \\ I_m - P \end{pmatrix} - r\begin{pmatrix} A \\ AP \end{pmatrix}. \end{aligned}$$

In view of the above equalities, then (8) is evident. Similarly, (9)-(11) can be derived. □

Theorem 4.3. *Let $A \in \mathbb{C}^{k\times m}$, $B \in \mathbb{C}^{k\times n}$, $C \in \mathbb{C}^{s\times m}$, $D \in \mathbb{C}^{n\times l}$, and $E \in \mathbb{C}^{s\times l}$ be given. Then*
(i) Consistent equations $AX = B$ and $CXD = E$ have a common solution $X \in \mathbb{C}_r^{m\times n}(P,Q)$ if and only if $r\begin{pmatrix} A & B \\ AP & BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}$, and

$$r[(I_n - Q)D] + r\begin{pmatrix} C(I_m + P) \\ A(I_m + P) \end{pmatrix} = r\begin{pmatrix} E & C(I_m + P) \\ BD & A(I_m + P) \\ (I_n - Q)D & 0 \end{pmatrix}, \tag{12}$$

$$r[(I_n + Q)D] + r\begin{pmatrix} C(I_m - P) \\ A(I_m - P) \end{pmatrix} = r\begin{pmatrix} E & C(I_m - P) \\ BD & A(I_m - P) \\ (I_n + Q)D & 0 \end{pmatrix}. \tag{13}$$

(ii) Consistent equations $AX = B$ and $CXD = E$ have a common solution $X \in \mathbb{C}_a^{m\times n}(P,Q)$ if and only if $r\begin{pmatrix} A & B \\ AP & -BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}$, and

$$r[(I_n + Q)D] + r\begin{pmatrix} C(I_m + P) \\ A(I_m + P) \end{pmatrix} = r\begin{pmatrix} E & C(I_m + P) \\ BD & A(I_m + P) \\ (I_n + Q)D & 0 \end{pmatrix},$$

$$r[(I_n - Q)D] + r\begin{pmatrix} C(I_m - P) \\ A(I_m - P) \end{pmatrix} = r\begin{pmatrix} E & C(I_m - P) \\ BD & A(I_m - P) \\ (I_n - Q)D & 0 \end{pmatrix}.$$

Proof. Since $AX = B$ has a solution $X \in \mathbb{C}_r^{m\times n}(P,Q)$ if and only if $r\begin{pmatrix} A & B \\ AP & BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}$. In this case, a general solution X can be written as

$$X = A_1^\dagger(P)B_1(Q) + F_{A_1(P)}V,$$

where $V \in \mathbb{C}_r^{m\times n}(P,Q)$ is arbitrary. Substituting the above X into $CXD = E$ yields

$$CF_{A_1(P)}VD = E - CA_1^\dagger(P)B_1(Q)D. \tag{14}$$

Since $CXD = E$ is consistent, then $CF_{A_1(P)}YD = E - CA_1^\dagger(P)B_1(Q)D$ is also consistent for general solution Y. It follows from [Lemma 2.1, 21] that,

$$\begin{aligned} A_1^\dagger(P) &= \begin{pmatrix} [A(I_m + P)]^\dagger & [A(I_m - P)]^\dagger \end{pmatrix}, \\ F_{A_1(P)} &= I_m - [A(I_m + P)]^\dagger A(I_m + P) - [A(I_m - P)]^\dagger A(I_m - P). \end{aligned}$$

Hence, $F_{A_1(P)}$ are commutative with $I_m + P$ and $I_m - P$ respectively. Thus, by Corollary 2.3 and Lemma 1.6, simple computations show that

$$\begin{aligned} r[CF_{A_1(P)}(I_m + P)] &= r[C(I_m + P)F_{A_1(P)}] = r[C(I_m + P)F_{A(I_m+P)}] \\ &= r\begin{pmatrix} C(I_m + P) \\ A(I_m + P) \end{pmatrix} - r[A(I_m + P)], \\ r[CF_{A_1(P)}(I_m - P)] &= r\begin{pmatrix} C(I_m - P) \\ A(I_m - P) \end{pmatrix} - r[A(I_m - P)], \end{aligned}$$

and

$$\begin{aligned}
& r\begin{pmatrix} E - CA_1^{\dagger}(P)B_1(Q)D & CF_{A_1(P)}(I_m+P) \\ (I_n-Q)D & 0 \end{pmatrix} \\
= \; & r\begin{pmatrix} E - CA_1^{\dagger}(P)B_1(Q)D & C(I_m+P)F_{A(I_m+P)} \\ (I_n-Q)D & 0 \end{pmatrix} \\
= \; & r\begin{pmatrix} E - CA_1^{\dagger}(P)B_1(Q)D & C(I_m+P) \\ (I_n-Q)D & 0 \\ 0 & A(I_m+P) \end{pmatrix} - r[A(I_m+P)] \\
= \; & r\begin{pmatrix} \begin{matrix} E - C[A(I_m+P)]^{\dagger}B(I_n+Q)D \\ -C[A(I_m-P)]^{\dagger}B(I_n-Q)D \end{matrix} & C(I_m+P) \\ (I_n-Q)D & 0 \\ 0 & A(I_m+P) \end{pmatrix} - r[A(I_m+P)] \\
= \; & r\begin{pmatrix} E - C[A(I_m+P)]^{\dagger}B(I_n+Q)D & C(I_m+P) \\ (I_n-Q)D & 0 \\ 0 & A(I_m+P) \end{pmatrix} - r[A(I_m+P)] \\
= \; & r\begin{pmatrix} E - \frac{1}{2}C(I_m+P)[A(I_m+P)]^{\dagger}B(I_n+Q)D & C(I_m+P) \\ (I_n-Q)D & 0 \\ 0 & A(I_m+P) \end{pmatrix} - r[A(I_m+P)] \\
= \; & r\begin{pmatrix} E & C(I_m+P) \\ (I_n-Q)D & 0 \\ \frac{1}{2}B(I_n+Q)D & A(I_m+P) \end{pmatrix} - r[A(I_m+P)] \\
= \; & r\begin{pmatrix} E & C(I_m+P) \\ (I_n-Q)D & 0 \\ BD & A(I_m+P) \end{pmatrix} - r[A(I_m+P)] \\
= \; & r\begin{pmatrix} E & C(I_m+P) \\ BD & A(I_m+P) \\ (I_n-Q)D & 0 \end{pmatrix} - r[A(I_m+P)] \\
& r\begin{pmatrix} E - CA_1^{\dagger}(P)B_1(Q)D & CF_{A_1(P)}(I_m-P) \\ (I_n+Q)D & 0 \end{pmatrix} \\
= \; & r\begin{pmatrix} E & C(I_m-P) \\ BD & A(I_m-P) \\ (I_n+Q)D & 0 \end{pmatrix} - r[A(I_m-P)].
\end{aligned}$$

Therefore, (12) and (13) are followed. Similarly, we can prove statement (*ii*) by a same method, and the details are omitted. □

Corollary 4.4. *Let $A \in \mathbb{C}^{k\times m}$, $B \in \mathbb{C}^{k\times n}$, $C \in \mathbb{C}^{n\times l}$, and $D \in \mathbb{C}^{m\times l}$ be given. Without loss of generality, suppose that consistent equations $AX = B$ and $XC = D$ have a common solution, i.e., $BC = AD$. Then*
(*i*) *$AX = B$ and $XC = D$ have a common solution $X \in \mathbb{C}_r^{m\times n}(P,Q)$ if and only if*

$$r\begin{pmatrix} A & B \\ AP & BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}, \text{ and } r\begin{pmatrix} C & QC \end{pmatrix} = r\begin{pmatrix} C & QC \\ D & PD \end{pmatrix}.$$

(*ii*) *$AX = B$ and $XC = D$ have a common solution $X \in \mathbb{C}_a^{m\times n}(P,Q)$ if and only if*

$$r\begin{pmatrix} A & B \\ AP & -BQ \end{pmatrix} = r\begin{pmatrix} A \\ AP \end{pmatrix}, \text{ and } r\begin{pmatrix} C & QC \end{pmatrix} = r\begin{pmatrix} C & QC \\ D & -PD \end{pmatrix}.$$

Proof. (*i*) On account of Theorem 4.3, (12) and (13) reduce to

$$\begin{aligned} r[(I_n - Q)C] + r(I_m + P) &= r\begin{pmatrix} D & I_m + P \\ BC & A(I_m + P) \\ (I_n - Q)C & 0 \end{pmatrix} = r\begin{pmatrix} D & I_m + P \\ (I_n - Q)C & 0 \end{pmatrix} \\ &= r\begin{pmatrix} (I_m - P)D & I_m + P \\ (I_n - Q)C & 0 \end{pmatrix} = r\begin{pmatrix} (I_m - P)D \\ (I_n - Q)C \end{pmatrix} + r(I_m + P), \\ r[(I_n + Q)C] + r(I_m - P) &= r\begin{pmatrix} D & I_m - P \\ BC & A(I_m - P) \\ (I_n + Q)C & 0 \end{pmatrix} = r\begin{pmatrix} (I_m + P)D \\ (I_n + Q)C \end{pmatrix} + r(I_m - P). \end{aligned}$$

Hence, $r[(I_n - Q)C] = r\begin{pmatrix} (I_m - P)D \\ (I_n - Q)C \end{pmatrix}$ and $r[(I_n + Q)C] = r\begin{pmatrix} (I_m + P)D \\ (I_n + Q)C \end{pmatrix}$, which are equivalent to

$$r\begin{pmatrix} (I_n - Q)C & (I_n + Q)C \end{pmatrix} = r\begin{pmatrix} (I_n - Q)C & (I_n + Q)C \\ (I_m - P)D & (I_n + P)D \end{pmatrix},$$

i.e.,

$$r\begin{pmatrix} C & QC \end{pmatrix} = r\begin{pmatrix} C & QC \\ D & PD \end{pmatrix}.$$

Similarly, statement (*ii*) is obtained. □

Remark 4.5. *It is important to point out that the conditions in statement* (*i*) *(or* (*ii*)*) of Corollary 4.4 ensure that both $AX = B$ and $XC = D$ have a reflexive (or anti-reflexive) solution. So, in this case that these conditions are satisfied, and if there exists a common solution to $AX = B$ and $XC = D$, then there also exists a common reflexive (or anti-reflexive) solution to them. Moreover, comparing with the results in [Theorem 2.1, 22], we note that the condition $BQC = APD$ (or $BQC = -APD$) is not necessary in statement* (*i*) *(or* (*ii*)*).*

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