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Normal Ω-Subgroups

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Abstract. Subgroups, congruences and normal subgroups are investigated for Ω-groups. These are lattice-valued algebraic structures, defined on crisp algebras which are not necessarily groups, and in which the classical equality is replaced by a lattice-valued one. A normal Ω-subgroup is defined as a particular class in an Ω-congruence. Our main result is that the quotient groups over cuts of a normal Ω-subgroup of an Ω-group $\overline{\mathcal{G}}$, are classical normal subgroups of the corresponding quotient groups over $\overline{\mathcal{G}}$. We also describe the minimal normal Ω-subgroup of an Ω-group, and some other constructions related to Ω-valued congruences.

1. Introduction

We introduce a concept of a normal subgroup in the framework of Ω-groups, introduced in [7]. Ω is a complete lattice, hence we deal with lattice-valued structures. In this case, the underlying algebra is not necessarily a group, and the classical equality is replaced by a lattice-valued one. Therefore algebraic (group) identities hold as particular lattice-valued formulas.

1.1. Historical background

First we recall particular basic references for fuzzy groups and related structures, not pretending to present an extensive list of such references. Chronologically, fuzzy groups and related notions (semigroups, rings etc.), were introduced early within the fuzzy era (e.g., Rosenfeld [23] and Das [10], then also Mordeson and Malik [22]). Since then, fuzzy groups remain among the most studied fuzzy structures (e.g., Malik, Mordeson and Kuroki [20], Mordeson, Bhutani, and Rosenfeld [21] and [26]). Investigations of notions from general algebra followed these first studies (see e.g., Di Nola, Gerla [13] and [24, 25]). The universe of an algebra was fuzzified, while the operations remained crisp. The set of truth values was either the unit interval, or a complete, sometimes residuated lattice; generalized co-domains were also used (lattice ordered monoids, Li and Pedrycz, [19], posets or relational systems, [25]). An analysis of different co-domain lattices in the framework of fuzzy topology is presented by Höhle and Šostak in [17]. The notion of a fuzzy equality was introduced by Höhle ([15]) and then used by many others. Using sheaf theory

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([14]), in [16], Höhle was dealing with Ω-valued sets and equalities (Ω being a complete Heyting algebra), representing many fields of fuzzy set theory in this framework. Demirci (see e.g., [11, 12]) introduced the new approach to fuzzy structures. He considered particular algebraic structures equipped with fuzzy equality relations and fuzzy operations. In this framework he developed detailed studies of fuzzy groups (vague groups, smooth subgroups) and related topics. Bělohlávek (papers [1, 3], [5] with Vychodil, the books [2, 4], the second with Vychodil) introduced and investigated algebras with fuzzy equalities. These are defined as classical algebras in which the crisp equality is replaced by a fuzzy one being compatible with the fundamental operations of the algebra. Bělohlávek develops and investigates main fuzzified universal algebraic topics. Some aspects of universal algebra in a fuzzy framework were also investigated by Kuraoka and Suzuki, [18].

1.2. *Organization of the paper*

Throughout the paper, the co-domain of all mappings is a complete lattice, denoted by Ω, so we deal with Ω-valued structures. Our approach is order-theoretic and algebraic, and in addition, we use techniques developed in basics of fuzzy sets. The reason for not using a residuated or related lattices for the membership values is that our results essentially depend on cut structures and their properties. Indeed, it is known that classical set-theoretic and algebraic properties which are fuzzified are preserved on cut-structures, but only if the co-domain structure is a basic complete lattice, without additional operations.

After Introduction, in Preliminaries we introduce necessary known notions and their properties: lattices, lattice-valued structures, some topics from universal algebra. Then we list the relevant results about Ω-algebras (omega algebras), in particular concerning Ω-groups. For each of these we refer to the corresponding literature. This is the framework for our results in the present work. The basic property we use is mentioned: an Ω-algebra is an Ω-group if and only if the cut structures over the corresponding cuts of the Ω-valued equality are classical groups. Finally, section Results contains our contribution. After proving that the property of being an Ω-subgroup is preserved under the corresponding quotient groups over cuts, we introduce the notion of a normal Ω-subgroup of an Ω-group. For this we use an Ω-valued congruence and the Ω-function being its block containing the neutral element. The main theorem which explains our definition is: A subgroup of an Ω-group is normal if and only if every quotient subgroup constructed over cuts of this Ω-subgroup is a normal subgroup of the corresponding quotient subgroup in the starting Ω-group. We explicitly describe the smallest normal Ω-subgroup and analyze its cut structures. We also present a construction of other normal Ω-subgroups, when an Ω-valued congruence is given.

2. Preliminaries

2.1. *Lattices, universal algebra*

We use a **complete lattice** as a partially ordered set $(\Omega, \leqslant)$, where every subset A has both a meet $\bigwedge A$ and a join $\bigvee A$. In addition, $\bigvee \emptyset = 0$, and $\bigwedge \emptyset = 1$, where 0 and 1 are the least and greatest elements of Ω, respectively. We use ordinary properties of lattices, given in every textbook dealing with this topic, e.g., [9].

Some basic notions from Universal Algebra are given in the sequel. For more, see e.g., [8].

A **language** or a **type** $\mathcal{L}$ is a set $\mathcal{F}$ of functional symbols, together with a set of natural numbers (arities) associated to these symbols. As usually, an **algebra** of type $\mathcal{L}$ is a pair (A, F) denoted by $\mathcal{A}$, where A is a nonempty set and F is a set of (fundamental) operations on A. Each operation in F corresponds to some symbol in the language; if the symbol is n-ary, then the arity of the operation is n. A **subalgebra** of $\mathcal{A}$ is an algebra of the same type, defined on a non-empty subset of A, closed under the operations in F. **Terms** in a language are usual regular expressions constructed by the variables and operational symbols. If $t(x_1, \ldots, x_n)$ is a term in the language of an algebra $\mathcal{A}$, then we denote in the same way the corresponding term-operation on $\mathcal{A}$. An **identity** in a language is a formula $t_1 = t_2$, where t_1, t_2 are terms in the same language. An identity $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$ is said to be **valid** on an algebra $\mathcal{A} = (A, F)$, or that $\mathcal{A}$ **satisfies** this identity, if for all $a_1, \ldots, a_n \in A$, the equality $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$ holds. An equivalence relation ρ on A which is compatible with respect to all fundamental operations ($x_i \rho y_i, i = 1, \ldots, n$ imply $f(x_1, \ldots, x_n) \rho f(y_1, \ldots, y_n)$) is a **congruence** on $\mathcal{A}$.

The following known properties of congruences and quotient structures are used throughout this text.

If ρ is a congruence on $\mathcal{A}$, then for $a \in A$, the **congruence class** of a, $[a]_\rho$, and the quotient algebra $\mathcal{A}/\rho$ are defined respectively by

$$[a]_\rho := \{x \in A \mid (a, x) \in \rho\}; \quad \mathcal{A}/\rho := (A/\rho, F),$$

where $A/\rho = \{[a]_\rho \mid a \in A\}$, and the operation on classes are defined by representatives. Next, let ϕ and θ be congruences on an algebra $\mathcal{A}$, and $\theta \subseteq \phi$. Then, *the relation*

$$\phi/\theta := \{([a]_\theta, [b]_\theta) \mid (a, b) \in \phi\}$$

is a congruence on $\mathcal{A}/\theta$.

Theorem 2.1 (Second Isomorphism Theorem). *If ϕ and θ are congruences on an algebra $\mathcal{A}$ and $\theta \subseteq \phi$, then ϕ/θ is a congruence on $\mathcal{A}/\theta$.*

Let $\mathcal{A}$ be an algebra, θ a congruence on $\mathcal{A}$ and $B \subseteq A$. Let

$$B^\theta := \{x \in A \mid B \cap [a]_\theta \neq \emptyset\},$$

and $\mathcal{B}^\theta$ the subalgebra of $\mathcal{A}$ generated by B^θ. We denote

$$\theta\restriction_{B^\theta} := \theta \cap B^2$$

(the restriction of θ to B). Now, *the universe of $\mathcal{B}^\theta$ is B^θ and $\theta\restriction_{B^\theta}$ is a congruence on $\mathcal{B}$.*

Theorem 2.2 (Third Isomorphism Theorem). *If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and θ a congruence on $\mathcal{A}$, then*

$$\mathcal{B}/\theta\restriction_B \cong \mathcal{B}^\theta/\theta\restriction_{B^\theta}.$$

2.2. Ω-valued functions and relations

An **Ω-valued function** μ on a nonempty set A, is a function $\mu : A \to \Omega$, where $(\Omega, \leqslant)$ is a complete lattice. This notion can be identified with the one of a **fuzzy set** on A. An Ω-valued function μ on A is said to be **nonempty**, if $\mu(x) > 0$ for some $x \in A$. If μ and ν are Ω-valued functions of A, then we say that μ is a **fuzzy subset** of ν, we write $\mu \subseteq \nu$ if for every $x \in A$, $\mu(x) \leqslant \nu(x)$.

For $p \in \Omega$, a **cut set** or a p-**cut** of $\mu : A \to \Omega$ is a subset μ_p of A which is the inverse image of the principal filter in Ω, generated by p:

$$\mu_p = \{x \in X \mid \mu(x) \geqslant p\}.$$

An **Ω-valued** (binary) **relation** ρ **on** A is an Ω-valued function on A^2, i.e., it is a mapping $\rho : A^2 \to \Omega$.

$$\rho \text{ is } \textbf{symmetric} \text{ if } \rho(x, y) = \rho(y, x) \text{ for all } x, y \in A; \tag{1}$$

$$\rho \text{ is } \textbf{transitive} \text{ if } \rho(x, y) \geqslant \rho(x, z) \wedge \rho(z, y) \text{ for all } x, y, z \in A. \tag{2}$$

We say that a symmetric and transitive relation ρ on A is an **Ω-valued equality**, or an **Ω-valued equality** on A.

Observe that an Ω-valued equality ρ on a set A fulfills the **strictness** property (see [16]):

$$\rho(x, y) \leqslant \rho(x, x) \wedge \rho(y, y). \tag{3}$$

Similarly as in [16], we say that an Ω-valued equality ρ on A is **separated**, if it satisfies the property

$$\rho(x, y) = \rho(x, x) \text{ implies } x = y. \tag{4}$$

Next, we briefly connect the above notions with Ω-*valued relations on* Ω-*valued sets*.

Let $\mu : A \to \Omega$ be an Ω-valued function on A and let $\rho : A^2 \to \Omega$ be an Ω-valued relation on A. If for all $x, y \in A$, ρ satisfies

$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y), \tag{5}$$

then we say that ρ is an **Ω-valued relation on** μ (see e.g., [28]).

An Ω-valued relation ρ on $\mu : A \to \Omega$ is said to be **reflexive on** μ if

$$\rho(x, x) = \mu(x) \text{ for every } x \in A. \tag{6}$$

Observe that a reflexive Ω-valued relation on μ is *strict* on A, in the sense of (3).

A symmetric and transitive Ω-valued relation ρ on A, which is reflexive on $\mu : A \to \Omega$ is an **Ω-valued equality** on μ. In addition, if ρ is separated on A, then we say that it is a **separated Ω-valued equality** on μ.

A **lattice-valued subalgebra** of an algebra $\mathcal{A} = (A, F)$, here an **Ω-valued subalgebra** of $\mathcal{A}$ is a function $\mu : A \to \Omega$ which is not constantly equal to 0, and which fulfils the following: For any operation f from F with arity greater than 0, $f : A^n \to A$, $n \in \mathbb{N}$, and for all $a_1, \dots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \dots, a_n)), \tag{7}$$

$$\text{and for a nullary operation } c \in F,\ \mu(c) = 1. \tag{8}$$

Proposition 2.3. *Let $\mu : A \to \Omega$ be an Ω-valued subalgebra of an algebra $\mathcal{A}$ and let $t(x_1, \dots, x_n)$ be a term in the language of $\mathcal{A}$. If $a_1, \dots, a_n \in A$, then the following holds:*

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(t^A(a_1, \dots, a_n)). \tag{9}$$

□

An Ω-valued relation $R : A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F)$ is **compatible** with the operations in F if the following two conditions holds: for every n-ary operation $f \in F$, for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \dots, a_n), f(b_1, \dots, b_n)); \tag{10}$$

$$R(c, c) = 1. \tag{11}$$

2.3. Ω-set

An **Ω-set** (as defined in [14]) is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A which may be separated if indicated.

For an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x). \tag{12}$$

We say that μ is *determined by* E. Clearly, by the strictness property, E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ. That is why we say that in an Ω-set (A, E), E is an **Ω-valued equality**.

Lemma 2.4. *If (A, E) is an Ω-set and $p \in \Omega$, then the cut E_p is an equivalence relation on the corresponding cut μ_p of μ.*

2.4. Ω-algebra

Next we introduce a notion of a lattice-valued algebra with a lattice valued equality.

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F. Then we say that $(\mathcal{A}, E)$ is an **Ω-algebra**. Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

Now we present some cut properties of Ω-algebras. These have been proved in [7], in the framework of groups.

Proposition 2.5. *Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold:*

(i) The function $\mu : A \to \Omega$ determined by E ($\mu(x) = E(x, x)$ for all $x \in A$), is an Ω-valued subalgebra of A.

(ii) For every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$, and

(iii) For every $p \in \Omega$, the cut E_p of E is a congruence relation on μ_p.

Remark 2.6. Observe the difference between an Ω-valued subalgebra μ of an algebra $\mathcal{A}$, and an Ω-algebra $(\mathcal{A}, E)$: the former is a function compatible with the operations on $\mathcal{A}$ in the sense of (9), and the latter is a pair $(\mathcal{A}, E)$, consisting of an algebra $\mathcal{A}$ and an Ω-equality E. Relationship among these two is given in the above Proposition 2.5.

2.5. Identities

Next we define how identities hold on Ω-algebras, according to the approach in [27].

Let and $u(x_1, \dots, x_n) \approx v(x_1, \dots, x_n)$ (briefly $u \approx v$) be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume, as usual, that variables appearing in terms u and v are from $x_1, \dots, x_n$ Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (i.e., this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)), \tag{13}$$

for all $a_1, \dots, a_n \in A$.

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity need not hold on $\mathcal{A}$. On the other hand, if the supporting algebra fulfills an identity then also the corresponding Ω-algebra does.

Proposition 2.7. *If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω-algebra $(\mathcal{A}, E)$.*

2.6. Ω-groups

The definitions and propositions in this section are from [7].

Definition 2.8. Let

$$\overline{\mathcal{G}} = (\mathcal{G},\ E^{\mu})$$

be an Ω-algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\cdot)$, unary operation $(^{-1})$ and a constant (e), and

$\mu : G \to \Omega$, such that $\mu(x) = E^{\mu}(x, x)$.

Then $\overline{\mathcal{G}}$ is an **Ω-group** if it satisfies the known group identities:

$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$

$x \cdot e \approx x, \quad e \cdot x \approx x$

$x \cdot x^{-1} \approx e, \quad x^{-1} \cdot x \approx e.$

By formula (13), the above identities hold if the following lattice-theoretic formulas are satisfied; observe that by (8), $\mu(e) = 1$:

(*i*) $\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E^{\mu}(x \cdot (y \cdot z), (x \cdot y) \cdot z)$;

(*ii*) $\mu(x) \leqslant E^{\mu}(x \cdot e, x)$, $\mu(x) \leqslant E^{\mu}(e \cdot x, x)$;

(*iii*) $\mu(x) \wedge \mu(x^{-1}) \leqslant E^{\mu}(x \cdot x^{-1}, e)$, $\mu(x) \wedge \mu(x^{-1}) \leqslant E^{\mu}(x^{-1} \cdot x, e)$.

Element e is said to be the **unit** in $\overline{\mathcal{G}}$, and x^{-1} is the **inverse** of element x in $\overline{\mathcal{G}}$. We also say that $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is the **underlying algebra** of Ω-group $\overline{\mathcal{G}}$.

Observe that according to the definition, μ has the following properties: for all $x, y \in G$

$$\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y),$$
$$\mu(x^{-1}) \geqslant \mu(x),$$
$$\mu(e) = 1.$$

Theorem 2.9. *Let $\mathcal{G} = (G, \cdot, ^{-1}, e)$ be a group, and E^{μ} an Ω-valued equality on $\mathcal{G}$. Then, $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ is an Ω-group.*

Theorem 2.10. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group with separated E^{μ}. Let also $t(x)$ be a term depending on a variable x only. Then the Ω-valued identity $E^{\mu}(t(x), x)$ holds on $\overline{\mathcal{G}}$ if and only if the corresponding crisp identity $t(x) = x$ holds on $\mathcal{G}$.*

Corollary 2.11. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group where E^{μ} is separated. Then the underlying algebra $\mathcal{G} = (G, \cdot, ^{-1}, e)$ fulfils:*
(*a*) *e is a neutral and a unique idempotent element with respect to binary operation $\cdot$,*
(*b*) *unary operation $^{-1}$ is an involution, and*
(*c*) *identity $(x \cdot x^{-1}) \cdot x = x$ holds.*

Let $\nu : G \to \Omega$ be a nonempty Ω-valued subset of an Ω-valued set $\mu : G \to \Omega$, R an Ω-valued relation on μ, and $S : G^2 \to \Omega$ an Ω-valued relation on G. Then, S is a **restriction** of R to ν if

$$S(x, y) = R(x, y) \wedge \nu(x) \wedge \nu(y). \tag{14}$$

Let A be a nonempty set and (A, E^{μ}) an Ω-set on A. If E^{μ_1} is the restriction of E^{μ} to a nonempty Ω-valued subset μ_1 of μ (where μ is determined by E^{μ}). Then clearly (A, E^{μ_1}) is also an Ω-set on A. An analogue property holds for Ω-algebras:

Proposition 2.12. *If $(\mathcal{A}, E^{\mu})$ is an Ω-algebra on an algebra $\mathcal{A} = (A, F)$ and μ_1 is an Ω-valued subset of μ and a subalgebra of $\mathcal{A}$, then also $(\mathcal{A}, E^{\mu_1})$ is an Ω-algebra on $\mathcal{A}$, where E^{μ_1} is the restriction of E^{μ} to μ_1.*

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ and $\overline{\mathcal{G}}_1 = (\mathcal{G}, E^{\mu_1})$ be Ω-groups over the same algebra $\mathcal{G} = (G, \cdot, ^{-1}, e)$. We say that $\overline{\mathcal{G}}_1$ is an Ω-**subgroup** of Ω-group $\overline{\mathcal{G}}$, if E^{μ_1} is a restriction of E^{μ} to the Ω-valued subalgebra μ_1 of $\mathcal{G}$, determined by E^{μ_1}.

Theorem 2.13. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group and $E^{\mu_1} : G^2 \to \Omega$ an Ω-valued relation on G, satisfying the formula:*

$$E^{\mu_1}(x, y) = E^{\mu}(x, y) \wedge E^{\mu_1}(x, x) \wedge E^{\mu_1}(y, y). \tag{15}$$

Then the structure $\overline{\mathcal{G}}_1 = (\mathcal{G}, E^{\mu_1})$ is an Ω-subgroup of the Ω-group $\overline{\mathcal{G}}$ if and only if it satisfies:

$$E^{\mu_1}(x, x) \wedge E^{\mu_1}(y, y) \leqslant E^{\mu_1}(x \cdot y, x \cdot y), \tag{16}$$
$$E^{\mu_1}(x, x) \leqslant E^{\mu_1}(x^{-1}, x^{-1}), \tag{17}$$
$$E^{\mu_1}(e, e) = 1. \tag{18}$$

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-algebra. By known properties of Ω-valued structures, for every $p \in \Omega$, the cut μ_p of the Ω-valued subalgebra μ ($\mu(x) = E^\mu(x,x)$) of $\mathcal{G}$ is a subalgebra of $\mathcal{G}$. Further, the cut relation E_p of E^μ is a congruence relation on μ_p.

Theorem 2.14. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-algebra. Then, $\overline{\mathcal{G}}$ is an Ω-group if and only if for every $p \in \Omega$ the quotient structure μ_p/E_p is a group.*

The last theorem is a special case of the following general property.

Theorem 2.15. *Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all the identities in $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p/E_p satisfies the same identities.*

3. Results

First we deal with particular cut properties of Ω-subgroups. Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group. Observe that by Theorem 2.15, for every $p \in \Omega$, the quotient structure μ_p/E_p^μ is a classical group, where μ_p is a p-cut of $\mu : G \to \Omega$, with $\mu(x) = E^\mu(x,x)$, and E_p^μ is the corresponding cut of E^μ.

Theorem 3.1. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ an Ω-subgroup of $\overline{\mathcal{G}}$. Then, for every $p \in \Omega$, the group ν_p/E_p^ν is, up to an isomorphism, a subgroup of the group μ_p/E_p^μ.*

Proof. Consider the quotient groups ν_p/E_p^ν and μ_p/E_p^μ, for $p \in \Omega$. Observe that ν_p is a subalgebra of the algebra μ_p, and that E_p^ν is a restriction of E_p^μ to ν_p, in the sense of the starting algebras with a binary, a unary and a nullary operation.

Now, E_p^ν is a congruence on ν_p, and E_p^μ is a congruence on μ_p. We also have that E_p^ν is a restriction of E_p^μ to ν_p. Let

$$\nu_p^{E_p^\mu} = \{a \in \mu_p \mid \nu_p \cap [a]_{E_p^\mu} \neq \emptyset\}.$$

In other words, $\nu_p^{E_p^\mu}$ is a union of classes of congruence E_p^μ having nonempty intersection with ν_p.

It is clear that $\nu_p^{E_p^\mu}$ is a subalgebra of μ_p, and that the restriction of E_p^μ to $\nu_p^{E_p^\mu}$, $E_p^\mu \restriction \nu_p^{E_p^\mu}$, is a congruence on $\nu_p^{E_p^\mu}$.

By the Third isomorphism theorem, we have that

$$\nu_p/E_p^\nu \cong \nu_p^{E_p^\mu}/(E^{\mu_p} \restriction \nu_p^{E_p^\mu}).$$

Since ν_p/E_p^ν is a group, we also have that the quotient structure on the righthand side, $\nu_p^{E_p^\mu}/(E^{\mu_p} \restriction \nu_p^{E_p^\mu})$ is a group. In addition, $\nu_p^{E_p^\mu}/(E^{\mu_p} \restriction \nu_p^{E_p^\mu})$ is a subset of μ_p/E_p^μ, since the former consists of some equivalence classes of μ_p/E_p^μ. Finally, $\nu_p^{E_p^\mu}/(E^{\mu_p} \restriction \nu_p^{E_p^\mu})$ is a group, hence it is a subgroup of μ_p/E_p^μ. □

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group.

By μ we denote the mapping from G to Ω, defined by $\mu(x) = E^\mu(x,x)$.

An **Ω-valued congruence** on $\overline{\mathcal{G}}$ is an Ω-valued relation $\Theta : G^2 \to \Omega$ on G, which is μ-reflexive, symmetric, transitive and compatible with the operations in $\mathcal{G}$, and which also for all $x, y \in G$ fulfills $\Theta(x,y) \geqslant E^\mu(x,y)$.

Observe that μ-reflexivity of Θ means that for every $x \in G$, $\Theta(x,x) = E^\mu(x,x)$.

Let Θ be a congruence on a given Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$. Define $\nu : G \to \Omega$ by

$$\nu(x) := \Theta(e,x), \tag{19}$$

where e is a constant, neutral element in $\mathcal{G}$. Next, let $E^\nu : G^2 \to \Omega$ be defined by

$$E^\nu(x,y) := E^\mu(x,y) \wedge \nu(x) \wedge \nu(y). \tag{20}$$

Proposition 3.2. *If $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ is an Ω-group, then $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ is an Ω-subgroup of $\overline{\mathcal{G}}$.*

Proof. We prove that conditions given in Theorem 2.13 are fulfilled.
First, condition (15) is fulfilled:

$$E^\nu(x,y) = E^\nu(x,y) := E^\mu(x,y) \wedge E^\nu(x,x) \wedge E^\nu(y,y),$$

by the definition of E^ν, since $E^\nu(x,x) = E^\mu(x,x) \wedge \Theta(e,x) = \Theta(e,x)$, and similarly for $E^\nu(y,y)$.
Further, by compatibility of Θ,

$$E^\nu(x,x) \wedge E^\nu(y,y) = \Theta(e,x) \wedge \Theta(e,y) \leqslant \Theta(e, x\cdot y) = E^\nu(x\cdot y, x\cdot y),$$

and (16) holds. Analogously, conditions (17) and (18) are satisfied.
Therefore, by Theorem 2.13, $\overline{\mathcal{N}}$ is an Ω-subgroup of $\overline{\mathcal{G}}$. $\square$

Remark 3.3. Observe that in the case of crisp, classical groups, (19) gives a characteristic function of a normal subgroup.

The above considerations motivates the following definition.
Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ an Ω-subgroup of $\overline{\mathcal{G}}$. Then, $\overline{\mathcal{N}}$ is a **normal Ω-subgroup** of $\overline{\mathcal{G}}$, if there is an Ω-valued congruence Θ on $\overline{\mathcal{G}}$, such that for all $x, y \in G$,

$$E^\nu(x,y) = E^\mu(x,y) \wedge \Theta(e,x) \wedge \Theta(e,y). \tag{21}$$

The following result is the main argument for the definition of a normal Ω-subgroup.

Theorem 3.4. *An Ω-subgroup $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$, if and only if for every $p \in \Omega$, ν_p/E^ν_p is a normal subgroup of the group μ_p/E^μ_p.*

Proof. Let $\overline{\mathcal{N}}$ be a normal Ω-subgroup of the Ω-group $\overline{\mathcal{G}}$. Then, by the definition, there is an Ω-valued congruence Θ on $\overline{\mathcal{G}}$, such that for all $x, y \in G$, $\theta(x,y) \geqslant E^\mu(x,y)$ and

$$E^\nu(x,y) = E^\mu(x,y) \wedge \Theta(e,x) \wedge \Theta(e,y).$$

Now, for $p \in \Omega$, we consider the cut Θ_p, which is, clearly, a congruence on the subalgebra μ_p of the underlying algebra $\mathcal{G}$, since for every $x \in G$, $\Theta(x,x) = E^\mu(x,x)$, and $E^\mu_p \subseteq \Theta_p$.
By the above, all conditions for the Second isomorphism theorem are fulfilled. Therefore, the relation Θ_p/E^μ_p, defined by

$$([x]_{E^\mu_p}, [y]_{E^\mu_p}) \in \Theta_p/E^\mu_p \ \text{ if and only if } \ (x,y) \in \Theta_p, \tag{22}$$

is a congruence on μ_p/E^μ_p (it is well defined since Θ_p is a congruence by the assumption).
In the above formula,

$$(x,y) \in \Theta_p \ \text{ if and only if } \ \Theta(x,y) \geqslant p.$$

Further, by the Second isomorphism theorem,

$$\mu_p/E^\mu_p/\Theta_p/E^\mu_p \cong \mu_p/\Theta_p.$$

Now, μ_p/E^μ_p is a group, Θ_p/E^μ_p is a congruence on this group, hence μ_p/Θ_p is a group.
Next, by the definition, for every $x \in G$, $\nu(x) = \Theta(e,x)$, hence for $p \in \Omega$, $x \in \nu_p$ if and only if $\Theta(e,x) \geqslant p$.
By Theorem 3.1, ν_p/E^ν_p is, up to an isomorphism, a subgroup of μ_p/E^μ_p. By the definition, ν_p/E^ν_p consists exactly of some equivalence classes of μ_p/E^μ_p, so it is indeed a subgroup of μ_p/E^μ_p.

Now we show that ν_p/E_p^ν is a normal subgroup of μ_p/E_p^μ. In other words, we prove that ν_p/E_p^ν is a class of a congruence on μ_p/E_p^μ, containing the neutral element.

Indeed, we have already noted that Θ_p/E_p^μ is a congruence on μ_p/E_p^μ and now we see that the class of this congruence containing the neutral element is exactly ν_p/E_p^ν.

Conversely, suppose that $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ is an Ω-subgroup of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$. By assumption, for every $p \in \Omega$, ν_p/E_p^ν is a normal subgroup of the group μ_p/E_p^μ which means that elements in ν_p/E_p^ν are exactly some classes of μ_p/E_p^μ. Now, for every $p \in \Omega$, we define a relation θ_p on μ_p/E_p^μ by

$$[x]_{E_p^\mu}\theta_p[y]_{E_p^\mu} \quad \text{if and only if} \quad [x]_{E_p^\mu} \cdot [y]_{E_p^\mu}^{-1} \in \nu_p/E_p^\nu.$$

Since ν_p/E_p^ν is a normal subgroup, θ_p is a congruence on μ_p/E_p^μ.

$[x]_{E_p^\mu} \cdot [y]_{E_p^\mu}^{-1} \in \nu_p/E_p^\nu$ is equivalent with $[x \cdot y^{-1}]_{E_p^\mu} \in \nu_p/E_p^\nu$, which is further equivalent with $x \cdot y^{-1} \in \nu_p$, which is equivalent with $\nu(x \cdot y^{-1}) \geqslant p$.

Now we consider a family of congruences $\{\theta_i \mid i \in I \subseteq \Omega\}$. Since $[x]_{E_i^\mu}\theta_i[y]_{E_i^\mu}$ is equivalent with $\nu(x \cdot y^{-1}) \geqslant i$, we have that $[x]_{E_i^\mu}\theta_i[y]_{E_i^\mu}$ for every $i \in I$ is equivalent with $\nu(x \cdot y^{-1}) \geqslant \bigvee_{i \in I} i$, this is further equivalent with $[x]_{E_i^\mu}\theta_{\bigvee_{i\in I} i}[y]_{E_i^\mu}$. Hence, we have that the family of all congruences $\{\theta_i \mid i \in \Omega\}$ is a closure system, since

$$\bigcap_{i \in I} \theta_i = \theta_{\bigvee_{i \in I} i}.$$

Now, we define a relation:

$$\Theta : G^2 \to \Omega \quad \text{by} \quad \Theta(x, y) = \bigvee \{p \mid ([x]_{E_p^\mu}, [y]_{E_p^\mu}) \in \theta_p\}.$$

Note that if (x, y) does not belong to any θ_p for $p \in \Omega$, then $\Theta(x, y) = 0$ by the definition of the supremum of $\emptyset$ in the complete lattice Ω.

Now, it is straightforward to prove that θ is a symmetric, transitive and compatible relation on $\overline{\mathcal{G}}$. It is also μ-reflexive: for $x \in G$

$$\Theta(x, x) = \bigvee \{p \mid ([x]_{E_p^\mu}, [x]_{E_p^\mu}) \in \theta_p\} = \bigvee \{p \mid x \in \mu_p\} = \mu(x) = E^\mu(x, x),$$

since $\mu(x)$ is one of the values over which the supremum run.

Finally, we prove that for all $x, y \in G$, $E^\mu(x, y) \leqslant \Theta(x, y)$. Let $E^\mu(x, y) = p$. Then $(x, y) \in E_p$ and hence $[x]_{e_p^\mu} = [y]_{e_p^\mu}$. Since θ_p is a congruence on μ_p/E_p^μ, it is obvious that we have $([x]_{e_p^\mu}, [y]_{e_p^\mu}) \in \theta_p$. By the definition of Θ, we get $\Theta(x, y) \geqslant p$.

Hence Θ is an Ω-valued congruence on $\overline{\mathcal{G}}$, and by the construction $\Theta(x, e) = \nu(x) = E^\nu(x, x)$. By the definition (21), $\overline{\mathcal{N}}$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$. □

Corollary 3.5. *If $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ is a commutative Ω-group, then every Ω-subgroup of $\overline{\mathcal{G}}$ is normal.*

Proof. Indeed, commutativity of an Ω-group is hereditary for quotient subgroups on cuts by Theorem 2.15. Therefore, if $\overline{\mathcal{G}}$ is commutative, then every quotient structure μ_p/E_p, $p \in \Omega$ is an Abelian group. All subgroups of these are normal, hence by Theorem 3.4, every Ω-subgroup of $\overline{\mathcal{G}}$ is normal. □

Example 1. The structure $(\mathcal{G}, E^\mu)$, where $\mathcal{G} = (G, \cdot, {}^{-1}, e)$ with a binary operation $\cdot$ on $G = \{e, a, b, c, d, f, g, h, i, j\}$ is given in Table 1; unary operation ${}^{-1}$ is the identity function, and neutral element is e. The lattice Ω is given by the diagram in Figure 1. The Ω-valued equality E^μ is presented in Table 2.

$\cdot$	e	a	b	c	d	f	g	h	i	j
e	e	a	b	c	d	f	g	h	i	j
a	a	e	b	c	d	f	h	g	i	j
b	b	b	e	e	g	h	f	d	i	j
c	c	c	e	e	h	g	d	f	i	j
d	d	f	h	g	e	e	c	b	i	j
f	f	d	g	h	e	e	b	c	i	j
g	g	g	d	f	b	c	e	e	i	j
h	h	h	f	d	c	b	e	e	i	j
i	i	a	b	c	d	f	g	h	e	e
j	j	a	b	c	d	f	g	h	e	e

Table 1: Binary operation on G

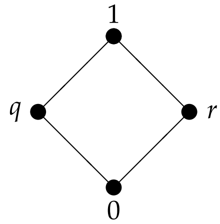

Figure 1: Lattice Ω

E^μ	e	a	b	c	d	f	g	h	i	j
e	1	r	0	0	0	0	0	0	0	0
a	r	r	0	0	0	0	0	0	0	0
b	0	0	r	r	0	0	0	0	0	0
c	0	0	r	r	0	0	0	0	0	0
d	0	0	0	0	r	r	0	0	0	0
f	0	0	0	0	r	r	0	0	0	0
g	0	0	0	0	0	0	r	r	0	0
h	0	0	0	0	0	0	r	r	0	0
i	0	0	0	0	0	0	0	0	q	q
j	0	0	0	0	0	0	0	0	q	q

Table 2: Ω-valued equality on G

The function $\mu : G \to \Omega$ is determined by E^μ: $\mu(x) = E^\mu(x, x)$.

x	e	a	b	c	d	f	g	h	i	j
$\mu(x)$	1	r	r	r	r	r	r	r	q	q

$(\mathcal{G}, E^\mu)$ is an Ω-group. Quotient cut-subgroups are:

$$\mu_r/E^\mu_r = \{\{e, a\}, \{b, c\}, \{d, f\}, \{g, h\}\} \text{ and } \mu_q/E^\mu_q = \{\{i, j\}\}.$$

An Ω-valued congruence Θ on $(\mathcal{G}, E^\mu)$ is given in Table 3.
By the definition we have $\nu(x) = \Theta(e, x)$:

x	e	a	b	c	d	f	g	h	i	j
$\nu(x)$	1	r	r	r	0	0	0	0	0	0

Therefore, $\nu_1 = \{e\}$, and $\nu_r = \{e, a, b, c\}$ and the remaining cut ν_q is the empty set.

Consequently, $\nu_r/E_r^\nu = \{\{e, a\}, \{b, c\}\}$ is a normal subgroup of μ_r/E_r^μ, and this is the only nonempty and non-trivial cut structure.

Θ	e	a	b	c	d	f	g	h	i	j
e	1	r	r	r	0	0	0	0	0	0
a	r	r	r	r	0	0	0	0	0	0
b	r	r	r	r	0	0	0	0	0	0
c	r	r	r	r	0	0	0	0	0	0
d	0	0	0	0	r	r	r	r	0	0
f	0	0	0	0	r	r	r	r	0	0
g	0	0	0	0	r	r	r	r	0	0
h	0	0	0	0	r	r	r	r	0	0
i	0	0	0	0	0	0	0	0	q	q
j	0	0	0	0	0	0	0	0	q	q

Table 3: Ω-valued congruence on G

Following (21) i.e., by

$$E^\nu(x, y) = E^\mu(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y),$$

we have E^ν presented in Table 4.

E^ν	e	a	b	c	d	f	g	h	i	j
e	1	r	0	0	0	0	0	0	0	0
a	r	r	0	0	0	0	0	0	0	0
b	0	0	r	r	0	0	0	0	0	0
c	0	0	r	r	0	0	0	0	0	0
d	0	0	0	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	0	0	0	0	0
h	0	0	0	0	0	0	0	0	0	0
i	0	0	0	0	0	0	0	0	0	0
j	0	0	0	0	0	0	0	0	0	0

By Theorem 3.4, the structure $(\mathcal{G}, E^\nu)$ is a normal Ω-subgroup of the Ω-group $(\mathcal{G}, E^\mu)$. □

Continuing with the general properties of normal Ω-subgroups, we use the fact that E^μ is also an Ω-valued congruence on $\overline{\mathcal{G}}$. Therefore, we examine a particular case when $\Theta = E^\mu$.

Theorem 3.6. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group, and $E^\epsilon : G^2 \to \Omega$ defined by*

$$E^\epsilon(x, y) = E^\mu(e, x) \wedge E^\mu(e, y), \tag{23}$$

with $\epsilon : G \to \Omega$, $\epsilon(x) := E^\epsilon(x, x)$. Then, $\overline{\mathcal{E}} = (\mathcal{G}, E^\epsilon)$ is the smallest normal Ω-subgroup of $\overline{\mathcal{G}}$.

Proof. By (21), E^ϵ is an Ω-congruence on $\overline{\mathcal{G}}$:

$$E^\epsilon(x, y) = E^\mu(e, x) \wedge E^\mu(e, y) = E^\mu(x, y) \wedge E^\mu(e, x) \wedge E^\mu(e, y),$$

since by symmetry and transitivity of E^μ

$$E^\mu(e, x) \wedge E^\mu(e, y) \leqslant E^\mu(x.y).$$

Therefore, $\overline{\mathcal{E}}$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$. We prove that it is the smallest one. Namely, let $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ be an arbitrary normal Ω-subgroup of $\overline{\mathcal{G}}$; we show that $(\mathcal{G}, E^\epsilon)$ is an Ω-subgroup of $\overline{\mathcal{N}}$. Indeed, E^ϵ is a restriction of E^ν to ϵ, where $\epsilon(x) = E^\mu(e,x)$, and $E^\nu(x,y) = E^\mu(x,y) \wedge \Theta(e,x) \wedge \Theta(e,y)$, for an Ω-congruence Θ on $\overline{\mathcal{G}}$, $E^\mu(x,y) \leqslant \Theta(x,y)$. So, we have

$E^\epsilon(x,y) = E^\mu(x,y) \wedge E^\mu(e,x) \wedge E^\mu(e,y) =$
$E^\mu(x,y) \wedge E^\mu(e,x) \wedge E^\mu(e,y) \wedge \Theta(e,x) \wedge \Theta(e,y) =$
$E^\nu(x,y) \wedge E^\mu(e,x) \wedge E^\mu(e,y) = E^\nu(x,y) \wedge \epsilon(x) \wedge \epsilon(y)$,

and E^ϵ is a restriction of E^ν to ϵ. By Proposition 2.12, $\overline{\mathcal{E}}$ is an Ω-subgroup of an arbitrary normal Ω-subgroup $\overline{\mathcal{N}}$ of $\overline{\mathcal{G}}$, hence it is the smallest one. □

The following is an explicit description of $\overline{\mathcal{E}}$ in terms of cut relations.

Corollary 3.7. *Let $\overline{\mathcal{E}} = (\mathcal{G}, E^\epsilon)$ be the subgroup of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$, with E^ϵ being defined by* (23). *Then, for every $p \in \Omega$, the cut E^ϵ_p is the diagonal relation (equality) on the quotient group μ_p/E^μ_p.*

Proof. By (22), the relation E^ϵ_p/E^μ_p, defined by

$$([x]_{E^\mu_p}, [y]_{E^\mu_p}) \in E^\epsilon_p/E^\mu_p \text{ if and only if } (x,y) \in E^\epsilon_p,$$

is a congruence on μ_p/E^μ_p. By the definition of E^ϵ and by transitivity of E^μ we have

$(x,y) \in E^\epsilon_p$ if and only if $E^\epsilon(x,y) \geqslant p$
which implies $E^\mu(x,y) \geqslant E^\mu(e,x) \wedge E^\mu(e,y) \geqslant p$.
Obviously, this is equivalent with $[x]_{E^\mu_p} = [y]_{E^\mu_p}$, hence E^ϵ_p is a classical equality on μ_p/E^μ_p. □

Next we prove that an Ω-valued congruence on an Ω-group, acting as an Ω-valued equality, generates an Ω-group itself. Recall that an Ω-valued congruence Θ on an Ω-group $(\mathcal{G}, E^\mu)$ is an Ω-valued equivalence on G, compatible with the group operations and satisfying $\Theta(x,y) \geqslant E^\mu(x,y)$.

Theorem 3.8. *Let $\Theta : G^2 \to \Omega$ be an Ω-valued congruence on an Ω-group $(\mathcal{G}, E^\mu)$. Then $(\mathcal{G}, \Theta)$ is an Ω-group as well. In addition, for every $p \in \Omega$, the mapping $f : \mu_p/E^\mu_p \to \mu_p/\Theta_p$, defined by $f([x]_{E^\mu_p}) = [x]_{\Theta_p}$ is a classical surjective group homomorphism.*

Proof. It is obvious that $(\mathcal{G}, \Theta)$ is an Ω-algebra. We prove that the group identities are fulfilled. This follows by the fact that for every $x \in G$, $\mu(x) = \Theta(x,x)$. Hence, e.g., for Ω-associativity of the binary operation on G, we have

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E^\mu(x \cdot (y \cdot z), (x \cdot y) \cdot z) \leqslant \Theta(x \cdot (y \cdot z), (x \cdot y) \cdot z),$$

similarly with other group identities.

Next, let $f : \mu_p/E^\mu_p \to \mu_p/\Theta_p$, be such that $f([x]_{E^\mu_p}) = [x]_{\Theta_p}$. Then, for $x, y \in \mu_p$,

$$f([x \cdot y]_{E^\mu_p}) = [x \cdot y]_{\Theta_p} = [x]_{\Theta_p} \cdot [y]_{\Theta_p} = f([x]_{E^\mu_p}) \cdot f([y]_{E^\mu_p}),$$

hence f is a homomorphism. Analogously, one can check that f is compatible with the unary operation $^{-1}$, and that $f([e]_{E^\mu_p}) = [e]_{\Theta_p}$. It is surjective, since every class $[x]_{\Theta_p}$ is the image of $[x]_{E^\mu_p}$ under f. □

4. Conclusion

In the framework of Ω-groups we have introduced and investigated normal Ω-subgroups. We have also shown the connection to classical groups and normal subgroups, which appear as quotient subgroups over the cuts.

As a continuing investigation, we intend to deal with particular important notions related to groups and normal subgroups, all in the framework of Ω-groups. These topics are chains of subgroups, subnormal subgroups and in particular lattices of these structures.

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