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Large Deviations for Stochastic Integrodifferential Equations of the Itô Type with Multiple Randomness

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Abstract. A Freidlin-Wentzell type large deviation principle is derived for a class of Itô type stochastic integrodifferential equations driven by a finite number of multiplicative noises of the Gaussian type. The weak convergence approach is used here to prove the Laplace principle, equivalently large deviation principle.

1. Introduction

Integrodifferential equations arise quite naturally in many applied problems of engineering, fluid mechanics, life sciences and other fields of nonlinear science. The most common integrodifferential equations are kinetic equations which describe the time evolution of a distribution function of certain interacting particles like ions, aersols, gas molecules, electrons etc. Equations of this kind also occur in the formulation of problems in reactor dynamics and in the study of the growth of biological population models. Most of these are intrinsically nonlinear, complex in nature and depends on random excitations of a Gaussian white noise type. Therefore it is ideal to consider them in a stochastic framework and when mathematically modelled, result in stochastic integrodifferential equations (SIDEs). Several authors have rigorously studied the SIDEs of the Itô type. For example, Jovanović and Janković studied the existence and uniqueness of solutions for a general SIDE of the Itô type in [20]. In paper [27] by Murge and Pachpatte, sufficient conditions for infinite explosion time and asymptotic behavior of the solutions of Itô type stochastic integrodifferential equations were derived. Controllability results of the same have been established in [1]. For more works on the Itô type SIDEs we refer to papers [17–19, 28] and references therein. Suvinthra and Balachandran in their paper [31] have proved the large deviation estimates for SIDEs of the Itô type. This is perhaps the first work on the large deviation estimates for SIDEs of the Itô type. The authors have considered a nonlinear SIDE in the Euclidean space $\mathbb{R}^n$ perturbed by a single Brownian motion in $\mathbb{R}^d$. In this work also we consider a nonlinear SIDE in $\mathbb{R}^n$, but with a finite number of independent Brownian motions in $\mathbb{R}^d$.

Large deviation theory is the study of exponential decay of probabilities of rare events with respect to an associated parameter. In recent years, there has been increased interest in the topic of large deviations

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as the large deviations estimates have proved to be the crucial tool required to handle many questions in engineering, statistical mechanics, population biology, mathematical finance and applied probability [9]. For instance, it is helpful to calculate the entropy in statistical mechanics, for both equilibrium and non-equilibrium systems [13]. The paper [22] by Klebaner and Liptser establishes the large deviation principle for a stochastic Lotka-Volterra model and is used to obtain a bound for the asymptotics of the time to extinction of prey population. Bertini et al. [2] applied large deviations to study the current fluctuations in lattice gases in the hydrodynamic sealing limit.

The general abstract framework for the Large Deviation Principle (LDP) was first proposed by Varadhan [34] in 1966. Subsequently, their applications to Stochastic Differential Equations (SDEs) driven by finitely many Brownian motions were first studied by Freidlin and Wentzell [15]. Da Prato and Zabczyk [8] and Peszat [29] extended the theory to infinite dimensional diffusions and stochastic partial differential equations under global Lipschitz condition on the nonlinear term. The idea of Freidlin-Wentzell type LDP usually relies on first approximating the original problem by time discretization so that LDP can be shown for the resulting simpler problems via contraction principle and then showing that LDP holds in the limit. The reader may find many works on the large deviation estimates for a class of infinite dimensional SDEs [5, 8, 21, 26] following the work of Freidlin and Wentzell. The most difficult part of large deviation analysis based on the standard approximation method is establishing the exponential continuity in probability and exponential tightness. Later Dupuis and Ellis [12] combined weak convergence methods to the stochastic control approach developed earlier by Fleming [14] to the large deviation theory. The advantage of this method is that one can avoid the difficulties in proving the large deviation estimates based on discretization and approximation arguments. Several recent papers have studied the LDP using weak convergence approach for the distribution of solution of infinite dimensional stochastic differential equations: Sritharan and Sundar for two-dimensional Navier-Stokes Equation in [30], Liu for stochastic evolution equations with small multiplicative noise in [23], Chiarini and Fischer for small noise Itô processes in [7] (to mention a few). See also [4, 16, 25, 32]. In recent years fractional order models are used as a successful tool for describing complex dynamical systems that cannot be well illustrated using ordinary differential and integral operators. If the effect of uncertainity is also considered, such systems can be modelled by stochastic fractional differential equations and stochastic fractional integro-differential equations. A Freidlin-Wentzell type LDP for a stochastic fractional integro-differential equations is studied in [33] using weak convergence approach.

The aim of this paper is to establish the LDP using weak convergence approach for a stochastic nonlinear integrodifferential equation of the Itô type perturbed by a finite number of independent Gaussian noise terms. In that sense the equation here is more general and considerably more difficult to study. For more works on the LDP for stochastic equations with multiple randomness the reader may refer to papers [6, 10, 36].

This paper is organized as follows. In Section 2, we introduce stochastic integrodifferential equations of the Itô type with multiple randomness. The existence and uniqueness of strong solution to the equation is also discussed here. We give the basic definitions and theorems of the large deviations theory in Section 3. The main result of this paper, the large deviation estimates for a general SIDE of the Itô type with multiple randomness is established in the last section. A simple example is also given at the end to illustrate the result proved in the present work.

2. Stochastic Integrodifferential Equations of the Itô type

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with an increasing family $\{\mathcal{F}_t\}_{0\leq t\leq T}$ of sub σ-algebras of $\mathcal{F}$ satisfying the usual conditions of right continuity and $\mathbf{P}$-completeness. Here we consider a general class of

Itô type SIDEs of the form:

$$\begin{cases} \mathrm{d}X^\epsilon(t) = b\Big(t, X^\epsilon(t), \int_0^t f_1(t,s,X^\epsilon(s))\mathrm{d}s, \dots, \int_0^t f_p(t,s,X^\epsilon(s))\mathrm{d}s\Big)\mathrm{d}t \\ \qquad\qquad + \sqrt{\epsilon}\sigma\Big(t, X^\epsilon(t),\ \sqrt{\epsilon}\int_0^t g_1(t,s,X^\epsilon(s))\mathrm{d}W_1(s), \dots,\ \sqrt{\epsilon}\int_0^t g_q(t,s,X^\epsilon(s))\mathrm{d}W_q(s)\Big)\mathrm{d}W(t), t \in (0,T], \\ X^\epsilon(0) = X_0, \end{cases} \tag{2.1}$$

where $W_1(t), W_2(t), \dots, W_q(t), W(t)$ are independent d-dimensional Brownian motions on $(\Omega, \mathcal{F}, \mathbf{P})$, ϵ is positive and $X_0 \in \mathbb{R}^n$ is deterministic. Also let $J = [0,T]$ and $\|\cdot\|$ denote the norm in the respective spaces. We assume the functions

$$b : J \times \mathbb{R}^n \times (\mathbb{R}^m)^p \rightarrow \mathbb{R}^n, \tag{2.2}$$
$$\sigma : J \times \mathbb{R}^n \times (\mathbb{R}^m)^q \rightarrow \mathbb{R}^{n\times d}, \tag{2.3}$$
$$f_i : J \times J \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \tag{2.4}$$
$$g_j : J \times J \times \mathbb{R}^n \rightarrow \mathbb{R}^{m\times d}, \tag{2.5}$$

$i = 1,2,\dots,p;\ j = 1,2,\dots,q$ are Borel measurable functions satisfying Lipschitz condition and the standard linear growth condition. i.e., there exist positive constants $L_b, L_\sigma, L_{f_i}, L_{g_j}$,
$K_b, K_\sigma, K_{f_i}, K_{g_j}$ such that, for all $x, y \in \mathbb{R}^n,\ x_i,\ y_i,\ x_j,\ y_j \in \mathbb{R}^m,\ i = 1,2,\dots,p,\ j = 1,2,\dots,q$ and $s, t \in J$,

$$\left.\begin{aligned} \|b(t,x,x_1,\dots,x_p) - b(t,y,y_1,\dots,y_p)\| &\le L_b\Big(\|x-y\| + \sum_{i=1}^p \|x_i - y_i\|\Big), \\ \|\sigma(t,x,x_1,\dots,x_q) - \sigma(t,y,y_1,\dots,y_q)\| &\le L_\sigma\Big(\|x-y\| + \sum_{j=1}^q \|x_j - y_j\|\Big), \\ \|f_i(t,s,x) - f_i(t,s,y)\| &\le L_{f_i}(\|x-y\|), \\ \|g_j(t,s,x) - g_j(t,s,y)\| &\le L_{g_j}(\|x-y\|), \end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned} \|b(t,x,x_1,\dots,x_p)\|^2 &\le K_b\Big(1 + \|x\|^2 + \sum_{i=1}^p \|x_i\|^2\Big), \\ \|\sigma(t,x,x_1,\dots,x_q)\|^2 &\le K_\sigma\Big(1 + \|x\|^2 + \sum_{j=1}^q \|x_j\|^2\Big), \\ \|f_i(t,s,x)\|^2 &\le K_{f_i}\Big(1 + \|x\|^2\Big), \\ \|g_j(t,s,x)\|^2 &\le K_{g_j}\Big(1 + \|x\|^2\Big). \end{aligned}\right\} \tag{2.7}$$

Under the assumptions (2.6) and (2.7) the existence and uniqueness of strong solution to equation (2.1) can be established by implementing the Picard's iteration technique as in [28] and we denote the solution process by $\{X^\epsilon(t) : \epsilon > 0\}$, which is defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

3. Large Deviation Principle

In this section we review some basic concepts and results from the large deviations theory. Let $\{Y^\epsilon : \epsilon > 0\}$ be a family of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in a Polish space E (i.e., a complete separable metric space).

Definition 3.1. *A **rate function** I is a lower semicontinuous mapping $I : E \rightarrow [0,\infty]$. A **good rate function** is a rate function for which the level sets $K_M = \{x \in E : I(x) \le M\}$ are compact subsets of E for each $M \in [0,\infty)$.*

Definition 3.2. *Let I be a rate function on E. We say the family $\{Y^\epsilon : \epsilon > 0\}$ satisfies the **large deviation principle (LDP)** with rate function I if the following two conditions hold:*

(i) For each closed subset F of E,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbf{P}(Y^\epsilon \in F) \le -\inf_{x\in F} I(x).$$

(ii) For each open subset G of E,

$$\liminf_{\epsilon\to 0} \epsilon \log \mathbf{P}(Y^\epsilon \in G) \ge -\inf_{x\in G} I(x).$$

Definition 3.3. *Let I be a rate function on E. A family $\{Y^\epsilon : \epsilon > 0\}$ is said to satisfy the* ***Laplace principle*** *on E with rate function I if for each real valued bounded continuous functions h defined on E,*

$$\lim_{\epsilon\to 0} \epsilon \log \mathbf{E}\Big\{ \exp[-\frac{1}{\epsilon} h(X^\epsilon)]\Big\} = -\inf_{x\in X}\Big\{h(x) + I(x)\Big\}.$$

One of the main results in the theory of large deviations is the equivalence between the Laplace principle and the large deviation principle.

Theorem 3.1. *The family $\{Y^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle with good rate function I on a Polish space if and only if $\{Y^\epsilon : \epsilon > 0\}$ satisfies the large deviation principle with the same rate function I.*

For a proof we refer the reader to Theorem 1.2.1 and Theorem 1.2.3 in [12].
Let

$$\mathcal{A} = \Big\{v : v \text{ is } (\mathbb{R}^d)^{q+1}\text{- valued } \mathcal{F}_t \text{ - predictable process and } \int_0^T \|v(s,\omega)\|^2 \mathrm{d}s < \infty \text{ a.s.}\Big\}.$$

Define the set S_N of bounded deterministic controls as

$$S_N = \Big\{v \in L^2([0,T]; (\mathbb{R}^d)^{q+1}) : \int_0^T \|v(s)\|^2 \mathrm{d}s \le N\Big\}.$$

Define $\mathcal{A}_N$ as the set of bounded stochastic controls by

$$\mathcal{A}_N = \Big\{v \in \mathcal{A} : v(\omega) \in S_N \ \mathbf{P} - a.s.\Big\}.$$

Here $L^2([0,T]; (\mathbb{R}^d)^{q+1})$ is the space of all $(\mathbb{R}^d)^{q+1}$ - valued square integrable functions on $J = [0,T]$. Then S_N endowed with the weak topology in $L^2(J; (\mathbb{R}^d)^{q+1})$ is a compact Polish Space (see [11]). For $\epsilon > 0$, let $\mathcal{G}^\epsilon : C(J; (\mathbb{R}^d)^{q+1}) \to C(J; \mathbb{R}^n)$ be a measurable map. Define $Y^\epsilon = \mathcal{G}^\epsilon(\beta(\cdot))$, where $\beta : J \to (\mathbb{R}^d)^{q+1}$ is given by $\beta(t) = (W_1(t), W_2(t), \ldots, W_q(t), W(t))$. We are interested in the large deviation principle for Y^ϵ as $\epsilon \to 0$. We formulate (following from Theorem 4.4 in [3]) the following sufficient condition for the Laplace principle of $\{Y^\epsilon\}$ as $\epsilon \to 0$.

Assumption 1. *Suppose that there exist a measurable map $\mathcal{G}^0 : C(J; (\mathbb{R}^d)^{q+1}) \to C(J; \mathbb{R}^n)$ such that the following two conditions hold:*

(i) Let $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. If v^ϵ converge in distribution as S_N-valued random elements to v, then

$$\mathcal{G}^\epsilon\Big(\beta(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v^\epsilon(s)\mathrm{d}s\Big) \to \mathcal{G}^0\Big(\int_0^\cdot v(s)\mathrm{d}s\Big)$$

in distribution as $\epsilon \to 0$.

(ii) For each $N < \infty$, the set

$$K_N = \Big\{\mathcal{G}^0\Big(\int_0^\cdot v(s)\mathrm{d}s\Big) : v \in S_N\Big\}$$

is a compact subset of $C(J; \mathbb{R}^n)$.

4. Large Deviations Result for SIDE of the Itô Type

The aim of this section is to prove the LDP for the solution processes $\{X^\epsilon : \epsilon > 0\}$ of the equation (2.1). The small noise coefficients occuring in the equation are multiplicative and the variational method developed by Budhiraja and Dupuis [3] is implemented here to establish the LDP. It follows from the Yamada-Watanabe theorem ([35]) that there exists a Borel measurable function $G^\epsilon : C(J;(\mathbb{R}^d)^{q+1}) \to C(J;\mathbb{R}^n)$ such that $X^\epsilon = G^\epsilon(\beta(\cdot))$ a.s. We show that the function G^ϵ satisfies Assumption 1 by proving the following lemmas. For verifying condition (ii) of Assumption 1 we introduce the skeleton equation associated with equation (2.1):

$$\begin{cases} dz_\nu(t) = b\Big(t, z_\nu(t), \int_0^t f_1(t,s,z_\nu(s))\mathrm{d}s, \ldots, \int_0^t f_p(t,s,z_\nu(s))\mathrm{d}s\Big)\mathrm{d}t \\ \qquad\qquad +\sigma\Big(t, z_\nu(t), \int_0^t g_1(t,s,z_\nu(s))v_1(s)\mathrm{d}s, \ldots, \\ \qquad\qquad\qquad \int_0^t g_q(t,s,z_\nu(s))v_q(s)\mathrm{d}s\Big)v(t)\mathrm{d}t, t \in (0,T], \\ z_\nu(0) = X_0, \end{cases} \tag{4.1}$$

with solution $z_\nu(t)$, where $\nu = (v_1, v_2, \ldots, v_q, v) \in L^2(J;(\mathbb{R}^d)^{q+1})$. It may be noted that as a result of the Lipschitz continuity and linear growth conditions in (2.6) and (2.7), existence and uniqueness of solution of equation (4.1) is standard.

Lemma 4.1. *(Compactness)*
Define $G^0 : C(J;(\mathbb{R}^d)^{q+1}) \to C(J;\mathbb{R}^n)$ *by*

$$G^0(g) = \begin{cases} z_\nu, & if\ g = \int_0^\cdot \nu(s)\mathrm{d}s \ \textit{for some}\ \nu \in L^2(J;(\mathbb{R}^d)^{q+1}), \\ 0, & otherwise, \end{cases}$$

where z_ν *denotes the solution of the equation (4.1). Then, for each* $N < \infty$*, the set*

$$K_N = \left\{ G^0\Big(\int_0^\cdot \nu(s)\mathrm{d}s\Big) : \nu \in S_N \right\}$$

is a compact subset of $C(J;\mathbb{R}^n)$*.*

Proof. Let (ν_k) be a sequence in S_N such that $\nu_k \to \nu$ weakly in $L^2(J;(\mathbb{R}^d)^{q+1})$. Viewing ν_k, ν as elements of $L^2(J;(\mathbb{R}^d)^{q+1})$, we write

$$\nu_k(t) = (v_{k_1}(t), v_{k_2}(t), \ldots, v_{k_q}(t), v_k(t)),$$
$$\nu(t) = (v_1(t), v_2(t), \ldots, v_q(t), v(t)),$$

where $v_{k_i}(t), v_k(t), v_i(t), v(t) \in \mathbb{R}^d, i = 1, 2, \ldots, q$.
We prove that the map $\nu \to z_\nu$ from S_N to $C(J;\mathbb{R}^n)$ is continuous. Now,

$$\begin{aligned} z_{\nu_k}(t) - z_\nu(t) = &\int_0^t \Big[b\Big(s, z_{\nu_k}(s), \int_0^s f_1(s,u,z_{\nu_k}(u))\mathrm{d}u, \ldots, \int_0^s f_p(s,u,z_{\nu_k}(u))\mathrm{d}u\Big) \\ &\quad -b\Big(s, z_\nu(s), \int_0^s f_1(s,u,z_\nu(u))\mathrm{d}u, \ldots, \int_0^s f_p(s,u,z_\nu(u))\mathrm{d}u\Big)\Big]\mathrm{d}s \\ &+ \int_0^t \Big[\sigma\Big(s, z_{\nu_k}(s), \int_0^s g_1(s,u,z_{\nu_k}(u))v_{k_1}(u)\mathrm{d}u, \ldots, \int_0^s g_q(s,u,z_{\nu_k}(u))v_{k_q}(u)\mathrm{d}u\Big) \\ &\quad - \sigma\Big(s, z_\nu(s), \int_0^s g_1(s,u,z_\nu(u))v_1(u)\mathrm{d}u \ldots, \int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)\Big]v_k(s)\mathrm{d}s \\ &+ \int_0^t \Big[\sigma\Big(s, z_\nu(s), \int_0^s g_1(s,u,z_\nu(u))v_1(u)\mathrm{d}u, \ldots, \int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)\Big](v_k(s) - v(s))\mathrm{d}s. \end{aligned}$$

Define

$$\kappa^k(t) = \sup_{0\le t_1\le t} \|z_{\nu_k}(t_1) - z_\nu(t_1)\|.$$

Using the Lipschitz continuity of the functions $b(\cdot)$ and $\sigma(\cdot)$, we get

$$\begin{aligned}\|z_{\nu_k}(t_1) - z_\nu(t_1)\| \le & \int_0^t L_b\Big[\|z_{\nu_k}(s) - z_\nu(s)\| \\ & + \sum_{i=1}^{p} \Big\|\int_0^s f_i(s,u,z_{\nu_k}(u))\mathrm{d}u - \int_0^s f_i(s,u,z_\nu(u))\mathrm{d}u\Big\|\Big]\mathrm{d}s \\ & + \|\zeta^k(t_1)\| + \int_0^t L_\sigma\Big[\|z_{\nu_k}(s) - z_\nu(s)\| \\ & + \sum_{j=1}^{q} \Big\|\int_0^s g_j(s,u,z_{\nu_k}(u))v_{k_j}(u)\mathrm{d}u - \int_0^s g_j(s,u,z_\nu(u))v_j(u)\mathrm{d}u\Big\|\Big]\|v_k(s)\|\mathrm{d}s \end{aligned} \tag{4.2}$$

where,

$$\zeta^k(t) = \int_0^t \sigma\Big(s, z_\nu(s), \ldots, \int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)(v_k(s) - v(s))\mathrm{d}s.$$

Now,

$$\begin{aligned}&\Big\|\int_0^s g_j(s,u,z_{\nu_k}(u))v_{k_j}(u)\mathrm{d}u - \int_0^s g_j(s,u,z_\nu(u))v_j(u)\mathrm{d}u\Big\| \\ &\qquad\le \Big\|\int_0^s g_j(s,u,z_{\nu_k}(u))v_{k_j}(u)\mathrm{d}u - \int_0^s g_j(s,u,z_\nu(u))v_{k_j}(u)\mathrm{d}u\Big\| \\ &\qquad + \Big\|\int_0^s g_j(s,u,z_\nu(u))v_{k_j}(u)\mathrm{d}u - \int_0^s g_j(s,u,z_\nu(u))v_j(u)\mathrm{d}u\Big\|.\end{aligned}$$

Define

$$\zeta^k_{g_j}(t) = \int_0^t g_j(t,u,z_\nu(u))(v_{k_j}(u) - v_j(u))\mathrm{d}u, \;\; j = 1, 2, \ldots, q.$$

Then $\zeta^k_{g_j}(t)$ is differentiable w.r.t.t and hence continuous on J. Since $\nu_k \in S_N$ for all k, an application of Hölder's inequality gives

$$\int_0^s \|\nu_k(u)\|\mathrm{d}u \le \sqrt{TN}.$$

Also $\|v_{k_j}(u)\| \le \|\nu_k\|$ for all $j = 1, 2, \ldots, q$ and therefore,

$$\int_0^s \|v_{k_j}(u)\|\mathrm{d}u \le \sqrt{TN}.$$

Then, using the Lipschitz continuity of the functions $f_i(\cdot)$ and $g_j(\cdot)$ we obtain

$$\begin{aligned}
\|z_{v_k}(t_1) - z_v(t_1)\| \leq & \int_0^t L_b(1 + pL_f T)\kappa^k(s)\mathrm{d}s \\
& + \int_0^t L_\sigma(1 + qL_g \sqrt{TN})\kappa^k(s)\|v_k(s)\|\mathrm{d}s \\
& + L_\sigma \sqrt{TN} \sum_{j=1}^{q} \sup_{0\leq s\leq t} \|\zeta^k_{g_j}(s)\| + \sup_{0\leq t_1\leq t} \|\zeta^k(t_1)\| \\
\leq & \int_0^t C_1\kappa^k(s)\mathrm{d}s + \int_0^t C_2\kappa^k(s)\|v_k(s)\|\mathrm{d}s + L_\sigma \sqrt{TN} \sum_{j=1}^{q} \sup_{0\leq s\leq t} \|\zeta^k_{g_j}(s)\| \\
& + \sup_{0\leq t_1\leq t} \|\zeta^k(t_1)\|,
\end{aligned}$$

where $L_f = \max_{i=1,2,\ldots,p} L_{f_i}, L_g = \max_{j=1,2,\ldots,q} L_{g_j}, C_1 = L_b(1 + pL_f T)$ and $C_2 = L_\sigma(1 + qL_g \sqrt{TN})$.
Therefore,

$$\kappa^k(t) \leq \int_0^t \Big[C_1 + C_2\|v_k(s)\|\Big]\kappa^k(s)\mathrm{d}s + \Big[L_\sigma \sqrt{TN} \sum_{j=1}^{q} \sup_{0\leq s\leq t} \|\zeta^k_{g_j}(s)\| + \sup_{0\leq t_1\leq t} \|\zeta^k(t_1)\|\Big].$$

By Gronwall's inequality,

$$\kappa^k(t) \leq \Big[L_\sigma \sqrt{TN} \sum_{j=1}^{q} \sup_{0\leq s\leq t} \|\zeta^k_{g_j}(s)\| + \sup_{0\leq t_1\leq t} \|\zeta^k(t_1)\|\Big] \exp \int_0^t [C_1 + C_2\|v_k(s)\|]\mathrm{d}s.$$

Thus,

$$\begin{aligned}
\|z_{v_k} - z_v\|_{C(J;\mathbb{R}^n)} &= \sup_{t\in J} \|z_{v_k}(t) - z_v(t)\| \\
&\leq \Big[L_\sigma \sqrt{TN} \sum_{j=1}^{q} \sup_{t\in J} \|\zeta^k_{g_j}(t)\| + \sup_{t\in J} \|\zeta^k(t)\|\Big] \exp[C_1 T + C_2 \sqrt{TN}]. \qquad (4.3)
\end{aligned}$$

By the linear growth of σ, we have

$$\begin{aligned}
\sup_{t\in J} \left\|\zeta^k(t)\right\| \leq & \int_0^T \left\|\sigma\Big(s, z_v(s), \int_0^s g_1(s, u, z_v(u))v_1(u)\mathrm{d}u, \ldots, \int_0^s g_q(s, u, z_v(u))v_q(u)\mathrm{d}u\Big)(v_k(s) - v(s))\right\| \mathrm{d}s \\
\leq & \Big(\int_0^T \left\|\sigma\Big(s, z_v(s), \int_0^s g_1(s, u, z_v(u))v_1(u)\mathrm{d}u, \ldots, \int_0^s g_q(s, u, z_v(u))v_q(u)\mathrm{d}u\Big)\right\|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \\
& \times \Big(\int_0^T \|v_k(s) - v(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}} \\
\leq & \, C < \infty. \qquad (4.4)
\end{aligned}$$

for some positive constant C independent of k. For $t_1, t_2 \in J$ with $t_2 \leq t_1$, we also have

$$\|\zeta^k(t_1) - \zeta^k(t_2)\| \leq K \sqrt{t_1 - t_2}, \qquad (4.5)$$

where K is a fixed constant. From equations (4.4) and (4.5), it follows that $\{\zeta^k(t)\}$ is a family of functions on J that satisfies a Hölder condition of order $\frac{1}{2}$ and are uniformly bounded by C. So by Arzela- Ascoli theorem there exist a subsequence which converges uniformly in $C(J;\mathbb{R}^n)$. Also $\{\zeta^k(t)\}$ converges to zero for each t as ν_k converges weakly to ν in $L^2(J;(\mathbb{R}^d)^{q+1})$. Hence

$$\lim_{k\to\infty}\sup_{t\in J}\|\zeta^k(t)\| = 0. \tag{4.6}$$

Following the same argument we get,

$$\lim_{k\to\infty}\sup_{t\in J}\|\zeta^k_{g_j}(t)\| = 0;\;\; j = 1,2,\dots,q. \tag{4.7}$$

From equations (4.3), (4.6) and (4.7) it follows that the map $\nu \to z_\nu$ is continuous. The space S_N is compact. Therefore K_N is compact for each $N < \infty$. □

Now it remains to prove the condition (i) of Assumption 1. For this consider the controlled stochastic equation with control $\nu^\epsilon = (v_1^\epsilon, v_2^\epsilon, \dots, v_q^\epsilon, v^\epsilon) \in L^2(J;(\mathbb{R}^d)^{q+1}), \epsilon > 0$,

$$\begin{aligned}
dX^\epsilon_{\nu^\epsilon}(t) =& b\Big(t, X^\epsilon_{\nu^\epsilon}(t), \int_0^t f_1(t,s,X^\epsilon_{\nu^\epsilon}(s))ds, \dots, \int_0^t f_p(t,s,X^\epsilon_{\nu^\epsilon}(s))ds\Big)dt\\
&+\sigma\Big(t, X^\epsilon_{\nu^\epsilon}(t),\ \sqrt{\epsilon}\int_0^t g_1(t,s,X^\epsilon_{\nu^\epsilon}(s))dW_1(s) + \int_0^t g_1(t,s,X^\epsilon_{\nu^\epsilon}(s))v_1^\epsilon(s)ds, \dots,\\
&\quad \sqrt{\epsilon}\int_0^t g_q(t,s,X^\epsilon_{\nu^\epsilon}(s))dW_q(s) + \int_0^t g_q(t,s,X^\epsilon_{\nu^\epsilon}(s))v_q^\epsilon(s)ds\Big)v^\epsilon(t)dt\\
&+\sqrt{\epsilon}\sigma\Big(t, X^\epsilon_{\nu^\epsilon}(t),\ \sqrt{\epsilon}\int_0^t g_1(t,s,X^\epsilon_{\nu^\epsilon}(s))dW_1(s) + \int_0^t g_1(t,s,X^\epsilon_{\nu^\epsilon}(s))v_1^\epsilon(s)ds, \dots,\\
&\quad \sqrt{\epsilon}\int_0^t g_q(t,s,X^\epsilon_{\nu^\epsilon}(s))dW_q(s) + \int_0^t g_q(t,s,X^\epsilon_{\nu^\epsilon}(s))v_q^\epsilon(s)ds\Big)dW(t), t\in(0,T],\\
X^\epsilon_{\nu^\epsilon}(0) =& X_0.
\end{aligned} \tag{4.8}$$

The following lemma asserts the existence of unique strong solution of the above equation and is a direct consequence of Girsanov's theorem.

Lemma 4.2. *Let $\{\nu^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. For $\epsilon > 0$, define*

$$X^\epsilon_{\nu^\epsilon} = G^\epsilon\Big(\beta(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^\cdot \nu^\epsilon(s)ds\Big).$$

Then $X^\epsilon_{\nu^\epsilon}$ is the unique solution of (4.8).

Lemma 4.3. *(Weak Convergence)*
Let $\{\nu^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. Assume ν^ϵ converge to ν in distribution as S_N-valued random elements, then

$$G^\epsilon\Big(\beta(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^\cdot \nu^\epsilon(s)ds\Big) \to G^0\Big(\int_0^\cdot \nu(s)ds\Big)$$

in distribution as $\epsilon \to 0$.

Proof. Applying Itô's formula to $\|X^\epsilon_{\nu^\epsilon(t)} - z_\nu(t)\|^2$, we get

$$\|X^\epsilon_{\nu^\epsilon(t)} - z_\nu(t)\|^2 = I_1(t) + I_2(t) + I_3(t) + I_4(t), \tag{4.9}$$

where

$$I_1(t) = 2\int_0^t \Big(X^{\epsilon}_{\nu^\epsilon}(s) - z_\nu(s)\Big)\cdot\Big\{b\Big(s, X^{\epsilon}_{\nu^\epsilon}(s), \dots, \int_0^s f_p(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}u\Big) - b\Big(s, z_\nu(s), \int_0^s f_1(s,u,z_\nu(u))\mathrm{d}u, \dots, \int_0^s f_p(s,u,z_\nu(u))\mathrm{d}u\Big)\Big\}\mathrm{d}s,$$

$$\begin{aligned}I_2(t) = 2\int_0^t \Big(X^{\epsilon}_{\nu^\epsilon}(s) - z_\nu(s)\Big)\cdot\Big\{\sigma\Big(s, X^{\epsilon}_{\nu^\epsilon}(s),\ \sqrt{\epsilon}\int_0^s g_1(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}W_1(u)+ \\ \int_0^s g_1(s,u,X^{\epsilon}_{\nu^\epsilon}(u))v^{\epsilon}_1(u)\mathrm{d}u, \dots, \sqrt{\epsilon}\int_0^s g_q(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}W_q(u) \\ + \int_0^s g_q(s,u,X^{\epsilon}_{\nu^\epsilon}(u))v^{\epsilon}_q(u)\mathrm{d}u\Big)v^{\epsilon}(s) \\ -\sigma\Big(s, z_\nu(s), \int_0^s g_1(s,u,z_\nu(u))v_1(u)\mathrm{d}u, \dots, \int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)v(s)\Big\}\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}I_3(t) =\epsilon\int_0^t \Big\|\sigma\Big(s, X^{\epsilon}_{\nu^\epsilon}(s),\ \sqrt{\epsilon}\int_0^s g_1(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}W_1(u) + \int_0^s g_1(s,u,X^{\epsilon}_{\nu^\epsilon}(u))v^{\epsilon}_1(u)\mathrm{d}u, \\ \dots, \sqrt{\epsilon}\int_0^s g_q(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}W_q(u) + \int_0^s g_q(s,u,X^{\epsilon}_{\nu^\epsilon}(u))v^{\epsilon}_q(u)\mathrm{d}u\Big)\Big\|^2 \mathrm{d}s,\end{aligned}$$

$$\begin{aligned}I_4(t) = 2\sqrt{\epsilon}\int_0^t \Big(X^{\epsilon}_{\nu^\epsilon}(s) - z_\nu(s)\Big)\cdot\sigma\Big(s, X^{\epsilon}_{\nu^\epsilon}(s), \dots, \\ \sqrt{\epsilon}\int_0^s g_q(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}W_q(u) + \int_0^s g_q(s,u,X^{\epsilon}_{\nu^\epsilon}(u))v^{\epsilon}_q(u)\mathrm{d}u\Big)\mathrm{d}W(s).\end{aligned}$$

Define

$$\kappa^{\epsilon}(s) = \sup_{0\le u\le s}\|X^{\epsilon}_{\nu^\epsilon}(u) - z_\nu(u)\|^2.$$

Since the function $b(\cdot)$ is Lipschitz continuous, we can write

$$\begin{aligned}I_1(t) \le& 2\int_0^t \|X^{\epsilon}_{\nu^\epsilon}(s) - z_\nu(s)\|\cdot L_b\Big\{\|X^{\epsilon}_{\nu^\epsilon}(s) - z_\nu(s)\| \\ &+ \sum_{i=1}^p \Big\|\int_0^s f_i(s,u,X^{\epsilon}_{\nu^\epsilon}(u))\mathrm{d}u - \int_0^s f_i(s,u,z_\nu(u))\mathrm{d}u\Big\|\Big\}\mathrm{d}s \\ \le& 2L_b\int_0^t (1+pL_fT)\kappa^{\epsilon}(s)\mathrm{d}s.\end{aligned}$$

Hence,

$$\mathbf{E}(I_1(t)) \le C_1\mathbf{E}\Big(\int_0^t \kappa^{\epsilon}(s)\mathrm{d}s\Big), \tag{4.10}$$

where $C_1 = 2L_b(1+pL_fT)$.
For simplicity, we write

$$I_2(t) = I_{21}(t) + I_{22}(t),$$

where

$$
\begin{aligned}
I_{21}(t) =& 2\int_0^t (X^\epsilon_{\nu^\epsilon}(s) - z_\nu(s)) \cdot \Big\{\sigma\Big(s, X^\epsilon_{\nu^\epsilon}(s), \dots, \\
&\qquad \sqrt{\epsilon}\int_0^s g_q(s,u,X^\epsilon_{\nu^\epsilon}(u))\mathrm{d}W_q(u) + \int_0^s g_q(s,u,X^\epsilon_{\nu^\epsilon}(u))v^\epsilon_q(u)\mathrm{d}u\Big) \\
&-\sigma\Big(s, z_\nu(s), \int_0^s g_1(s,u,z_\nu(u))v_1(u)\mathrm{d}u, \dots, \int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)\Big\}v^\epsilon(s)\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
I_{22}(t) = 2\int_0^t (X^\epsilon_{\nu^\epsilon}(s) - z_\nu(s)) \cdot \Big\{\sigma\Big(s, z_\nu(s), \int_0^s g_1(s,u,z_\nu(u))v_1(u)\mathrm{d}u, \dots, \\
\int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)\Big\}(v^\epsilon(s) - v(s))\mathrm{d}s.
\end{aligned}
$$

Define

$$
\zeta^\epsilon(t) = \int_0^t \sigma\Big(s, z_\nu(s), \dots, \int_0^s g_q(s,u,z_\nu(u))v_q(u)\mathrm{d}u\Big)(v^\epsilon(s) - v(s))\mathrm{d}s,
$$

$$
\zeta^\epsilon_{g_j}(t) = \int_0^t g_j(t,u,z_\nu(u))(v^\epsilon_j(u) - v_j(u))\mathrm{d}u.
$$

Using the Lipschitz continuity of the function $\sigma(\cdot)$ and applying Young's inequality we get,

$$
\begin{aligned}
I_{21}(t) \le& 2L_\sigma \int_0^t \Big[\frac{1}{16L_\sigma N}\|X^\epsilon_{\nu^\epsilon}(s) - z_\nu(s)\|^2\|v^\epsilon(s)\|^2 + 4L_\sigma N\Big\{\|X^\epsilon_{\nu^\epsilon}(s) - z_\nu(s)\| \\
&+ \sum_{j=1}^q \|\zeta^\epsilon_{g_j}(s)\| + \sum_{j=1}^q \Big\|\sqrt{\epsilon}\int_0^s g_j(s,u,X^\epsilon_{\nu^\epsilon}(u))\mathrm{d}W_j(u)\Big\| \\
&+ \sum_{j=1}^q \int_0^s \Big\|\Big[g_j(s,u,X^\epsilon_{\nu^\epsilon}(u)) - g_j(s,u,z_\nu(u))\Big]v^\epsilon_j(u)\mathrm{d}u\Big\|\Big\}^2\Big]\mathrm{d}s.
\end{aligned}
$$

Applying Tchebychef's inequality and Burkholder-Davis-Gundy inequality in the above results in,

$$
\begin{aligned}
\mathbf{E}(I_{21}(t)) \le& \frac{1}{8}\mathbf{E}(\kappa^\epsilon(t)) + 24L_\sigma^2 NC_*\mathbf{E}\int_0^t \kappa^\epsilon(s)\mathrm{d}s + 24L_\sigma^2 Nq\sum_{j=1}^q \mathbf{E}\Big(\sup_{0\le s\le t}\|\zeta^\epsilon_{g_j}(s)\|^2\Big) \\
&+ 96L_\sigma^2 NT\epsilon qK_g\mathbf{E}\Big(\int_0^t \Big(1 + \|X^\epsilon_{\nu^\epsilon}(u)\|^2\Big)\mathrm{d}s\Big), \qquad (4.11)
\end{aligned}
$$

where C_* is a positive constant. Then,

$$
\begin{aligned}
I_{22}(t) = 2(X^{\epsilon}_{\nu^{\epsilon}}(t) - z_{\nu}(t))\zeta^{\epsilon}(t) - 2\int_0^t \Big[b\Big(s, X^{\epsilon}_{\nu^{\epsilon}}(s), \dots, \int_0^s f_p(s,u,X^{\epsilon}_{v^{\epsilon}}(u))\mathrm{d}u\Big)\mathrm{d}s \\
- b\Big(s, z_{\nu}(s), \int_0^s f_1(s,u,z_{\nu}(u))\mathrm{d}u, \dots, \int_0^s f_p(s,u,z_{\nu}(u))\mathrm{d}u\Big)\Big]\zeta^{\epsilon}(s)\mathrm{d}s \\
-2\int_0^t \Big[\sigma\Big(s, X^{\epsilon}_{\nu^{\epsilon}}(s), \sqrt{\epsilon}\int_0^s g_1(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))\mathrm{d}W_1(u) + \int_0^s g_1(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))v^{\epsilon}_1(u)\mathrm{d}u, \\
\dots, \sqrt{\epsilon}\int_0^s g_q(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))\mathrm{d}W_q(u) + \int_0^s g_q(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))v^{\epsilon}_q(u)\mathrm{d}u\Big)v^{\epsilon}(s) \\
- \sigma\Big(s, z_{\nu}(s), \dots, \int_0^s g_q(s,u,z_{\nu}(u))v_q(u)\mathrm{d}u\Big)v(s)\Big]\zeta^{\epsilon}(s)\mathrm{d}s \\
-2\sqrt{\epsilon}\int_0^t \sigma\Big(s, X^{\epsilon}_{\nu^{\epsilon}}(s), \sqrt{\epsilon}\int_0^s g_1(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))\mathrm{d}W_1(u) + \int_0^s g_1(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))v^{\epsilon}_1(u)\mathrm{d}u, \\
\dots, \sqrt{\epsilon}\int_0^s g_q(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))\mathrm{d}W_q(u) + \int_0^s g_q(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))v^{\epsilon}_q(u)\mathrm{d}u\Big)\zeta^{\epsilon}(s)\mathrm{d}W(s).
\end{aligned}
$$

The functions $\sigma(\cdot)$, $g_j(\cdot)$, $j = 1, 2, \dots, q$ satisfy standard linear growth conditions. The arithmetic mean of non negative numbers is never less their geometric mean. Using these facts along with Burkhölder-Davis-Gundy inequality, we get the following estimate:

$$
\begin{aligned}
\mathbf{E}(I_{22}(t)) \leq &\frac{1}{4}\mathbf{E}(\kappa^{\epsilon}(t)) + 4\mathbf{E}(\|\zeta^{\epsilon}(t)\|^2) \\
&+2\mathbf{E}\Big\{\sup_{0\leq s\leq t}\zeta^{\epsilon}(s)\Big[C_1\int_0^t (1 + \|X^{\epsilon}_{\nu^{\epsilon}}(s)\|)\mathrm{d}s + C_1\int_0^t (1 + \|X^{\epsilon}_{\nu}(s)\|)\mathrm{d}s\Big]\Big\} \\
&+2\mathbf{E}\Big\{\sup_{0\leq s\leq t}\zeta^{\epsilon}(s)\Big[N + C_{31}\int_0^t (1 + \|z_{\nu}(s)\|^2)\mathrm{d}s \\
&\quad + C_{32}\int_0^t (1 + \|X^{\epsilon}_{\nu^{\epsilon}}(s)\|^2)\mathrm{d}s + \epsilon C_{33}\int_0^t (1 + \|X^{\epsilon}_{\nu^{\epsilon}}(s)\|^2)\mathrm{d}s\Big]\Big\} \\
&+ \sqrt{2\epsilon}\mathbf{E}\Big\{\sup_{0\leq s\leq t}\zeta^{\epsilon}(s)\Big[C_{41}\int_0^t (1 + \|X^{\epsilon}_{\nu^{\epsilon}}(s)\|^2)\mathrm{d}s + \epsilon C_{42}\int_0^t (1 + \|X^{\epsilon}_{\nu^{\epsilon}}(s)\|^2)\mathrm{d}s\Big]\Big\} \\
&+ \sqrt{2\epsilon}\mathbf{E}(\sup_{0\leq s\leq t}\zeta^{\epsilon}(s)),
\end{aligned} \tag{4.12}
$$

the constants in (4.12) all being positive.
Using the linear growth condition of $\sigma(\cdot)$,

$$
\begin{aligned}
I_3(t) \leq &\epsilon K_{\sigma}\int_0^t [1 + \|X^{\epsilon}_{\nu^{\epsilon}}(s)\|^2]\mathrm{d}s \\
&+\epsilon K_{\sigma}\sum_{j=1}^{q}\int_0^t \left\| \sqrt{\epsilon}\int_0^s g_j(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))\mathrm{d}W_j(u) + \int_0^s g_j(s,u,X^{\epsilon}_{\nu^{\epsilon}}(u))v^{\epsilon}_j(u)\mathrm{d}u\right\|^2 \mathrm{d}s,
\end{aligned}
$$

and from Lemma 2 in [24] and using the fact that the functions $g_j(\cdot)$ satisfies the standard linear growth property we get,

$$\begin{aligned}\mathbf{E}(I_3(t)) \leq & \epsilon K_\sigma \mathbf{E}\int_0^t [1+\|X^\epsilon_{\nu^\epsilon}(s)\|]^2 \mathrm{d}s \\ & + \epsilon K_\sigma \sum_{j=1}^{q}\int_0^t M_j N \mathbf{E}\Big(\int_0^s K_{g_j}(1+\|X^\epsilon_{\nu^\epsilon}(u)\|)^2\mathrm{d}u\Big)\mathrm{d}s \\ \leq & \epsilon C_3 \mathbf{E}\int_0^t \Big(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2\Big)\mathrm{d}s, \end{aligned} \tag{4.13}$$

where $M_j, j = 1,2,\dots,q$ are the constants obtained on the application of the above mentioned lemma, $M := \max_{j=1,2,\dots,q} M_j, K_g := \max_{j=1,2,\dots,q} K_{g_j}$ and $C_3 = K_\sigma + qK_\sigma K_g MNT$.
Applying Burkhölder-Davis-Gundy inequality and using again the fact that the arithmetic mean (AM) of a list of non-negative real numbers is greater than or equal to their geometric mean (GM) of the same list we have,

$$\begin{aligned}\mathbf{E}\Big(\sup_{0\leq s\leq t}|I_4(s)|\Big) \leq & \, 2\sqrt{2\epsilon}\mathbf{E}\Big(\sup_{0\leq s\leq t}\|(X^\epsilon_{\nu^\epsilon}(s)-z_\nu(s))\|^2\int_0^t\Big\|\sigma\Big(s, X^\epsilon_{\nu^\epsilon}(s), \sqrt{\epsilon}\int_0^s g_1(s,u,X^\epsilon_{\nu^\epsilon}(u))\mathrm{d}W_1(u) \\ & + \int_0^s g_1(s,u,X^\epsilon_{\nu^\epsilon}(u))v^\epsilon_1(u)\mathrm{d}u, \dots, \sqrt{\epsilon}\int_0^s g_q(s,u,X^\epsilon_{\nu^\epsilon}(u))\mathrm{d}W_q(u) + \int_0^s g_q(s,u,X^\epsilon_{\nu^\epsilon}(u))v^\epsilon_q(u)\mathrm{d}u\Big)\Big\|^2\mathrm{d}s\Big)^{\frac{1}{2}} \\ \leq & \frac{1}{2}\mathbf{E}(\kappa^\epsilon(t)) + 4\epsilon C_3 \mathbf{E}\int_0^t(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2)\mathrm{d}s. \end{aligned} \tag{4.14}$$

Combining the estimates (4.10)-(4.14), we have from (4.9) the following estimate:

$$\begin{aligned}\mathbf{E}(\kappa^\epsilon(t)) \leq & C_1'\mathbf{E}\Big(\int_0^t \kappa^\epsilon(s)\mathrm{d}s\Big) + \Big[C_2'\sum_{j=1}^{q}\mathbf{E}(\sup_{0\leq s\leq t}\|\zeta^\epsilon_{g_j}(s)\|) \\ & + \epsilon C_3'\mathbf{E}\Big(\int_0^t(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2)\mathrm{d}s\Big) + C_4'\mathbf{E}\Big(\sup_{0\leq s\leq t}\|\zeta^\epsilon(s)\|^2\Big) + C_5'\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\Big) \\ & + C_6'\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\int_0^t(1+\|z_\nu(s)\|^2)\mathrm{d}s\Big) + C_7'\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\int_0^t(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2)\mathrm{d}s\Big) \\ & + \epsilon C_8'\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\int_0^t(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2)\mathrm{d}s\Big) + \sqrt{2\epsilon}\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\Big) \\ & + \sqrt{2\epsilon}C_9'\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\int_0^t(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2)\mathrm{d}s\Big) \\ & + \epsilon\sqrt{2\epsilon}C_{10}'\mathbf{E}\Big(\sup_{0\leq s\leq t}\zeta^\epsilon(s)\int_0^t(1+\|X^\epsilon_{\nu^\epsilon}(s)\|^2)\mathrm{d}s)\Big)\Big], \end{aligned} \tag{4.15}$$

where the constants in the equation (4.15) are all positive.

Now applying Gronwall's inequality,

$$
\begin{aligned}
\mathbf{E}(\kappa^{\epsilon}(t)) \leq \Big[& \epsilon C_3' \mathbf{E}\Big(\int_0^t (1 + \|X_{\nu^\epsilon}^{\epsilon}(s)\|^2)\mathrm{d}s\Big) \\
& \qquad + \epsilon C_8' \mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s) \int_0^t (1 + \|X_{\nu^\epsilon}^{\epsilon}(s)\|^2)\mathrm{d}s\Big) \\
& + \sqrt{2\epsilon}\mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s)\Big) + \sqrt{2\epsilon} C_9' \mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s) \int_0^t (1 + \|X_{\nu^\epsilon}^{\epsilon}(s)\|^2)\mathrm{d}s\Big) \\
& + \epsilon\sqrt{2\epsilon} C_{10}' \mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s) \int_0^t (1 + \|X_{\nu^\epsilon}^{\epsilon}(s)\|^2)\mathrm{d}s\Big) + C_2' \sum_{j=1}^{q} \mathbf{E}\Big(\sup_{0\leq s\leq t} \|\zeta_{g_j}^{\epsilon}(s)\|\Big) \\
& + C_4' \mathbf{E}\Big(\sup_{0\leq s\leq t} \|\zeta^{\epsilon}(s)\|^2\Big) + C_5' \mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s)\Big) \\
& + C_6' \mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s) \int_0^t (1 + \|z_\nu(s)\|^2)\mathrm{d}s\Big) \\
& + C_7' \mathbf{E}\Big(\sup_{0\leq s\leq t} \zeta^{\epsilon}(s) \int_0^t (1 + \|X_{\nu^\epsilon}^{\epsilon}(s)\|^2)\mathrm{d}s\Big)\Big] \exp(C_1' T).
\end{aligned} \tag{4.16}
$$

If we prove the convergence of the last terms on the right hand side of the equation (4.16) we get $\kappa^{\epsilon}(t) \to 0$ in distribution as $\epsilon \to 0$. To prove the convergence of $\zeta^{\epsilon}(s)$ define

$$f(u) = \int_0^{\cdot} \sigma\Big(s, z_\nu(s), \ldots, \int_0^s g_q(s, u, z_\nu(u)) v_q(u)\mathrm{d}u\Big) u(s)\mathrm{d}s.$$

The above mapping $f : S_N \to C(J; \mathbb{R}^n)$ is bounded and continuous by the linear growth of the functions $\sigma(\cdot), g_j(\cdot), j = 1, 2, \ldots, q$. Since $\nu^\epsilon, \nu \in S_N$ and ν^ϵ converges to ν in distribution as S_N- valued random elements, by Theorem A.3.6 in [12] $\zeta^\epsilon \to 0$ in distribution as $\epsilon \to 0$. In a similar way, we can show that $\zeta_{g_j}^{\epsilon}$, $j = 1, 2, \ldots, q$ also tends to zero as $\epsilon \to 0$. Thus $\kappa^\epsilon(t) \to 0$ in distribution as $\epsilon \to 0$. □

Now we state the main result of this paper.

Theorem 4.1. *The family $\{X^\epsilon(t)\}$ satisfies the Laplace principle in $C(J; \mathbb{R}^n)$ with good rate function*

$$I(f) = \inf\Big\{\frac{1}{2}\int_0^T \|v(t)\|^2\mathrm{d}t : z_\nu = f; \nu \in L^2(0, T; (\mathbb{R}^d)^{q+1})\Big\},$$

where z_ν is the solution of the equation (4.1).

We conclude our work by providing an example to illustrate the theory in the paper.
Example:
Consider the following stochastic integrodifferential equation of the Itô type

$$
\begin{cases}
\mathrm{d}X_{v^\nu}^{\nu}(t) = \Big[\frac{1}{\sqrt{1+|X_{v^\nu}^{\nu}(t)|}} + \sin\frac{2^{-t}\nu}{1+|X_{\nu^\epsilon}^{\nu}(t)|}\Big]\mathrm{d}t \\
\qquad\qquad + \Big[\ln\Big(e^{-t}|\int_0^t X_{v^\nu}^{\nu}(s) + (2+s)^{\frac{1}{\nu}}\mathrm{d}W_1(s)| + 1\Big) + \ln\Big(1 + \frac{\nu}{1+t}\Big)^{\frac{1}{2}}\Big]\mathrm{d}W(t), \\
X(0) = X_0 + \nu.
\end{cases} \tag{4.17}
$$

All the functions in the equation satisfy the global Lipschitz condition and the growth condition. If $\mathbf{E}|X_0|^{2m} < \infty$, then the above SIDE has a.s. continuous solution. If both the noises are small, we may replace

$dW_1(t)$ by $\sqrt{\epsilon}dW_1(t)$ and $dW(t)$ by $\sqrt{\epsilon}dW(t)$. We employ the method developed in this paper to find the rate function of the LDP. For each $v = (v_1, v_2) \in L^2(0, T; \mathbb{R}^2)$, the associated controlled equation is:

$$\begin{aligned} dz_v(t) &= \Big[\frac{1}{\sqrt{1+|z_v(t)|}} + \sin\frac{2^{-t}\nu}{1+|z_v(t)|}\Big]dt \\ &\quad + \Big[\ln\Big(e^{-t}\Big|\int_0^t [z_v(s) + (2+s)^{\frac{-1}{\nu}}]v_1(s)ds\Big| + 1\Big) + \ln\Big(1 + \frac{\nu}{1+t}\Big)^{\frac{1}{2}}\Big]v_2(t)dt, \\ z_v(0) &= X_0, \end{aligned}$$

with unique solution $z_v(t)$.
The rate function $I : C(J; R) \to [0, \infty]$ is then given by

$$I(f) = \inf\Big\{\frac{1}{2}\int_0^T \|v(t)\|^2 dt : z_v = f\Big\}.$$

Readers may refer to [19] for more details on the above example. The problem considered here is more general than that in [19] as it includes the effect of two independent Brownian motions $W_1(t)$ and $W(t)$.

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