Filomat 32:2 (2018), 643–652
https://doi.org/10.2298/FIL1802643P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Variational Problem in Lagrange Spaces Endowed with a Special Type of (α, β)-Metrics

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Abstract. In this paper, we will continue our investigation on the new recently introduced (α, β)-metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ in [12]; where α is a Riemannian metric; β is an 1-form and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. We will investigate the variational problem in Lagrange spaces endowed with this type of metrics. Also, we will study the dually local flatness for this type of metric and we will proof that this kind of metric can be reduced to a locally Minkowskian metric. Finally, we will introduce the 2-Killing equation in Finsler spaces.

1. Introduction

The purpose of this paper is twofold. On the one hand, we will investigate the locally dually flatness; the variational problem in Lagrange spaces endowed with the (α, β)-metric

$$F = \beta + \frac{a\alpha^2 + \beta^2}{\alpha}, \tag{1}$$

where α is a Riemannian metric; β is an 1-form and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar; and on the other hand, we will investigate the 2-Killing equation in Finsler geometry. We introduced this class of (α, β)-metrics in [12] and we have analyzed the S-curvature and other important properties of this class of metrics in [13].

The variational problem of Lagrange spaces endowed with (α, β)-metrics is very important and worth to be studied not only in Finsler geometry, but also in physics. Some papers in which the variational problem is presented, are ([7], [8], [2]).

Another important topic investigated in this paper, is the dually locally flatness for the (α, β)-metric (1). This notion was introduced in Finsler geometry by Z. Shen in [14] where he extend the previous work of

2010 *Mathematics Subject Classification*. Primary 53B40
Keywords. Finsler (α, β)-metric; 2-Killing equation; variational problem.
Received: 15 March 2017; Revised: 23 March 2017; Accepted: 17 April 2017
Communicated by Mića Stanković
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S.I. Amari and H. Nagaoka from Riemannian geometry (see [1]). We will investigate for our metric (1) the dually locally flatness because this notion play an important role to the study of flat Finsler structures. This we will give us information about the locally flatness of a Finsler spaces endowed with this kind of metric. In some previous works the dually locally flatness was investigated for some (α, β)-metrics (see [3], [4]) and this encouraged us to study this notion for our metric (1). Moreover, we will study the 2-Killing equation in Finsler geometry, which is a new topic and worth to be study. The Finsler spaces endowed with (α, β)-metrics were investigated in a lot of papers (see [7], [8], [2]).

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \qquad u, v \in T_xM.$$

The following notion can be found in [7]:

Definition 2.1. *A Lagrange space is a pair $L^n = (M, L(x, y))$ formed by a smooth real, n-dimensional manifold M and a regular differentiable Lagrangian $L(x, y)$, for which the d-tensor field g_{ij} has constant signature over the manifold $\widetilde{TM}$.*

As we know from [17] and [9], Finsler spaces endowed with (α, β)-metrics were applied succefully to the study of gravitational magnetic fields. Other important results from [7] are presented as follows:
Let $F^n = (M, F(x, y))$ be a Finsler space. It has an (α, β)-metric if the fundamental function can be expressed in the following form: $F(x, y) = \check{F}(\alpha(x, y), \beta(x, y))$, where $\check{F}$ is a differentiable function of two variables with: $\alpha^2(x, y) = a_{ij}(x)y^iy^j$; $\beta(x, y) = b_i(x)y^i$.
$a = a_{ij}(x)dx^idx^j$ is a pseudo-Riemannian metric on the base manifold M and $b_i(x)dx^i$ is the electromagnetic 1-form on M. As we know from , if we denote by $L^n = (M, L)$ a Lagrange space; the fundamental tensor $g_{ij}(x, y)$ of L^n is: $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$ and this tensor can be written as follows for (α, β)-Lagrangians:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}\left(b_i\mathcal{Y}_j + b_j\mathcal{Y}_i\right) + \rho_{-2}\mathcal{Y}_i\mathcal{Y}_j$$

where $b_i = \frac{\partial\beta}{\partial y^i}$; $\mathcal{Y}_i = a_{ij}y^j = \alpha\frac{\partial\alpha}{\partial y^i}$.
$\rho; \rho_0; \rho_{-1}; \rho_{-2}$ are invariants of the space L^n.
Here, $\rho; \rho_0; \rho_{-1}; \rho_{-2}$ are given by (see [7]):

$$\rho = \frac{1}{2\alpha}L_\alpha; \rho_0 = \frac{1}{2}L_{\beta\beta};$$

$$\rho_{-1} = \frac{1}{2\alpha}L_{\alpha\beta}; \rho_{-2} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha\right). \tag{2}$$

where $L_\alpha = \frac{\partial L}{\partial\alpha}$; $L_\beta = \frac{\partial L}{\partial\beta}$; $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}$; $L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}$ and $L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}$.
Shimada and Sabău in [16], have proved that the system of covectors $\{b_i, \mathcal{Y}_i\}$ is independent. The following formulae holds (see [7]):

$$y_i = \frac{1}{2}\frac{\partial L}{\partial y^i} = \rho_1 b_i + \rho\mathcal{Y}_i; \rho_1 = \frac{1}{2}L_\beta;$$

$$\frac{\partial\rho_1}{\partial y^i} = \rho_0 b_i + \rho_{-1}\mathcal{Y}_i; \frac{\partial\rho}{\partial y^i} = \rho_{-1}b_i + \rho_{-2}\mathcal{Y}_i$$

$$\frac{\partial\rho_0}{\partial y^i} = r_{-1}b_i + r_{-2}\mathcal{Y}_i; \frac{\partial\rho_{-1}}{\partial y^i} = r_{-2}b_i + r_{-3}\mathcal{Y}_i \tag{3}$$

$$\frac{\partial\rho_{-2}}{\partial y^i} = r_{-3}b_i + r_{-4}\mathcal{Y}_i$$

with $r_{-1} = \frac{1}{2}L_{\beta\beta\beta}$; $r_{-2} = \frac{1}{2\alpha}L_{\beta\beta\beta}$; $r_{-3} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha\beta} - \frac{1}{\alpha}L_{\alpha\beta}\right)$ and $r_{-4} = \frac{1}{2\alpha^3}\left(L_{\alpha\alpha\alpha} - \frac{3}{\alpha}L_{\alpha\alpha} + \frac{3}{\alpha^2}L_\alpha\right)$.
The Cartan tensor in such of space can be computed as follows(see [7]):

$$2C_{ijk} = \underset{(i,j,k)}{\sigma}\left\{\rho_{-1}a_{ij}b_k + \rho_{-2}a_{ij}\mathcal{Y}_k + \frac{1}{3}r_{-1}b_ib_jb_k + r_{-2}b_ib_j\mathcal{Y}_k + r_{-3}b_i\mathcal{Y}_j\mathcal{Y}_k + \frac{1}{3}r_{-4}\mathcal{Y}_i\mathcal{Y}_j\mathcal{Y}_k\right\} \tag{4}$$

where $\sigma_{(i,j,k)}$ is the cyclic sum in the indices i, j, k.
The variational problem for Finsler spaces endowed with (α,β)-metrics is an important topic in Finsler geometry. For such spaces, the Euler-Lagrange equations $E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)=0$, can be give in the following way:

$$E_i(L) = E_i(\alpha^2) + 2\frac{\rho_1}{\rho}E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i} \tag{5}$$

Remark 2.2. *From paper [6], we know the following:*
The function $F = \alpha\phi(s)$ is a Finsler function if and only if three conditions are satisfied:
I) $\phi(s) > 0$,
II) $\phi(s) - s\phi'(s) > 0$,
III) $[\phi(s) - s\phi'(s)] + (b^2 - s^2)\phi''(s) > 0$.
In our case, one obtains for $\phi(s) = s^2 + s + a$:
I) $\phi(s) > 0 \Leftrightarrow a \in (\frac{1}{4}, +\infty)$,
II) $\phi(s) - s\phi'(s) > 0 \Leftrightarrow s \in (-\sqrt{a}, +\sqrt{a})$,
III) $[\phi(s) - s\phi'(s)] + (b^2 - s^2)\phi''(s) > 0 \Leftrightarrow 3s^2 < a + 2b^2$ for $|s| < b < \sqrt{a}$.

The above conditions for general (α,β)-metrics can be also found in [6]. The following result is very important:

Theorem 2.3. *([7]) In the natural parametrization, $t = s$; the Euler-Lagrange equations of the Lagrangian $L(\alpha,\beta)$, are given by:*

$$E_i(\alpha^2) + 2\frac{\rho_1}{\rho}F_{ij}(x)y^j = 0;\ y^i = \frac{dx^i}{ds} \tag{6}$$

Remark 2.4. *If we use the following equations $E_i(\beta) = F_{ij}(x)\frac{dx^j}{ds}$;*

$$F_{ij} = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j} = b_{j|i} - b_{i|j},$$

then (5) can be rewritted in the following way:

$$E_i(\alpha^2) + 2\frac{\rho_1}{\rho}\left(b_{j|i} - b_{i|j}\right) = 0;\ y^i = \frac{dx^i}{ds} \tag{7}$$

Some interesting results in the theory of Lagrange spaces are also presented in [5]. Let's recall now some notions regarding 2-Killing vector fields on Riemannian manifolds from paper [11].

Definition 2.5. *Let (M, g) a Riemannian manifold. A vector field $X \in \chi(M)$ is called 2-Killing, if $L_X L_X g = 0$, where L is the Lie derivative.*

In this paper we will extend this notion for the case of Finsler spaces. First we will recall the notion of Killing vectors in Finsler spaces. We will follow the results from [15]. We will consider the coordinate transformation:

$$\overline{x}^i = x^i + \epsilon V^i;\ \overline{y}^i = y^i + \epsilon \frac{\partial V^i}{\partial x^j} y^j$$

Under this change of coordinates, a Finsler structure became (see [15]):

$$\overline{F}(\overline{x}, \overline{y}) = \overline{F}(x, y) + \epsilon V^i \frac{\partial F}{\partial x^i} + \epsilon y^j \frac{\partial V^i}{\partial x^j} \frac{\partial F}{\partial y^i}.$$

where $\overline{F}(\overline{x}, \overline{y})$ must be equal with $F(x, y)$. Using this remark, the authors of paper [15], concluded that the Killing equation in Finsler space is:

$$K_V(F) = V^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial F}{\partial y^i} = 0 \tag{8}$$

Also, for an (α, β)-metric, the authors of paper [15], remarked that the Killing equation for the Finsler spaces endowed with this kind of metrics, is given as follows:

$$0 = K_V(\alpha)\phi(s) + \alpha K_V(\phi(s))$$

$$\Rightarrow 0 = \left(\phi(s) - s\frac{\partial \phi(s)}{\partial s}\right) K_V(\alpha) + \frac{\partial \phi(s)}{\partial s} K_V(\beta) \tag{9}$$

with

$$K_V(\alpha) = \frac{1}{2\alpha}\left(V_{i|j} - V_{j|i}\right) y^i y^j$$

$$K_V(\beta) = \left(V^i \frac{\partial b_j}{\partial x^i} + b_i \frac{\partial V_i}{\partial x^j}\right) y^j$$

where "|" denotes the covariant derivative with respect to Riemannian metric α. As we presented in previous section, in Introduction, the dually locally flatness on Finsler spaces is an important topic and worth to be studied because give us important informations about the flatness of the space. An important result obtained in [18], is the following one:

Theorem 2.6. *([18]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β)-metric on an n-dimensional manifold $M^n, (n \geq 3)$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i \neq 0$ is an 1-form on M. Suppose that F is not Riemannian and $\phi'(s) \neq 0$; $\phi'(0) \neq 0$; $\beta \neq 0$. Then F is a locally dually flat on M if and only if α, β and $\phi = \phi(s)$, satisfy:*

- $1.s_{lo} = \frac{1}{3}(\beta\theta_l - \theta b_l)$,
- $2.r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\theta\beta^2$,
- $3.G^l_\alpha = \frac{1}{3}\left[2\theta + (3k_1 - 2)\theta\beta\right] y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l$,
- $4.\tau\left[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')\right] = 0$,

where $\tau = \tau(x)$ is a scalar function; $\theta = \theta_i(x) y^i$ is an 1-form on M, $\theta^l = a^{lm}\theta_m$,

$$k_1 = \Pi(0); k_2 = \frac{\Pi'(0)}{Q(0)}; k_3 = \frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)\right], \tag{10}$$

and $Q = \frac{\phi'}{\phi - s\phi'}$; $\Pi = \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}$.

3. Main Results

In the following lines, we will give our first main result. Using our metric (1), in which $\phi(s) = s^2 + s + a$, with $s = \frac{\beta}{\alpha}$, we will compute $Q(s), \Pi(s)$ and k_1, k_2, k_3 given in (10) for our metric. After tedious computations, one obtains:

$$Q(s) = \frac{2s+1}{a-s^2};\ Q(0) = \frac{1}{a};\ Q'(0) = \frac{2}{a}.$$

$$\Pi(s) = \frac{6s^2+6s+2a+1}{(s^2+s+a)(a-s^2)};\ \Pi'(0) = \frac{6}{a^2} - \frac{1+2a}{a^3}.$$

$$k_1 = \Pi(0) = \frac{1+2a}{a^2} \tag{11}$$

$$k_2 = \frac{\Pi'(0)}{Q(0)} = \frac{4a-1}{a^2} \tag{12}$$

$$k_3 = \frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)\right] = \frac{2(1-4a)}{a^3} \tag{13}$$

Now, using Theorem 2.6 and the above relations (11)-(13), we will formulate the following:

Theorem 3.1. *Let $M^n, (n \geq 3)$ an n-dimensional manifold and let $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$; where α is a Riemannian metric; β is an 1-form and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar; given in (1). Supposing that F is not Riemannian, then F is locally dually flat on M, if and only if α, β and $\phi(s) = s^2 + s + a$, satisfy:*

- $1. s_{lo} = \frac{1}{3}(\beta\theta_l - \theta b_l)$,
- $2. r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2 + \left[\left(\frac{4a-1}{a^2}\right)\left(1 + \frac{2b^2}{a}\right) - \frac{2}{3}\right]\tau\beta^2$,
- $3. G^l_\alpha = \frac{1}{3}\left[2\theta + \left(\frac{-2a^2+6a+3}{a^2}\right)\tau\beta\right]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 - \frac{4a-1}{a^3}\tau\beta^2 b^l$,
- $4. \tau\left[\frac{4a-1}{a^2}s\left(1 + \frac{2s^2}{a}\right)(a - 3s^3 - 3s^2) - (6s^2 + 6s + 2a + 1) + \frac{1+2a}{a^3}(s^2 + s + a)(a - s^2)\right] = 0$,

where $\tau = \tau(x)$ is a scalar function; $\theta = \theta_i(x)y^i$ is an 1-form on M, $\theta^l = a^{lm}\theta_m$.

Proof. Using the above relations (11)-(13) computed for our metric (1) and replacing them in Theorem 2.6, we get easily the asertion of the theorem. □

Remark 3.2. *Using Theorem 2.6, in paper [19], is presented the following corollary:*

Corollary 3.3. *([19]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β)-metric on a manifold M of dimension $n \geq 3$ with the same assumptions as in Theorem 2.6. Let ϕ, satisfy:*

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi' - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0. \tag{14}$$

Then, F is locally dually flat on M if and only if:

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$$

$$r_{00} = \frac{2}{3}\left[\theta\beta - (\theta_l b^l)\alpha^2\right],$$

$$G^l_\alpha = \frac{1}{3}\left[2\theta y^l + \theta^l\alpha^2\right]$$

where $k_i, (1 \leq i \leq 3)$ are the same with those of Theorem 2.6.

Another important result from paper [19], is the following lemma:

Lemma 3.4. *([19]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β)-metric on a manifold and $b = \left\|\beta_x\right\|_\alpha$. Suppose that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for ny constant $c_1 > 0$; c_2 and c_3. Then F is of isotropic S-curvature, S=(n+1)cF, if and only if one of the following holds:*

(a) β satisfies:

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), s_j = 0,$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies:

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2},$$

where k is a constant. In this case, $S = (n + 1)cF$, with $c = k\epsilon$.
(b) β satisfies:

$$r_{ij} = 0; s_j = 0$$

In this case, $S = 0$, regardless of choices of a particular ϕ.

Next, using the above results, we are ready to proof the following theorem:

Theorem 3.5. *Let $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ our metric (1) with $\phi(s) = s^2 + s + a$; where α is a Riemannian metric; β is an 1-form and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar, on the manifold M with $n \geq 3$, on the same assumptions as in Theorem 2.6. Let ϕ, satisfy:*

$$s\left(\frac{4a-1}{a^2}\left(1 + \frac{2s^2}{a}\right)\right)(a - s^2) + (s^2 + s + a)\left(\frac{1+2a}{a^2}(a - s^2) - 6\right) + 4a - 1 \neq 0$$

Then F is locally dually flat on M and of isotropic S-curvature $S = (n + 1)cF$, if and only if:

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$$

$$r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2,$$

$$G^l_\alpha = \frac{1}{3}\left[2\theta + \left(\frac{-2a^2 + 6a + 3}{a^2}\right)\tau\beta\right]y^l$$

where $\theta = \theta_i(x)y^i$ is an 1-form on M.

Proof. We will use the same approach as in paper [19], but for our metric (1). In this case, $\phi(s) = s^2 + s + a$; $\phi'(s) = 2s + 1$; $\phi'(0) \neq 0$.
From (14), we know the condition:

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi' - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0.$$

This condition, for our metric (1), can be written in the following way:

$$s\left(\frac{4a-1}{a^2}\left(1 + \frac{2s^2}{a}\right)\right)(a - s^2) + (s^2 + s + a)\left(\frac{1+2a}{a^2}(a - s^2) - 6\right) + 4a - 1 \neq 0.$$

Now, from Theorem 3.1, we know:

$$r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2 + \left[\left(\frac{4a-1}{a^2}\right)\left(1 + \frac{2b^2}{a}\right) - \frac{2}{3}\right]\tau\beta^2, \quad (15)$$

$$G^l_\alpha = \frac{1}{3}\left[2\theta + \left(\frac{-2a^2+6a+3}{a^2}\right)\tau\beta\right]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 - \frac{4a-1}{a^3}\tau\beta^2 b^l, \tag{16}$$

By $s_0 = 0$; from previous Lemma 3.4, one obtains: $\theta = \frac{(b^l\theta_l)}{b^2}\beta$. Replacing this expresion in (15), we get:

$$r_{00} = \frac{2}{3}\frac{b^l\theta_l}{b^2}\beta^2 + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 - \left[\frac{4a-1}{a^2}\left(1+\frac{2b^2}{a}\right) - \frac{2}{3}\right]\tau\beta^2 = 0.$$

Differentiating this relation with respect to y^m and taking account that $r_{00} = 0$ by Lemma 3.4; one obtains:

$$\frac{4}{3}\frac{b^l\theta_l}{b^2}\beta b_m + 2\left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 - \left[\frac{4a-1}{a^2}\left(1+\frac{2b^2}{a}\right) - \frac{2}{3}\right]\tau\beta b_m = 0.$$

Multiplying with b^m, one obtains:

$$2\left[\frac{4a-1}{a^2}\left(1+\frac{2b^2}{a^2}\right) - \frac{2}{3}\right]\tau\beta = 0.$$

By assumption,

$$\left[\frac{4a-1}{a^2}\left(1+\frac{2b^2}{a^2}\right) - \frac{2}{3}\right]\tau\beta \neq 0.$$

So, we can deduce that $\tau = 0$. Taking into above relations (15) and (16), we get the desired result. □

3.1. The variational problem for Finsler spaces endowed with the (α, β)-metric (1)

Starting from (1), in this section, we will consider a Finsler space endowed with the fundamental function $L = F^2 = \left(\beta + \frac{a\alpha^2+\beta^2}{\alpha}\right)^2$, where α is a Riemannian metric; β is an 1-form and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. We will investigate the variational problem for this Finler space endowed with this (α, β)-metric. First, after tedious computations, one obtains:

$$L_\alpha = \frac{2(\beta\alpha + a\alpha^2 + \beta^2)(a\alpha^2 - \beta^2)}{\alpha^3}; \quad L_\beta = \frac{2(\beta\alpha + a\alpha^2 + \beta^2)(\alpha + 2\beta)}{\alpha^2}$$

$$L_{\alpha\beta} = \frac{2(-4\beta^3 - 3\beta^2\alpha + a\alpha^3)}{\alpha^3}; \quad L_{\beta\beta} = \frac{2(\alpha^2 + 6\beta\alpha + 6\beta^2 + 2a\alpha^2)}{\alpha^2}; \quad L_{\alpha\alpha} = \frac{2(a^2\alpha^4 + 3\beta^4 + 2\beta^3\alpha)}{\alpha^4}$$

$$L_{\alpha\alpha\alpha} = \frac{-12\beta^3(\alpha+2\beta)}{\alpha^5}; \quad L_{\alpha\beta\beta} = \frac{-(12(\alpha+2\beta))\beta}{\alpha^3}; \quad L_{\beta\beta\beta} = \frac{12(\alpha+2\beta)}{\alpha^2}; \quad L_{\alpha\alpha\beta} = \frac{12(\alpha+2\beta)\beta^2}{\alpha^4} \tag{17}$$

Next, we will compute for our metric (1):

$$\rho = \frac{1}{2\alpha}L_\alpha = \frac{(\beta\alpha + a\alpha^2 + \beta^2)(a\alpha^2 - \beta^2)}{\alpha^4}; \rho_0 = \frac{1}{2}L_{\beta\beta} = \frac{\alpha^2 + 6\beta\alpha + 6\beta^2 + 2a\alpha^2}{\alpha^2};$$

$$\rho_{-1} = \frac{1}{2\alpha}L_{\alpha\beta} = \frac{a\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4} \tag{18}$$

$$\rho_1 = \frac{1}{2}L_\beta = \frac{(\alpha\beta + a\alpha^2 + \beta^2)(\alpha + 2\beta)}{\alpha^2}$$

$$\rho_{-2} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha\right) = \frac{\beta(4\beta^3 + 3\alpha\beta^2 - a\alpha^3)}{\alpha^6}$$

Next, we compute:

$$r_{-1} = \frac{1}{2}L_{\beta\beta\beta} = \frac{6(\alpha+2\beta)}{\alpha^2};\ \ r_{-2} = \frac{1}{2\alpha}L_{\alpha\beta\beta} = -\frac{6(\alpha+2\beta)\beta}{\alpha^4}$$

$$r_{-3} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha\beta} - \frac{1}{\alpha}L_{\alpha\beta}\right);\ \ r_{-4} = \frac{1}{2\alpha^3}\left(L_{\alpha\alpha\alpha} - \frac{3}{\alpha}L_{\alpha\alpha} + \frac{3}{\alpha^2}L_{\alpha}\right) = \frac{3\beta(-8\beta^3 - 5\alpha\beta^2 + a\alpha^3)}{\alpha^8} \tag{19}$$

Next we will follow the same treatment as in [7], but this time for the metric (1). Using equations (3), we will get the following results:

$$y_i = \frac{(\alpha\beta + a\alpha^2 + \beta^2)(\alpha+2\beta)}{\alpha^2}b_i + \frac{(\alpha\beta + a\alpha^2 + \beta^2)(a\alpha^2 - \beta^2)}{\alpha^4}$$

$$\frac{\partial\rho_1}{\partial y^i} = \frac{\alpha^2 + 6\alpha\beta + 6\beta^2 + 2a\alpha^2}{\alpha^2}b_i + \frac{a\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4}\mathcal{Y}_i$$

$$\frac{\partial\rho}{\partial y^i} = \frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4}b_i + \frac{\beta(4\beta^3 + 3\alpha\beta^2 - a\alpha^3)}{\alpha^6}\mathcal{Y}_i$$

$$\frac{\partial\rho_0}{\partial y^i} = \frac{6(\alpha+2\beta)}{\alpha^2}b_i - \frac{6\beta(\alpha+2\beta)}{\alpha^4}\mathcal{Y}_i \tag{20}$$

$$\frac{\partial\rho_{-2}}{\partial y^i} = \frac{4\beta^3 - 6\beta\alpha^2 - 9\alpha^2\beta - 9\alpha\beta^2 - a\alpha^3}{\alpha^6}b_i + \frac{3\beta(-8\beta^3 - 5\alpha\beta^2 + a\alpha^3)}{\alpha^8}\mathcal{Y}_i$$

$$\frac{\partial\rho_{-1}}{\partial y^i} = \frac{-6\beta(\alpha+2\beta)}{\alpha^4}b_i + \frac{4\beta^3 - 6\beta\alpha^2 - 9\alpha\beta^2 - a\alpha^3}{\alpha^6}\mathcal{Y}_i$$

Now, we are ready to give the following result:

Theorem 3.6. *The Cartan tensor for the (α,β)-metric (1), has the following form:*

$$2C_{ijk} = \underset{(i,j,k)}{\sigma}\Bigg(\frac{a\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4}a_{ij}b_k + \frac{\beta(4\beta^3 + 3\alpha\beta^2 - a\alpha^3)}{\alpha^6}a_{ij}\mathcal{Y}_k + \frac{2(\alpha+2\beta)}{\alpha^2}b_ib_jb_k -$$

$$\frac{6(\alpha+2\beta)\beta}{\alpha^4}b_ib_j\mathcal{Y}_k + \frac{4\beta^3 - 6\beta\alpha^2 - 9\alpha\beta^2 - a\alpha^3}{\alpha^6}b_i\mathcal{Y}_j\mathcal{Y}_k + \frac{\beta(a\alpha^3 - 8\beta^3 - 5\alpha\beta^2)}{\alpha^8}\mathcal{Y}_i\mathcal{Y}_j\mathcal{Y}_k\Bigg) \tag{21}$$

where $\sigma_{(i,j,k)}$ is the cyclic sum of the indices.

Proof. From (4), we know the general form of the Cartan tensor for an (α,β)-metric. Replacing (15) and (16) in (4), we get the Cartan tensor for the metric (1). □

Theorem 3.7. *The Euler-Lagrange equations for the metric (1), can be expressed in the following way:*

$$E_i(\alpha^2) + 2\frac{\alpha+2\beta}{(a\alpha^2 - \beta^2)\alpha^2}(b_{j|i} - b_{i|j}) = 0,\ \ y^i = \frac{dx^i}{ds}$$

Proof. The proof is direct, using (5) and (15). □

Now, using (8), (9) and Definition 2.5, we will study the 2-Killing equation for an (α,β)-metric. In this regard, we can give now the following:

Theorem 3.8. *The* $2-$*Killing equation for an general* (α,β)*-metric,* $F=\alpha\phi(s)$*, where* $s=\frac{\beta}{\alpha}$*, has the following form:*

$$\begin{gathered}K_V\left(\frac{\partial\phi(s)}{\partial s}\right)(K_V(\beta)-K_V(\alpha))+\frac{\partial\phi(s)}{\partial s}\left(K_V(K_V(\beta))-sK_V(K_V(\alpha))\right)+\\ \phi(s)K_V(K_V(\alpha))+K_V(\alpha)K_V(\phi(s))-K_V(s)K_V(\alpha)\frac{\partial\phi(s)}{\partial s}=0\end{gathered}\tag{22}$$

with

$$K_V(\alpha)=\frac{1}{2\alpha}\left(V_{i|j}-V_{j|i}\right)y^iy^j$$

$$K_V(\beta)=\left(V^i\frac{\partial b_j}{\partial x^i}+b_i\frac{\partial V_i}{\partial x^j}\right)y^j$$

where "|" denotes the covariant derivative with respect to Riemannian metric α*.*

Proof. *We will start with the 2-Killing equation addapted this time for Finsler spaces endowed with* (α,β)*-metrics:*

$$K_V(K_V(\alpha\phi(s)))=0.$$

After computation, we get:

$$K_V(K_V(\alpha\phi(s)))=K_V\left(K_V(\alpha)\left(\phi(s)-s\frac{\partial\phi(s)}{\partial s}\right)+\frac{\partial\phi(s)}{\partial s}K_V(\beta)\right)=$$

$$K_V(K_V(\alpha))\left(\phi(s)-s\frac{\partial\phi(s)}{\partial s}\right)+K_V(\alpha)\left(K_V(\phi(s))-K_V(s\frac{\partial\phi(s)}{\partial s})-sK_V\left(\frac{\partial\phi(s)}{\partial s}\right)\right)+$$

$$K_V\left(\frac{\partial\phi(s)}{\partial s}K_V(\beta)+\frac{\partial\phi(s)}{\partial s}\right)K_V\left(K_V(\beta)\right)$$

and after we group the terms, we get the desired result. □

In the following lines, we will recall some notions from [10].

Proposition 3.9. *([10]) We have the relations:*

$$\partial_i\alpha=\frac{1}{\alpha}y_i;\ \ \partial_i\partial_j\alpha=\frac{1}{\alpha}\gamma_{ij}(x)-\frac{1}{\alpha^3};$$

$$\partial_i\beta=A_i(x);\ \ \partial_i\partial_j\beta=0$$

The moments of the Lagrangian $L(\alpha(x,y),\beta(x,y))$ (see [10]), is given by:

$$p_i=\frac{1}{2}\left(L_\alpha\partial_i\alpha+L_\beta\partial_i\beta\right)$$

Proposition 3.10. *([10]) The moments of the Lagrangian* $L(x,y)$*, are given by:* $p_i=\rho y_i+\rho_1A_i$*, where* $\rho=\frac{1}{2\alpha}L_\alpha$*;* $\rho_1=\frac{1}{2}L_\beta$*. The derivatives of the principal invariants of the Lagrange space are given by:*

$$\partial_i\rho=\rho_{-2}y_i+\rho_{-1}A_i;\ \ \partial_i\rho_1=\rho_{-1}y_i+\rho_0A_i$$

Now, we are able to give the following result for our metric (1):

Proposition 3.11. *The moments of the Lagrangian* $L(x,y)$ *for the metric (1), are given by:*

$$p_i=\frac{(\alpha\beta+a\alpha^2+\beta^2)(a\alpha^2-\beta^2)}{\alpha^4}y_i+\frac{(\alpha\beta+a\alpha^2+\beta^2)(\alpha+2\beta)}{\alpha^2}A_i$$

Proof. The proof is direct, using (15) and Proposition 3.10. □

Proposition 3.12. *The derivatives of the principal invariants of the Lagrange space endowed with metric (1), are given by:*

$$\partial_i \rho = \frac{\beta(4\beta^3 + 3\alpha\beta^2 - a\alpha^3)}{\alpha^6} y_i + \frac{a\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4} A_i$$

$$\partial_i \rho_1 = \frac{a\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4} y_i + \frac{\alpha^2 + 6\alpha\beta + 6\beta^2 + 2a\alpha^2}{\alpha^4} A_i$$

Proof. The proof is obvious and follow easily, using (18) and the above Proposition 3.10. □

Remark 3.13. *In this paper we used the Maple 13 program at computations.*

4. Conclusion

In this paper we have continued the investigations on the new introduced (α, β)-metric (1) and we succeed to investigate the dually locally flatness and the Cartan tensor for this type of metrics. Also we investigated the 2-Killing equation for Finsler spaces, which represent an important step for the study of Finsler spaces.

Acknowledgments

The authors express their sincere thanks to referees for their valuable suggestions and comments.

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