Filomat 32:2 (2018), 671–680
https://doi.org/10.2298/FIL1802671S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Common Fixed Point Theorems for Multi-Valued Weak Contractive Mappings in Metric Spaces with Graphs

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Abstract. In this paper, a new type of graph contractive multi-valued mappings in a metric space with a directed graph is introduced and studied. A common fixed point theorem of those two multi-valued mappings is established under some appropriate conditions. Moreover, some examples illustrating our main result are also given. The obtained result extends and generalizes several fixed point results of multi-valued mappings in the literature. We apply our main result to obtain common fixed point results for two multi-valued mappings in ε -chainable complete metric spaces and two cyclic contraction multi-valued mappings.

1. Introduction

Fixed point theory is the main tool in nonlinear analysis. It can be applied to solve the existence problems of solutions of various nonlinear equations in science and applied science. One of the most important principle is Banach contraction principle which plays a significant role in such study. This principle was extended and studied in various directions. In 1961, Edelstein extended a Banach contraction principle for uniformly locally contractive mappings in a ε-chainable complete metric space.

Definition 1.1. A metric space (X, d) is said to be a *ε-chainable metric space* for some $\varepsilon > 0$ if for any $u, v \in X$, there exists $n_0 \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^{n_0}$ such that

$$x_0 = u, \;\; x_{n_0} = v \;\; \text{and} \;\; d(x_{i-1}, x_i) < \varepsilon \quad \text{for } i = 1, ..., n_0.$$

Definition 1.2. Let (X, d) be a metric space, $\varepsilon > 0$, $0 \le \kappa < 1$ and $u, v \in X$. A mapping $f : X \to X$ is said to be (ε, κ) *uniformly locally contractive* if $0 < d(u, v) < \varepsilon \Rightarrow d(fu, fv) < \kappa d(u, v)$.

Edelstein [14] obtained the following theorem which was an extension of Banach contraction principle.

Theorem 1.3 ([14]). *Let (X, d) be a ε-chainable complete metric space. If $f : X \to X$ is a (ε, κ) uniformly locally contractive mapping, then f has a unique fixed point.*

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 47H04
Keywords. fixed point, directed graph, multi-valued mapping, weak contraction, graph contractive mapping
Received: 16 March 2017; Accepted: 03 June 2017
Communicated by Vasile Berinde
Research supported by the Thailand Research Fund under the project RTA 5780007, the Royal Golden Jubilee (RGJ) Ph.D. Programme with contract no. PHD/0093/2558 and Chiang Mai University
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Let (X, d) be a metric space and let $S : X \to 2^X$. A point $x \in X$ is a *fixed point* of S if $x \in Sx$. Let $Fix(S) := \{x \in X : x \in Sx\}$ denote the set of all fixed points of S. Let $CB(X)$ be the set of all nonempty closed bounded subsets of X. For $U, V \in CB(X)$, let

$$H(U, V) := max\{\sup_{v \in V} D(v, U), \sup_{u \in U} D(u, V)\},$$

where

$$D(u, V) := \inf_{v \in V} d(u, v).$$

We call the mapping H a *Pompeiu-Hausdorff metric* induced by d.

In 1969, Nadler [26] extended the Banach contraction principle for multi-valued mappings. A mapping $S : X \to CB(X)$ is called a *contraction* if there exists $k \in [0, 1)$ such that

$$H(Su, Sv) \leq kd(u, v) \quad \text{for all } u, v \in X.$$

He proved that in a complete metric space, every contraction multi-valued mapping has a fixed point. Moreover, he also generalized Edelstein's theorem for multi-valued mappings as the following theorem.

Theorem 1.4 ([26]). *Let (X, d) be a ε-chainable complete metric space for some $\varepsilon > 0$ and let $F : X \to C(X)$ be a multi-valued mapping such that Fx is a nonempty compact subset of X. If F satisfies the following condition:*

$$u, v \in X \text{ and } 0 < d(u, v) < \varepsilon \Rightarrow H(Fu, Fv) < \kappa d(u, v),$$

then F has a fixed point.

Nadler's theorem was also extended and generalized in several directions, see [1, 3, 6–9, 11, 13, 15, 16, 22, 24, 25, 27, 29–31, 36, 38]. Some of them were studied in partially ordered metric spaces and metric spaces with directed graphs, see [1, 3, 6–8, 11, 16, 27, 36, 38].

In 1989, Takahashi and Mizoguchi [25] extended Nadler's theorem by introducing a new contractive condition as follows:

$$H(Su, Sv) \leq \psi(d(u, v))d(u, v) \quad \text{for all } u, v \in X,$$

where $\psi : [0, \infty) \to [0, 1)$ satisfies the condition $\limsup_{t \to r^+} \psi(t) < 1$ for each $r \in [0, \infty)$, and proved an existence result for this type of mappings.

In 2007, Berinde and Berinde [9] introduced a new class of multi-valued mappings, called "weak contraction", which is more general than that of Nadler and Takahashi and Mizoguchi. A mapping $S : X \to CB(X)$ is said to be *weak contraction* or (θ, L)-*weak contraction* if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Su, Sv) \leq \theta d(u, v) + LD(v, Su) \quad \text{for all } u, v \in X. \tag{1.1}$$

They proved a fixed point theorem of multi-valued weak contractions in a complete metric space, see [9, Theorem 3].

Moreover, they also suggested a wider class of multi-valued mappings, called "generalized multi-valued (φ, L)-weak contraction", by replacing θ in (1.1) with $\varphi(d(u, v))$ where a function $\varphi : [0, \infty) \to [0, 1)$ satisfies $\limsup_{t \to r^+} \varphi(t) < 1$ for every $r \in [0, \infty)$.

By combination two concepts in fixed point theory and graph theory, Banach contraction principle and Nadler's theorem were extensively extended and studied in metric spaces with directed graphs. This direction was considered first by Jachymski [17]. He introduced the notion of G-contraction in a metric space endowed with a directed graph and established a fixed point result of this type of mappings. After that many mathematicians draw their attentions to investigate fixed point theorems of various kinds of contraction mappings in the setting of metric spaces with directed graphs, see for examples, [1, 3, 8, 11, 16, 27, 33, 36, 38].

Recently, Tiammee and Suantai [38] introduced a new concept of graph-preserving and weak G-contraction for multi-valued mappings and proved some fixed point theorems in a complete metric space with a directed graph. By setting a graph G to be a complete graph, they obtained Nadler's theorem and Takahashi and Mizoguchi's theorem as special cases.

The existence of common fixed point of nonlinear mappings is an important topic in fixed point theory. It plays a significant role in studying existence of solutions of system of equations. Many mathematicians investigated common fixed point theorems for various kinds of single and multi-valued mappings in various spaces, see [2, 5, 8, 10, 12, 18–20, 34, 35, 37], for examples. However, there are a few papers paying attention for existence of a common fixed point for multi-valued mappings because it is more difficult than that of single valued mappings. In 2013, by introducing the concept of graph contractive mapping, Beg and Butt [8] proved the existence result for a common fixed point of two multi-valued mappings in a complete metric space with a directed graph.

Definition 1.5 ([8]). Let (X, d) be a metric space and $S, T : X \to CB(X)$. The mappings S, T are said to be *graph contractive* if there exists $k \in (0, 1)$ such that

$$(u \neq v), \ \ (u, v) \in E(G) \ \ \Rightarrow \ \ H(Su, Tv) < kd(u, v),$$

and if $x \in Su$ and $y \in Tv$ are such that

$$d(x, y) < d(u, v),$$

then $(x, y) \in E(G)$.

They proved that graph contractive mappings $S, T : X \to CB(X)$ have a common fixed point if the set $X_S := \{u \in X : (u, x) \in E(G) \text{ for some } x \in Su\}$ is nonempty.

Motivated by these works, by using the idea given by Beg and Butt [8] and Berinde and Berinde [9], we aim to introduce a new class of graph contractive multi-valued mappings and investigate a common fixed point result in the setting of a metric space with a directed graph. Our main result generalize those of Berinde and Berinde [9], Tiammee and Suantai [38], and Beg and Butt [8].

2. Preliminaries

In this section, we recall and give some notions, definitions and useful results which will be used in the later sections.

The following three lemmas are useful for our main result and they can be found in [26], [25] and [4].

Lemma 2.1 ([26]). *Let (X, d) be a metric space. If $U, V \in CB(X)$ and $u \in U$, then, for each $\epsilon > 0$, there exists $v \in V$ such that*

$$d(u, v) \leq H(U, V) + \epsilon.$$

Lemma 2.2 ([25]). *Let (X, d) be a metric space, $\{U_k\}$ be a sequence in $CB(X)$ and $\{u_k\}$ be a sequence in X such that $u_k \in U_{k-1}$. Let $\varphi : [0, \infty) \to [0, 1)$ be a function satisfying* $\limsup_{t\to r^+} \varphi(t) < 1$ *for every $r \in [0, \infty)$. Suppose that $\{d(u_{k-1}, u_k)\}$ is a non-increasing sequence such that*

$$H(U_{k-1}, U_k) \leq \varphi(d(u_{k-1}, u_k))d(u_{k-1}, u_k),$$
$$d(u_k, u_{k+1}) \leq H(U_{k-1}, U_k) + [\varphi(d(u_{k-1}, u_k))]^{n_k},$$

where $n_1 < n_2 < ...$ and $k, n_k \in \mathbb{N}$. Then $\{u_k\}$ is a Cauchy sequence in X.

Lemma 2.3 ([4]). *Let $\{U_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} H(U_n, U) = 0$ for $U \in CB(X)$. If $u_n \in U_n$ and $\lim_{n\to\infty} d(u_n, u) = 0$, then $u \in U$.*

We now recall some concepts in graph theory. Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices and $E(G)$ is a set of its edges. Suppose that G has no parallel edges. Let Δ be a set of all loops, *i.e.*, $\Delta = \{(x, x) : x \in X\}$. The conversion of a graph G is denoted by G^{-1} where $G^{-1} = (V(G^{-1}), E(G^{-1}))$, $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}$. Let $\widetilde{G}$ be a directed graph such that

$$E(\widetilde{G}) := E(G) \cup E(G^{-1}).$$

We note that $\widetilde{G}$ can be seen as an undirected graph by ignoring the direction of edges.

Definition 2.4. Let $G = (V(G), E(G))$ be a directed graph and $u, v \in V(G)$. A *path* in G from u to v of length n_0 ($n_0 \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^{n_0}$ of $n_0 + 1$ vertices such that $x_0 = u$, $x_{n_0} = v$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, ..., n_0$.

A graph G is said to be *connected* if there is a path between any two vertices of G.

A graph G is *disconnected* if it is not connected. Furthermore, if $\widetilde{G}$ is connected, then G is said to be *weakly connected*.

For $u \in V(G)$, let G_u be the subgraph of G consisting of all edges and vertices which are contained in some path in G beginning at u. We call G_u the component of G containing u. We see that $V(G_u) = \{v \in X :$ there is a path from u to $v\}$. In the case that $E(G)$ is symmetric, we denote $[u]_G$ the equivalence class of u, where $[u]_G = \{v \in X :$ there is a path from u to $v\}$ and we see that $V(G_u) = [u]_G$.

The following property is also needed for our main result.

Property A: Let (X, d) be a metric space with a directed graph G. We say that G has *Property* A if there exists a positive integer n_0 such that $(u_n, u) \in E(G)$ for all $n \geq n_0$ whenever $\{u_n\}_{n\in\mathbb{N}} \subset X$ with $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$.

3. Main Results

We first introduce a new type of contractive multi-valued mappings in a metric space with a directed graph.

Definition 3.1. Let (X, d) be a metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and $S, T : X \to CB(X)$. The mappings S, T are said to be *Berinde graph contractive* if there exist a function $\varphi : [0, \infty) \to [0, 1)$ with $\limsup_{t\to r^+} \varphi(t) < 1$ for every $r \in [0, \infty)$ and $L \geq 0$ such that, for $u \neq v$, $(u, v) \in E(G)$,

(i) $H(Su, Tv) < \varphi(d(u, v))d(u, v) + LD(v, Su)$;

(ii) if $x \in Su$, $y \in Tv$ and $d(x, y) < d(u, v)$, then $(x, y) \in E(G)$.

Remark 3.2. In above definition, if $L = 0$ and $\varphi(t) = k$, $t \in [0, \infty)$, $k \in (0, 1)$, then The mappings S, T become graph contractive mappings defined by Beg and Butt [8, Definition 1.11]. Example of Berinde graph contractive multi-valued mappings can be seen in Example 3.5 and 3.7.

If $T : X \to CB(X)$, the graph of T, *Graph*(T), is defined by *Graph*(T) $:= \{(x, y) : x \in X, y \in Tx\}$.

We now prove a common fixed point theorem for Berinde graph contractive multi-valued mappings.

Theorem 3.3. *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph having Property A. Let $S, T : X \to CB(X)$ be Berinde graph contractive and set $X_S := \{u \in X : (u, x) \in E(G)$ for some $x \in Su\}$. Then we have the following:*

(1) *$S|_{V(G_u)}$ and $T|_{V(G_u)}$ have a common fixed point for all $u \in X_S$.*

(2) *If G is weakly connected and $X_S \neq \emptyset$, then S and T have a common fixed point in X.*

(3) *If $X' := \bigcup\{V(G_u) : u \in X_S\}$, then $S|_{X'}$ and $T|_{X'}$ have a common fixed point.*

(4) *If Graph(S) $\subseteq E(G)$ and $E(G)$ contains all loops, then S and T have a common fixed point.*

Proof. (1) Let $u_0 \in X_S$, then there exists $u_1 \in Su_0$ such that $(u_0, u_1) \in E(G)$. We can choose $n_1 \in \mathbb{N}$ such that

$$[\varphi(d(u_0, u_1))]^{n_1} \leq [1 - \varphi(d(u_0, u_1))]d(u_0, u_1). \tag{3.1}$$

From Lemma 2.1, there is a $u_2 \in Tu_1$ such that

$$d(u_1, u_2) \leq H(Su_0, Tu_1) + [\varphi(d(u_0, u_1))]^{n_1}. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$\begin{aligned} d(u_1, u_2) &\leq H(Su_0, Tu_1) + [\varphi(d(u_0, u_1))]^{n_1} \\ &< \varphi(d(u_0, u_1))d(u_0, u_1) + LD(u_1, Su_0) + [\varphi(d(u_0, u_1))]^{n_1} \\ &\leq \varphi(d(u_0, u_1))d(u_0, u_1) + [1 - \varphi(d(u_0, u_1))]d(u_0, u_1) \\ &= d(u_0, u_1). \end{aligned}$$

Since $(u_0, u_1) \in E(G)$, $u_1 \in Su_0$, $u_2 \in Tu_1$, and $d(u_1, u_2) < d(u_0, u_1)$, by Definition 3.1, we have $(u_1, u_2) \in E(G)$.

Next, choose $n_2 > n_1$ such that

$$[\varphi(d(u_1, u_2))]^{n_2} \leq [1 - \varphi(d(u_1, u_2))]d(u_1, u_2).$$

By Lemma 2.1, there is a $u_3 \in Su_2$ such that

$$d(u_2, u_3) \leq H(Tu_1, Su_2) + [\varphi(d(u_1, u_2))]^{n_2}.$$

Then

$$\begin{aligned} d(u_2, u_3) &\leq H(Tu_1, Su_2) + [\varphi(d(u_1, u_2))]^{n_2} \\ &< \varphi(d(u_1, u_2))d(u_1, u_2) + LD(u_2, Tu_1) + [\varphi(d(u_1, u_2))]^{n_2} \\ &\leq \varphi(d(u_1, u_2))d(u_1, u_2) + [1 - \varphi(d(u_1, u_2))]d(u_1, u_2) \\ &= d(u_1, u_2). \end{aligned}$$

Since $(u_1, u_2) \in E(G)$, $u_2 \in Tu_1$, $u_3 \in Su_2$, and $d(u_2, u_3) < d(u_1, u_2)$, by Definition 3.1 we have $(u_2, u_3) \in E(G)$.

By induction, we obtain a sequence $\{u_k\}$ in X and a sequence $\{n_k\}$ of positive integers such that for each $k \in \mathbb{N}$, $u_{2k+1} \in Su_{2k}$ and $u_{2k+2} \in Tu_{2k+1}$, $(u_k, u_{k+1}) \in E(G)$ and

$$d(u_k, u_{k+1}) < d(u_{k-1}, u_k),$$

$$[\varphi(d(u_{k-1}, u_k))]^{n_k} \leq [1 - \varphi(d(u_{k-1}, u_k))]d(u_{k-1}, u_k),$$

$$d(u_k, u_{k+1}) \leq H(Su_{k-1}, Tu_k) + [\varphi(d(u_{k-1}, u_k))]^{n_k}, \text{ when } k \text{ is odd},$$

and

$$d(u_k, u_{k+1}) \leq H(Tu_{k-1}, Su_k) + [\varphi(d(u_{k-1}, u_k))]^{n_k}, \text{ when } k \text{ is even}.$$

Then $\{d(u_{k-1}, u_k)\}$ is a strictly decreasing sequence. It follows by Lemma 2.2 that $\{u_k\}$ is Cauchy. So there exists $u \in X$ such that $\lim_{k\to\infty} u_k = u$.

Next, we show that $u \in Su \cap Tu$.

For k even, by property A, there exists a positive integer k_0 such that $(u_k, u) \in E(G)$ for all $k \geq k_0$. Since S, T are Berinde graph contractive, we have

$$\begin{aligned} H(Su_k, Tu) &< \varphi(d(u_k, u))d(u_k, u) + LD(u, Su_k) \\ &\leq \varphi(d(u_k, u))d(u_k, u) + Ld(u, u_{k+1}) \to 0. \end{aligned}$$

Since $u_{k+1} \in Su_k$ and $u_k \to u$, by Lemma 2.3, we get $u \in Tu$.

For k odd, we have

$$\begin{aligned} H(Tu_k, Su) &< \varphi(d(u_k, u))d(u_k, u) + LD(u, Tu_k) \\ &\leq \varphi(d(u_k, u))d(u_k, u) + Ld(u, u_{k+1}) \to 0. \end{aligned}$$

Again by Lemma 2.3, we get $u \in Su$.

Since $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $(u_k, u) \in E(G)$ for all $k \geq k_0$, we have $(u_0, u_1, ..., u_k, u)$ is a path in G and so $u \in V(G_{u_0})$.

(2) Let $u_0 \in X_S$. Since G is weakly connected, we have $V(G_{u_0}) = X$, and by (1), we can conclude that S and T have a common fixed point in X.

(3) It follows directly from (1).

(4) If $Graph(S) \subseteq E(G)$, then we have, for each $u \in X$, there is a point $x \in Su$ such that $(u, x) \in E(G)$, so $X_S = X$. Since $\Delta \subseteq E(G)$, it follows that $X' := \bigcup\{V(G_u) : u \in X_S\} = X$. By (3), we obtain that S and T have a common fixed point. $\square$

Remark 3.4. The result of Beg and Butt [8, Theorem 2.1] is obtained directly by Theorem 3.3 and Remark 3.2.

The following example provides two multi-valued mappings on a metric space with a directed graph satisfying all assumptions and conditions of Theorem 3.3.

Example 3.5. Let $X = \{\frac{1}{3^n} : n \in \mathbb{N}\} \cup \{0, 1, 2\}$ and $d(u, v) = |u - v|$ for all $u, v \in X$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G) = \{(0, 0), (0, \frac{1}{3^n}) : n \in \mathbb{N}\}$. Let $S, T : X \to CB(X)$ be defined by

$$S(u) = \begin{cases} \{0\} & \text{if } \ u = 0, \\ \{0, \frac{1}{3^{n+2}}, \frac{1}{3^{n+1}}\} & \text{if } \ u = \frac{1}{3^n}, n \in \mathbb{N}, \\ \{2\} & \text{if } \ u = 1, \\ \{1\} & \text{if } \ u = 2, \end{cases}$$

and

$$T(u) = \begin{cases} \{0\} & \text{if } \ u = 0, \\ \{\frac{1}{3^{n+3}}, \frac{1}{3^{n+2}}\} & \text{if } \ u = \frac{1}{3^n}, n \in \mathbb{N} \cup \{0\}, \\ \{2\} & \text{if } \ u = 2. \end{cases}$$

We shall show that S, T are Berinde graph contractive with $\varphi(d(u, v)) = \frac{1}{3}$ and $L = 3$.

Let $(u, v) \in E(G)$ such that $u \neq v$. Then $(u, v) = (0, \frac{1}{3^n})$ for $n \in \mathbb{N}$. We have $H(S0, T(\frac{1}{3^n})) = \frac{1}{3^{n+2}} < \frac{1}{3}d(0, \frac{1}{3^n}) + 3D(\frac{1}{3^n}, S0)$ and $H(T0, S(\frac{1}{3^n})) = \frac{1}{3^{n+1}} < \frac{1}{3}d(0, \frac{1}{3^n}) + 3D(\frac{1}{3^n}, T0)$.

Next, we show that (ii) of Definition 3.1 is satisfied. Let $(u, v) \in E(G)$ such that $u \neq v$. Then $(u, v) = (0, \frac{1}{3^n})$ for $n \in \mathbb{N}$. Thus $Su = S(0) = \{0\}$, $Tv = T(\frac{1}{3^n}) = \{\frac{1}{3^{n+3}}, \frac{1}{3^{n+2}}\}$, $Tu = T(0) = \{0\}$ and $Sv = S(\frac{1}{3^n}) = \{0, \frac{1}{3^{n+2}}, \frac{1}{3^{n+1}}\}$.

We note that if $x \in Su$, $y \in Tv$ and $d(x, y) < d(u, v)$, then (x, y) are $(0, \frac{1}{3^{n+3}}), (0, \frac{1}{3^{n+2}})$, so $(x, y) \in E(G)$, and if $x' \in Tu$, $y' \in Sv$ and $d(x', y') < d(u, v)$, then (x', y') are $(0, 0), (0, \frac{1}{3^{n+2}}), (0, \frac{1}{3^{n+1}})$, so $(x', y') \in E(G)$.

Hence S, T are Berinde graph contractive. It is easy to see that G has Property A. Therefore all conditions of Theorem 3.3 are satisfied, so for any $u \in X_S$, $S|_{V(G_u)}$ and $T|_{V(G_u)}$ have a common fixed point. We note that $X_S := \{u \in X : (u, x) \in E(G) \text{ for some } x \in Su\} = \{0\}$, while $V(G_0) = \{0, \frac{1}{3^n} : n \in \mathbb{N}\}$. Moreover, $Fix(S) = \{0\}$, $Fix(T) = \{0, 2\}$ and $Fix(S) \cap Fix(T) = \{0\}$.

Remark 3.6. In Theorem 3.3, the condition $X_S \neq \emptyset$ can be replaced by $X_T \neq \emptyset$ and the condition $Graph(S) \subseteq E(G)$ can be replaced by $Graph(T) \subseteq E(G)$.

The next example shows that $X_S = \emptyset$ but $X_T \neq \emptyset$.

Example 3.7. Let $X = \{\frac{1}{3^n} : n \in \mathbb{N}\} \cup \{0,1,2\}$ and $d(u,v) = |u - v|$ for all $u, v \in X$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G) = \{(0,0), (0, \frac{1}{3^n}) : n \in \mathbb{N}\}$. Let $S, T : X \to CB(X)$ be defined by

$$S(u) = \begin{cases} \{1\} & \text{if } u = 0, \\ \{\frac{1}{3}\} & \text{if } u = \frac{1}{3}, \\ \{\frac{1}{3^{n+2}}, \frac{1}{3^{n+1}}\} & \text{if } u = \frac{1}{3^n}, n \in \mathbb{N} \setminus \{1\}, \\ \{0,1\} & \text{if } u = 1, \\ \{2\} & \text{if } u = 2, \end{cases}$$

and

$$T(u) = \begin{cases} \{0\} & \text{if } u = 0, \\ \{\frac{1}{3}\} & \text{if } u = \frac{1}{3}, \\ \{\frac{1}{3^{n+3}}, \frac{1}{3^{n+2}}\} & \text{if } u = \frac{1}{3^n}, n \in \mathbb{N} \cup \{0\} \setminus \{1\}, \\ \{2\} & \text{if } u = 2. \end{cases}$$

We shall show that S, T are Berinde graph contractive with $\varphi(d(u,v)) = \frac{1}{3}$ and $L = 3$.

Let $(u,v) \in E(G)$ such that $u \neq v$. Then $(u,v) = (0, \frac{1}{3^n})$ for $n \in \mathbb{N}$. If $(u,v) = (0, \frac{1}{3^n})$ for $n \in \mathbb{N} \setminus \{1\}$, then we have $H(S0, T(\frac{1}{3^n})) = \frac{3^{n+3}-1}{3^{n+3}} < \frac{1}{3}d(0, \frac{1}{3^n}) + 3D(\frac{1}{3^n}, S0)$ and $H(T0, S(\frac{1}{3^n})) = \frac{1}{3^{n+1}} < \frac{1}{3}d(0, \frac{1}{3^n}) + 3D(\frac{1}{3^n}, T0)$. If $(u,v) = (0, \frac{1}{3})$, then we have $H(S0, T(\frac{1}{3})) = \frac{2}{3} < \frac{1}{3}d(0, \frac{1}{3}) + 3D(\frac{1}{3}, S0)$ and $H(T0, S(\frac{1}{3})) = \frac{1}{3} < \frac{1}{3}d(0, \frac{1}{3}) + 3D(\frac{1}{3}, T0)$.

Next, we show that (ii) of Definition 3.1 is satisfied. Let $(u,v) \in E(G)$ such that $u \neq v$. Then $(u,v) = (0, \frac{1}{3^n})$ for $n \in \mathbb{N}$. If $(u,v) = (0, \frac{1}{3^n})$ for $n \in \mathbb{N} \setminus \{1\}$, then $Su = S(0) = \{1\}$, $Tv = T(\frac{1}{3^n}) = \{\frac{1}{3^{n+3}}, \frac{1}{3^{n+2}}\}$, $Tu = T(0) = \{0\}$ and $Sv = S(\frac{1}{3^n}) = \{\frac{1}{3^{n+2}}, \frac{1}{3^{n+1}}\}$. We note that if $x \in Su$, $y \in Tv$ and $d(x,y) < d(u,v)$, then $(x,y) \in E(G)$, and if $x' \in Tu$, $y' \in Sv$ and $d(x', y') < d(u,v)$, then (x', y') are $(0, \frac{1}{3^{n+2}}), (0, \frac{1}{3^{n+1}})$, so $(x', y') \in E(G)$.

If $(u,v) = (0, \frac{1}{3})$, then $Su = S(0) = \{1\}$, $Tv = T(\frac{1}{3}) = \{\frac{1}{3}\}$, $Tu = T(0) = \{0\}$ and $Sv = S(\frac{1}{3}) = \{\frac{1}{3}\}$. We note that if $x \in Su$, $y \in Tv$ and $d(x,y) < d(u,v)$, then $(x,y) \in E(G)$, and if $x' \in Tu$, $y' \in Sv$ and $d(x', y') < d(u,v)$, then $(x', y') \in E(G)$.

Hence S, T are Berinde graph contractive. It is easy to see that G has Property A. We see that $X_S := \{u \in X : (u,x) \in E(G) \text{ for some } x \in Su\} = \emptyset$ but $X_T := \{u \in X : (u,x) \in E(G) \text{ for some } x \in Tu\} = \{0\} \neq \emptyset$. Therefore all conditions of Theorem 3.3 are satisfied, so for any $u \in X_T$, $S|_{V(G_u)}$ and $T|_{V(G_u)}$ have a common fixed point. We note that $V(G_0) = \{0, \frac{1}{3^n} : n \in \mathbb{N}\}$. Moreover, $Fix(S) = \{\frac{1}{3}, 1, 2\}$, $Fix(T) = \{0, \frac{1}{3}, 2\}$ and $Fix(S) \cap Fix(T) = \{\frac{1}{3}, 2\}$.

The following results are a direct consequence of Theorem 3.3.

Corollary 3.8. *Let (X,d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph having Property A. If G is weakly connected, then Berinde graph contractive mappings $S, T : X \to CB(X)$ such that $(x_0, x_1) \in E(G)$ for some $x_1 \in Sx_0$ have a common fixed point.*

Corollary 3.9. *Let (X,d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph having Property A. Let $S : X \to CB(X)$ be a mapping satisfying the following conditions: there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\limsup_{t\to r^+} \varphi(t) < 1$ for every $r \in [0,\infty)$ and $L \geq 0$ such that, for $u \neq v$, $(u,v) \in E(G) \Rightarrow H(Su, Sv) < \varphi(d(u,v))d(u,v) + LD(v, Su)$ and if $x \in Su$, $y \in Sv$ with $d(x,y) < d(u,v)$, then $(x,y) \in E(G)$, and let $X_S := \{u \in X : (u,x) \in E(G) \text{ for some } x \in Su\}$. Then we have*

(1) *$S|_{V(G_u)}$ has a fixed point for all $u \in X_S$.*
(2) *If G is weakly connected and $X_S \neq \emptyset$, then S has a fixed point in X.*
(3) *If $X' := \bigcup\{V(G_u) : u \in X_S\}$, then $S|_{X'}$ has a fixed point.*
(4) *If $Graph(S) \subseteq E(G)$ and $E(G)$ contains all loops, then S has a fixed point.*
(5) *If $X_S \neq \emptyset$, then $Fix(S) \neq \emptyset$.*

Proof. (1)-(4) are obtained directly by Theorem 3.3 by setting $S = T$ and (5) follows by (1). □

4. Some applications on ε-chainable metric spaces and cyclic contractions

In this section, we apply our main result, Theorem 3.3, to obtain common fixed point results for two multi-valued mappings in ε-chainable complete metric spaces and two cyclic contraction multi-valued mappings.

From Theorem 1.3, the result of Edelstein [14], we get the idea for the following theorem.

Theorem 4.1. *Let (X, d) be a ε-chainable complete metric space. Let $S, T : X \to CB(X)$ be such that there exist a function $\varphi : [0, \infty) \to [0, 1)$ with $\limsup_{t \to r^+} \varphi(t) < 1$ for every $r \in [0, \infty)$ and $L \geq 0$ such that*

$$0 < d(u, v) < \varepsilon \Rightarrow H(Su, Tv) < \varphi(d(u, v))d(u, v) + LD(v, Su).$$

There exists $x_0 \in Su_0$ such that $0 < d(u_0, x_0) < \varepsilon$. Then S and T have a common fixed point.

Proof. Set the graph G as $V(G) := X$ and

$$E(G) := \Delta \cup \{(u, v) \in X \times X : 0 < d(u, v) < \varepsilon\}.$$

The ε-chainability of (X, d) gives connectivity of G. If $(u, v) \in E(G)$, then

$$H(Su, Tv) < \varphi(d(u, v))d(u, v) + LD(v, Su).$$

Next, let $x \in Su$, $y \in Tv$ and $d(x, y) < d(u, v)$. Since $(u, v) \in E(G)$, then $0 < d(u, v) < \varepsilon$. We note that if $x \neq y$, then $0 < d(x, y) < d(u, v) < \varepsilon$, so $(x, y) \in E(G)$. Therefore S, T are Berinde graph contractive.

We also see that if $u_n \to u$ and $d(u_n, u_{n+1}) < \varepsilon$ for $n \in \mathbb{N}$, then there exists a positive integer n_0 such that $d(u_n, u) < \varepsilon$ for all $n \geq n_0$, so $(u_n, u) \in E(G)$. Thus G has Property A. Since there exists $x_0 \in Su_0$ such that $0 < d(u_0, x_0) < \varepsilon$, $(u_0, x_0) \in E(G)$. Then $u_0 \in X_S$, that is $X_S \neq \emptyset$. Therefore, by Theorem 3.3(2), S and T have a common fixed point. $\square$

In 2003, Kirk et. al. [23] introduced the notions of cyclic contractions. Then in 2005, Rus [32] introduced the notions of cyclic representations, suggested by Kirk et. al. [23], see [28], as the following notion. Let X be a nonempty set, m a positive integer and $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X and $f : \cup_{i=1}^m A_i \to \cup_{i=1}^m A_i$ be an operator. Then $X := \cup_{i=1}^m A_i$ is said to be *cyclic representation of X with respect to f* if;

$$f(A_1) \subset A_2, ..., f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$$

and operator f is said to be *cyclic operator*.

The following existence result for cyclic contraction was proved by Karapinar [21].

Theorem 4.2 ([21]). *Let (X, d) be a complete metric space. Let m be a positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that $\cup_{i=1}^m A_i$ is cyclic representation of Y w.r.t. f and there exists $\psi : [0, \infty) \to [0, \infty)$ where ψ is continuous, nondecreasing, positive on $(0, \infty)$, $\psi(0) = 0$ and the following holds:*

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for } x \in A_i, y \in A_{i+1}; A_{m+1} = A_1.$$

Then f has a unique fixed point $p \in \cap_{i=1}^m A_i$ and $f^n y \to p$ for all $y \in \cup_{i=1}^m A_i$.

Finally, we introduce the notions of cyclic representations for multi-valued mapping. Let X be a nonempty set, m a positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $X := \cup_{i=1}^m A_i$ and $T : X \to 2^X$ be a mapping. Then $X := \cup_{i=1}^m A_i$ is said to be *cyclic representation of X with respect to T* if

$$T : A_i \to CB(A_{i+1}) \text{ for } i = 1, ..., m; A_{m+1} = A_1$$

and operator T is said to be *cyclic multi-valued mapping*.

Next, we prove a common fixed point of cyclic multi-valued mappings.

Theorem 4.3. *Let (X, d) be a complete metric space. Let m be a positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $S, T : Y \to 2^Y$. Assume that $\cup_{i=1}^m A_i$ is cyclic representation of Y w.r.t. S, T and there exist a function $\varphi : [0, \infty) \to [0, 1)$ with $\limsup_{t \to r^+} \varphi(t) < 1$ for every $r \in [0, \infty)$ and $L \geq 0$ such that, for $u \neq v$,*

$$H(Su, Tv) < \varphi(d(u, v))d(u, v) + LD(v, Su) \text{ for } u \in A_i, v \in A_{i+1}; A_{m+1} = A_1.$$

Then S and T have a common fixed point.

Proof. Since A_i, $i \in \{1, ..., m\}$ are closed, (Y, d) is a complete metric space. Consider a graph G consisting of $V(G) = Y$ and $E(G) = \Delta \cup \{(u, v) \in Y \times Y : u \in A_i, v \in A_{i+1}; i = 1, ..., m, A_{m+1} = A_1\}$. Let $u, v \in Y$ be such that $u \neq v$ and $(u, v) \in E(G)$. Then $u \in A_i$, $v \in A_{i+1}$ for some $i \in \{1, ..., m\}$, so we have

$$H(Su, Tv) < \varphi(d(u, v))d(u, v) + LD(v, Su).$$

Suppose that $x \in Su$, $y \in Tv$ and $d(x, y) < d(u, v)$. Then $x \in Su \subseteq A_{i+1}$, $y \in Tv \subseteq A_{i+2}$, hence $(x, y) \in E(G)$. Therefore S, T are Berinde graph contractive.

Next, we will show that G has Property A. Assume $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in Y with $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. It follows that a sequence $\{u_n\}$ has infinitely many terms in each A_i, so that one can easily extract a subsequence of $\{u_n\}$ converging to u. Since A_i is closed for all $i \in \{1, ..., m\}$, it follows that $u \in \cap_{i=1}^m A_i$. This implies by definition of $E(G)$ that $(u_n, u) \in E(G)$ for all $n \in \mathbb{N}$.

We also note that $u_0 \in Y$, then $u_0 \in A_i$ for some $i \in \{1, ..., m\}$ and $Su_0 \subseteq A_{i+1}$. Choose $y_0 \in Su_0$. By definition of $E(G)$, we have $(u_0, y_0) \in E(G)$. This implies that $Y_S := \{u \in Y : (u, y) \in E(G) \text{ for some } y \in Su\} \neq \emptyset$.

By definition of $E(G)$, we see that $Graph(S) \subseteq E(G)$ and $Graph(T) \subseteq E(G)$. By Theorem 3.3, we can conclude that S and T have a common fixed point in Y. □

5. Acknowledgement

The authors would like to thank the referees for valuable comments and suggestions for improving this work and the Thailand Research Fund under the project RTA 5780007, the Royal Golden Jubilee (RGJ) Ph.D. Programme with contract no. PHD/0093/2558 and Chiang Mai University for the financial support.

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