Filomat 32:3 (2018), 921–931
https://doi.org/10.2298/FIL1803921A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Differential Operator Equations with Interface Conditions in Modified Direct Sum Spaces

Kadriye Aydemir[a], Hayati Olğar[b], Oktay Sh. Mukhtarov[b,c], Fahreddin Muhtarov[d]

[a]*Department of Mathematics, Faculty of Art and Science, Amasya University, 05100 Amasya, Turkey*
[b]*Department of Mathematics, Faculty of Arts and Science, Gaziosmanpaşa University, 60250 Tokat, Turkey*
[c]*Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan*
[d]*Odlar Yurdu University Baku, Baku, Azerbaijan*

Abstract. We investigate a new type boundary value problem consisting of a differential-operator equation, eigendependent boundary conditions, and two supplementary conditions so-called interface conditions. We give a characterisation of some spectral properties of the considered problem. Particularly, it is established such properties as isomorphism and coerciveness, discreteness of the spectrum and found asymptotic formulas for eigenvalues.

1. Introduction

The classical boundary value problems (BVP's) arise as a mathematical modeling of many systems and processes in the fields of physics, chemistry, aerodynamics, fluid dynamics, diffusion etc. But some mechanical and physical systems lead to various non-classical forms of BVP's. For example, Sturm-Liouville problems with eigenparameter appearing in the boundary conditions, and with supplementary interface conditions at some interior singular points arise in non-classical problems of physics, namely in vibrating string problems when the string loaded additionally with point masses, in problems involving heat conduction through a liquid-interface, in diffraction problems of water vapour through a porous membrane (for other examples see [14, 22–24]).

In this study, we consider new type of non-classical boundary value problems consisting of a "Sturm-Liouville equation" involving an abstract linear operator $\mathcal{A}$ given by

$$\ell f := -f'' + q(x)f + \mathcal{A}f|_x = \lambda f, \quad x \in [-\pi, 0) \cup (0, \pi] \tag{1}$$

together with eigenparameter-dependent boundary conditions given by

$$\ell_1 f := \delta_{10} f(-\pi) - \delta_{11} f'(-\pi) = 0, \tag{2}$$

$$\ell_2(\lambda) f := \delta_{20} f(\pi) - \delta_{21} f'(\pi) + \lambda(\delta'_{20} f(\pi) - \delta'_{21} f'(\pi)) = 0, \tag{3}$$

2010 *Mathematics Subject Classification*. Primary 34A36; Secondary 34B08, 34B24
Keywords. Sturm-Liouville problem, eigenvalues, abstract linear operator
Received: 28 December 2016; Revised: 23 May 2017; Accepted: 23 May 2017
Communicated by Allaberen Ashyralyev
Email addresses: kadriyeaydemr@gmail.com (Kadriye Aydemir), hayatiolgar@gmail.com (Hayati Olğar), omukhtarov@yahoo.com (Oktay Sh. Mukhtarov), fahreddinmuhtarov@gmail.com (Fahreddin Muhtarov)

and two supplementary interface conditions at one interior singular point $x = 0$, given by

$$\ell_3 f := \gamma_{11}^- f'(0-) + \gamma_{10}^+ f(0+) = 0, \tag{4}$$

$$\ell_4 f := \gamma_{21}^- f'(0-) + \gamma_{20}^- f(0-) + \gamma_{21}^+ f'(0+) + \gamma_{20}^+ f(0+) = 0, \tag{5}$$

where the function $q(x)$ is continuous in the intervals $[-\pi, 0)$ and $(0, \pi]$ for which there are finite left and right limits $q(0\pm)$ at the singular point $x = 0$, $\lambda \in \mathbb{C}$ is a eigenvalue parameter, δ_{ij}, δ'_{ij} $\gamma_{ij}^\pm$, $(i = 1, 2$ and $j = 0, 1)$ are real numbers, $\mathcal{A}$ is an abstract linear operator which is non-self-adjoint and unbounded in general in the Lebesgue space $L_2[-\pi, \pi]$. Naturally we shall assume that $|\delta_{10}| + |\delta_{11}| \neq 0$, $|\delta_{20}| + |\delta_{21}| + |\delta'_{20}| + |\delta'_{21}| \neq 0$, $|\gamma_{11}^-| + |\gamma_{10}^+| \neq 0$ and $|\gamma_{21}^-| + |\gamma_{20}^-| + |\gamma_{21}^+| + |\gamma_{20}^+| \neq 0$. Note that the considered boundary-value problem covered a wide class of non-standard Sturm-Liouville type problems. For example, the results of this study is applicable to the problem consisting of the equation

$$-f''(x) + q(x)f(x) + \sum_{k=1}^{n} u_k(x)f(c_k) + \sum_{k=1}^{m} v_k(x)f'(d_k) + \sum_{k=0}^{1}\Big(\int_{-\pi}^{0} R_k(x,t)f^{(k)}(t)dt$$

$$+ \int_{0}^{\pi} T_k(x,t)f^{(k)}(t)dt\Big) = \lambda f(x), \ \ x \in [-\pi, 0) \cup (0, \pi],$$

and the same boundary and interface conditions (2) – (5), where $u_k(x)$ and $v_k(x)$ are piecewise continuous functions on $[-\pi, \pi]$ having discontinuities only at the point $x = 0$ and only of the first kind, the kernels $R_k(x, t)$ and $T_k(x, t)$ are defined and continuous in $[-\pi, \pi]\times[-\pi, 0]$ and $[-\pi, \pi]\times[0, \pi]$, respectively. Note that some non-classical Sturm-Liouville differential operators have been investigate extensively in the recent years [1–5, 7–11, 13, 15–21, 25].

2. Construction of the Adequate Hilbert Spaces

Let us consider boundary value problems (1) – (5). For operator-treatment of this problem we shall introduce a new inner-products in the classical Sobolev spaces. To this we shall assume everywhere in below that

$$\theta := \begin{vmatrix} \delta_{21} & \delta'_{21} \\ \delta_{20} & \delta'_{20} \end{vmatrix} > 0 \quad \text{and} \quad \Delta := \begin{vmatrix} \gamma_{21}^- & \gamma_{21}^+ \\ \gamma_{20}^- & \gamma_{20}^+ \end{vmatrix} > 0.$$

Let $\Omega \subset \mathbb{R}$ be any closed bounded interval. Recall that the Sobolev space $W_2^k(\Omega)(k = 0, 1, 2, ...)$ is the Hilbert space consisting of all functions $f \in L_2(\Omega)$ that have generalized derivatives $f', f'', ..., f^{(k)} \in L_2(\Omega)$ with the inner product

$$< f, g >_{W_2^k(\Omega)} = \sum_{n=0}^{k} (< f^{(n)}, g^{(n)} >_{L_2(\Omega)},$$

where $L_2(\Omega)$ is the usual Lebesgue space, i.e. the Hilbert space of measurable and square-integrable complex valued functions on the interval Ω with the inner product

$$\langle f, g\rangle_{L_2(\Omega)} := \int_{\Omega} f(x)\overline{g(x)}dx.$$

Of course, here by $f^{(0)}, g^{(0)}$, and $W_2^0(\Omega)$, we mean f, g, and $L_2(\Omega)$, respectively.
The standard inner product in direct sum space $\mathcal{H}_0 = (L_2(-\pi, 0) \oplus L_2(0, \pi)) \oplus \mathbb{C}$ which is given by

$$< U, V >_{\mathcal{H}_0} := < u(.), v(.) >_{L_2} + u_1\overline{v_1}$$

for $U = (u(.), u_1),\ V = (v(.), v_1) \in \mathcal{H}_0$, we shall replace by the "weight" inner product on the direct sum space $\mathcal{H} = \mathcal{H}_1 \oplus \mathbb{C}$ by

$$\langle F, G\rangle_{\mathcal{H}} := \langle f, g\rangle_{\mathcal{H}_1} + \frac{\Delta}{\theta} f_1 \overline{g_1} \tag{6}$$

for $F = (f(x), f_1)$ and $G = (g(x), g_1) \in \mathcal{H}$, where by $\mathcal{H}_1$ we mean the linear space $L_2[-\pi, 0) \oplus L_2(0, \pi]$ equipped with modified the inner product

$$\langle f, g\rangle_{\mathcal{H}_1} := \Delta \int_{-\pi}^{0-} f(x)\overline{g(x)}dx + \int_{0+}^{\pi} f(x)\overline{g(x)}dx$$

and apply operator theory in the Hilbert space

$$\mathcal{H} := (L_2(-\pi, 0) \oplus L_2(0, \pi)) \oplus \mathbb{C}, < ., . >_{\mathcal{H}}).$$

Remark 2.1. It is readily seen that modified inner product (6) is equivalent to standard inner product of $(L_2(-\pi, 0) \oplus L_2(0, \pi)) \oplus C$, so $\mathcal{H}$ is also Hilbert space and can be seen as different realization of the Hilbert space $\mathcal{H}_0$.

3. Operator-Theoretical Interpretation of the Problem

Denoting

$$B_\pi[f] := \delta_{20} f(\pi) - \delta_{21} f'(\pi),$$

$$B'_\pi[f] := \delta'_{20} f(\pi) - \delta'_{21} f'(\pi),$$

and

$$\Phi u := -u'' + q(x)u,$$

we shall define the linear operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ with action low

$$\mathcal{L}(f(x), -B'_\pi[f]) := (\ell f, B_\pi[f])$$

and domain of definition

$$dom(\mathcal{L}) := \Big\{ F = (f(x), f_1) : f(x), f'(x) \in AC_{loc}(-\pi, 0) \cap AC_{loc}(0, \pi),\ \ell F \in L_2(-\pi, 0) \oplus L_2(0, \pi),$$
$$\text{there are finite limits} f(0\mp) \text{ and } f'(0\mp),\ \ell_1(f) = \ell_3(f) = \ell_4(f) = 0, f_1 = -B'_\pi[f] \Big\}$$

Then problems $(1) - (5)$ is acquired to the operator equation form

$$\mathcal{L}F = \lambda F,\ \ F = (f(x), -B'_\pi[f]) \in dom(\mathcal{L})$$

in the Hilbert space $\mathcal{H}$. Consequently the eigenvalues of the operator $\mathcal{L}$ and those of considered problems $(1) - (5)$ are coincide.

Lemma 3.1. *The set $dom(\mathcal{L})$ is dense in the Hilbert space $\mathcal{H}$.*

Proof. Let, $Y_0 = (y_0(.), y_1) \in \mathcal{H}$ be any element satisfying the orthogonality relation

$$\langle X, Y_0\rangle_{\mathcal{H}} := \Delta \int_{-\pi}^{0-} x(s)\overline{y_0(s)}ds + \int_{0+}^{\pi} x(s)\overline{y_0(s)}ds - \frac{\Delta}{\theta} B'_\pi[x]\overline{y_1} = 0 \tag{7}$$

for all $X = (x(.), -B'_\pi[x]) \in D(\mathcal{L})$. Let $f_1 \in C_0^\infty[-\pi, 0]$ and $f_2 \in C_0^\infty[0, \pi]$ be arbitrary functions and let $f = \begin{cases} f_1(x) & for\ x \in [-\pi, 0) \\ f_2(x) & for\ x \in (0, \pi] \end{cases}$.

Obviously $F := (f(.), 0) \in D(\mathcal{L})$. Putting in (7), we get

$$\Delta \int_{-\pi}^{0-} f_1(s)\overline{y_0(s)}ds + \int_{0+}^{\pi} f_2(s)\overline{y_0(s)}ds - \frac{\Delta}{\theta}B'_\pi[f]\overline{y_1} = 0.$$

By taking $f_2 = 0$, we see from the last equality that

$$\Delta \int_{-\pi}^{0-} f_1(s)\overline{y_0(s)}ds = 0$$

for all $f_1 \in C_0^\infty[-\pi, 0]$. Since $C_0^\infty[-\pi, 0]$ is dense in $L_2(-\pi, 0)$, this leads to $y_0(s) = 0$ on $[-\pi, 0)$. Similarly, by taking $f_1 = 0$, we have that

$$\int_{0+}^{\pi} f_2(s)\overline{y_0(s)}ds = 0$$

for all $f_2 \in C_0^\infty[0, \pi]$, from which it follows that $y_0(s) = 0$ on $(0, \pi]$.

We can choose an element $\widetilde{X}_0 := (\widetilde{x_0}(.), -B'_\pi(\widetilde{x_0})) \in D(\widetilde{\mathcal{L}})$ such that $-B'_\pi(\widetilde{x_0}) = -y_1$. Putting in (7) we get

$$\langle \widetilde{X}_0, Y_0 \rangle_{\mathcal{H}} = -\frac{\Delta}{\theta} \mid y_1 \mid^2 = 0$$

and so $y_1 = 0$. Consequently, $Y_0 = 0$ which proves that the orthogonal complement of $D(\mathcal{L})$ is null element of the space $\mathcal{H}$. Hence $D(\mathcal{L})$ is dense in $\mathcal{H}$. The proof is complete. □

4. Topological Isomorphism and Coerciveness

To establish the topological isomorphism and coerciveness we shall define a new inner product space $\mathcal{H}_2$ as the linear space

$$\left\{U = (\ u(.), u_1\ ) : u(.) \in W_2^2(-\pi, 0)\ \oplus W_2^2(0, \pi),\ \ell_1(u) = \ell_3(u) = \ell_4(u) = 0,\ u_1 = -B'_\pi(u)\right\}$$

equipped with inner product

$$< (u(.), u_1), (v(.), v_1) >_{\mathcal{H}_2} = < u(.), v(.) >_{W_2^2} \tag{8}$$

and corresponding norm

$$\|(u(.), u_1)\|_{\mathcal{H}_2} = \|u(.)\|_{W_2^2}.$$

Lemma 4.1. *$\mathcal{H}_2$ is a Hilbert space.*

Proof. Let $U_n = (u_n(.), -B'_\pi[u_n]), n = 1, 2, ...$ be any Cauchy sequence in the inner-product space $\mathcal{H}_2$. Since

$$\|u_n - u_m\|_{W_2^2} = \|U_n - U_m\|_{\mathcal{H}_2}$$

by (8) we see that the first components $(u_n(.))$ of the sequence (U_n) forms a Cauchy sequence of the Hilbert space $W_2^2(-\pi, 0) \oplus W_2^2(0, \pi)$, therefore is convergent. Let $u = u(x)$ be the limit of this sequence. By virtue of the well-known properties of the Sobolev spaces, the embeddings $W_2^2(-\pi, 0) \subset C[-\pi, 0]$ and $W_2^2(0, \pi) \subset C[0, \pi]$ are compact and therefore the reel sequences $\ell_1(u_n), \ell_2(u_n)$, and $\ell_4(u_n)$ converge to $\ell_1(u), \ell_2(u)$ and $\ell_4(u)$ respectively. By the definition of the space $\mathcal{H}_2$ we know that $\ell_1(u_n) = \ell_2(u_n) = \ell_4(u_n) = 0$ for all n. Consequently, $\ell_1(u) = \ell_2(u) = \ell_4(u) = 0$. Hence $U := (u(.), -B'_\pi[u]) \in \mathcal{H}_2$ and

$$\|U_n - U\|_{\mathcal{H}_2} = \|u_n - u\|_{W_2^2} \to 0\ \ as\ \ n \to \infty$$

so (U_n) is convergent. Since the sequence (U_n) is an arbitrary Cauchy sequence, $\mathcal{H}_2$ is complete. Thus this inner product space is a Hilbert Space. □

Now, consider nonhomogeneous boundary value transmission problem

$$\Phi u - \lambda u = f(x),\ x \in [-\pi, 0) \cup (0, \pi]\ ,\ \ell_1(\lambda)u = f_1,\ \ell_2 u = \ell_3 u = \ell_4 u = 0, \tag{9}$$

for $f \in L_2(-\pi, 0) \oplus L_2(0, \pi), f_1 \in \mathbb{C}$. Denote $U(x) := (u(x), -B'_\pi(u)) \in D(\mathcal{L})$ and $F := (f(x), f_1) \in \mathcal{H}$. Then problem (9) reduces to operator equation

$$(\lambda I - \mathcal{L})U = F,\ F \in \mathcal{H} \tag{10}$$

in the Hilbert space $\mathcal{H}$. For convenience, in below we use the notations

$$G_\epsilon = \{\lambda \in \mathbb{C} \mid \epsilon < arg\mu < 2\pi - \epsilon\},\ 0 < \epsilon < 2\pi,$$

and

$$U_\infty(r) = \{\lambda \in \mathbb{C} :\mid \lambda \mid > r\},\ r > 0.$$

Theorem 4.2. *Suppose that the operator $\mathcal{A}$ acted compactly from $W_2^2(-\pi, 0) \oplus W_2^2(0, \pi)$ into $L_2(-\pi, 0) \oplus L_2(0, \pi)$. Then, for any $\varepsilon > 0$ there exists sufficiently large $r_\varepsilon > 0$ such that for all $\lambda \in G_\varepsilon \cap U_\infty(r_\varepsilon)$ the operator $\mathcal{L} - \lambda I$ is an isomorphism from $\mathcal{H}_2$ onto $\mathcal{H}$ and following coercive estimate*

$$\|U(\lambda, F)\|_{\mathcal{H}_2} + |\lambda|\, \|U(\lambda, F)\|_{\mathcal{H}} \le C(\varepsilon)\, \|F\|_{\mathcal{H}} \tag{11}$$

holds for the solution $U = U(\lambda, F)$ of operator equation (10) where $C(\varepsilon)$ is a constant, which depend only of ε.

Proof. It is obvious that the operator $\mathcal{L} - \lambda I$ is bounded from $\mathcal{H}_2$ into $\mathcal{H}$ for all complex number λ. Applying the same argument from [19], we have that for arbitrary $\epsilon > 0$, small enough, there are positive numbers r_ϵ and C_ϵ such that for all $\lambda \in G_\epsilon \cap U_\infty(r_\epsilon)$ the linear operator $T(\lambda)$ defined by

$$T(\lambda)u = (\lambda u - \Phi(\lambda)u, \ell_1 u)$$

is an isomorphism between the Hilbert spaces $W_2^2(-\pi, 0) \oplus W_2^2(0, \pi)$ and $(L_2(-\pi, 0) \oplus L_2(0, \pi)) \oplus \mathbb{C}$ and for these λ, coercive estimate

$$\|u\|_{W_2^2} + |\lambda|(\, \|u\|_{L_2} + |B'_\pi(u)|) \le C_\epsilon(\|f\|_{L_2} + |f_1|) \tag{12}$$

holds for the solution of the nonhomogeneous boundary-value-transmission problem (9). This proves that the linear operator $\mathcal{L} - \lambda I$ is an isomorphism between the Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}$. The claimed inequality (11) follows immediately from estimate (12). The proof is complete. □

Remark 4.3. From coercive estimate (11), in particular follows the maximal decreasing of the resolvent operator $R(\lambda, \mathcal{L}) = (\lambda I - \mathcal{L})^{-1}$, namely the estimate

$$\|R(\lambda, \mathcal{L})\|_{\mathcal{H} \to \mathcal{H}} \le C(\varepsilon)\, |\lambda|^{-1}$$

holds for all complex numbers λ as in the formulation of the last Theorem.

Theorem 4.4. *If the operator $\mathcal{A}$ acts compactly from $W_2^2(-\pi, 0) \oplus W_2^2(0, \pi)$ into $L_2(-\pi, 0) \oplus L_2(0, \pi)$ then the spectrum of problems* (1) – (5) *consists of isolated eigenvalues.*

Proof. By virtue of Theorem 4.2 for any $\epsilon > 0$, small enough, there are a positive numbers r_ϵ and C_ϵ such that for all $F \in \mathcal{H}_2$ and for all $\lambda \in G_\epsilon$ with $|\lambda| > r_\epsilon$ the estimate

$$\|U(\lambda, F)\|_{\mathcal{H}_2} \le C_\epsilon \|F\|_{\mathcal{H}}$$

holds. Consequently, the resolvent operator $R(\lambda, \mathcal{L}) = (\lambda I - \mathcal{L})^{-1}$ acted continuously from $\mathcal{H}$ onto $\mathcal{H}_2$. Since the embedding operator $\mathcal{H}_2 \subset \mathcal{H}$ is compact, this resolvent operator acted compactly from $\mathcal{H}$ into $\mathcal{H}$. Then by virtue of the well-known theorems about linear operators with compact resolvent in the Hilbert space (see, [12], Chapter III, Section 6) the spectrum of $\mathcal{L}$ consist of isolated eigenvalues. The proof is complete. □

5. Asymptotics of the Eigenvalues

Consider the pure differential part (i.e. without operator $\mathcal{A}$) of the considered problem (1) – (5). Let $\mathcal{L}_0$ be linear differential operator in the Hilbert space $\mathcal{H}$ with domain $D(\mathcal{L}_0) = D(\mathcal{L})$ given by

$$\mathcal{L}_0(u(x), -B'_\pi[u]) = (\Phi u, B_\pi[u]).$$

Theorem 5.1. *$\mathcal{L}_0$ is a self-adjoint linear operator.*

Proof. Since the operator $\mathcal{L}_0$ is symmetric, we must show that $D(\mathcal{L}_0^*) = D(\mathcal{L}_0)$. Let $X \in D(\mathcal{L}_0^*)$ be any element and λ_0 be any non-real regular value of $\mathcal{L}_0$. Then we get

$$\langle(\lambda_0 I - \mathcal{L}_0)Y, X\rangle_{\mathcal{H}} = \langle Y, (\overline{\lambda_0} I - \mathcal{L}_0^*)X\rangle_{\mathcal{H}}, \quad \text{for all} \quad Y \in D(\mathcal{L}_0).$$

Denoting

$$X_0 := (\overline{\lambda_0} I - \mathcal{L}_0)^{-1}(\overline{\lambda_0} X - \mathcal{L}_0^* X)$$

we have that $X_0 \in D(\mathcal{L}_0)$ and

$$(\overline{\lambda_0} I - \mathcal{L}_0)X_0 = \overline{\lambda_0} X - \mathcal{L}_0^* X.$$

Taking in view these equalities and then applying the Theorem 4.4 we have that

$$\begin{aligned}
\langle(\lambda_0 I - \mathcal{L}_0)Y, X\rangle_{\mathcal{H}} &= \langle Y, (\overline{\lambda_0} I - \mathcal{L}_0^*)X\rangle_{\mathcal{H}} \\
&= \langle Y, \overline{\lambda_0} X_0 - \mathcal{L}_0 X_0\rangle_{\mathcal{H}} \\
&= \lambda_0\langle Y, X_0\rangle_{\mathcal{H}} - \langle Y, \mathcal{L}_0 X_0\rangle_{\mathcal{H}} \\
&= \langle \lambda_0 Y, X_0\rangle_{\mathcal{H}} - \langle \mathcal{L}_0 Y, X_0\rangle_{\mathcal{H}} \\
&= \langle(\lambda_0 I - \mathcal{L}_0)Y, X_0\rangle_{\mathcal{H}}
\end{aligned}$$

for all $Y \in D(\mathcal{L}_0)$. This shows that the equality

$$\langle(\lambda_0 I - \mathcal{L}_0)Y, X - X_0\rangle_{\mathcal{H}} = 0$$

holds for arbitrary $Y \in D(\mathcal{L}_0)$. Choosing $Y = (\lambda_0 I - \mathcal{L}_0)^{-1}(X - X_0)$ and putting in the last equality yields $\|X - X_0\|_{\mathcal{H}} = 0$. Thus $X = X_0 \in D(\mathcal{L}_0)$ which proves that $D(\mathcal{L}_0^*) = D(\mathcal{L}_0)$ as desired. The proof is complete. □

Corollary 5.2. *All eigenvalues of the differential operator $\mathcal{L}_0$ are real.*

Lemma 5.3. *The pure differential operator $\mathcal{L}_0$ has precisely denumerable many eigenvalues $\lambda_n(L_0)$, $n = 1, 2, ...,$ which are real and satisfy the asymptotic formula*

$$\lambda_n(\mathcal{L}_0) = \frac{n^2}{4} + O(n).$$

Proof. Let $\varphi_1(x, \lambda)$ be the solution of the differential equation

$$\tau f := -f'' + q(x)f = \lambda f, \quad x \in [-\pi, 0) \cup (0, \pi] \tag{13}$$

satisfying the initial conditions

$$f(-\pi) = \delta_{11} \quad , \quad f'(-\pi) = \delta_{10}.$$

Now we proceed from $\varphi_1(x, \lambda)$ to define the solution $\varphi_2(x, \lambda)$ of the same equation (13). Namely, we shall define by $\varphi_2(x, \lambda)$ the solution of equation (13) satisfying the initial conditions

$$f(0+) = -\frac{\gamma_{11}^-}{\gamma_{10}^+}\varphi_1(0-, \lambda),$$

$$f'(0+) = -\frac{\gamma_{20}^{+}\gamma_{11}^{-} - \gamma_{10}^{+}\gamma_{20}^{-}}{\gamma_{10}^{+}}\varphi_1(0-,\lambda) - \gamma_{21}^{-}\varphi_1'(0-,\lambda).$$

It is easy to see that the function $\varphi(x,\lambda)$ defined by

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda) & for\ x \in [-\pi, 0) \\ \varphi_2(x,\lambda) & for\ x \in (0,\pi] \end{cases}$$

satisfies equation (13), first boundary condition (2) and both interface conditions (4) and (5). Therefore, by substituting $\varphi(x,\lambda)$ in condition (3) we find the eigenvalues, i.e. the eigenvalues consist of the solutions of equation

$$\omega(\lambda) := \ell_2(\varphi_2(.,\lambda)) = 0. \tag{14}$$

Let $\lambda = \mu^2$. It is easy to verify that the solution $\varphi_i(x,\lambda)$ satisfies the next integral equations:

$$\varphi_i(x,\lambda) = \varphi_i(a_i,\lambda)\cos\left[\mu\left(x-a_i\right)\right] + \frac{1}{\mu}\varphi_i'(a_i,\lambda)\sin\left[\mu\left(x-a_i\right)\right] + \frac{1}{\mu}\int_{a_i}^{x}\sin\left[\mu\left(x-t\right)\right]q(t)\varphi_i(t,\lambda)dt$$

and

$$\varphi_i'(x,\lambda) = -\mu\varphi_i(a_i,\lambda)\sin\left[\mu\left(x-a_i\right)\right] + \varphi_i'(a_i,\lambda)\cos\left[\mu\left(x-a_i\right)\right] + \int_{a_i}^{x}\cos\left[\mu\left(x-t\right)\right]q(t)\varphi_i(t,\lambda)dt$$

for $i = 1,2;\ a_1 = -\pi,\ \ a_2 = 0+$. Then, by using the approach in [17] we can find

$$\varphi_2(x,\lambda) = -\frac{\gamma_{11}^{+}\gamma_{21}^{+}}{\gamma_{21}^{-}}\delta_{10}\mu\sin[\pi\mu]\cos[\mu x] + O(e^{|Re\mu|(x+\pi)}) \tag{15}$$

and

$$\varphi_2'(x,\lambda) = \frac{\gamma_{11}^{+}\gamma_{21}^{+}}{\gamma_{21}^{-}}\delta_{10}\mu^2\sin[\pi\mu]\sin[\mu x] + O(|\mu|e^{|Re\mu|(x+\pi)}) \tag{16}$$

as $|\lambda| \to \infty$. Substituting (15) and (16) in the equation (14) we arrive at the asymptotic equation

$$\mu^4\sin^2[\pi\mu] + O(|\mu|^3 e^{2\pi|Re\mu|}) = 0$$

Take a circle $\Gamma_n := \{\mu \in \mathbb{C} \mid |\mu| = n + \frac{1}{2}\}$ of radius $n + \frac{1}{2}$ in the $\mu-$ plane, where n is a natural number. By applying the well-known Rouche theorem, we have that there are as many zeros of $\Delta(\mu) := \omega(\mu^2)$ inside Γ_n as the function $\Delta_0(\mu) := \mu^4\sin^2[\pi\mu]$ for sufficiently large n, provided that each zero is counted according to its multiplicity, i.e., $4n+6$. Since the function $\Delta_0(\mu)$ is even, we only need consider its positive zeros. Consequently there are $2n+3$ positive roots μ_k of function $\Delta(\mu)$ less than $n + \frac{1}{2}$ for sufficiently large n. Then we have $\mu_n = \frac{n}{2} + O(1)$ as $n \to \infty$, from which it follows immediately the needed asymptotic formula (13). The proof is complete. □

Let us define a new operator $\mathcal{A}_0 : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}_0(F) = \left(\ (\mathcal{A}F)(x), 0\ \right) \tag{17}$$

and domain of definition $D(\mathcal{A}_0) = D(\mathcal{L}_0)$. Then, the main problem is acquired to the operator-equation form

$$(\mathcal{L}_0 + \mathcal{A}_0)U = \lambda U,\ \ U \in D(\mathcal{L}_0) \tag{18}$$

in the Hilbert space $\mathcal{H}$.

Remark 5.4. The eigenvalues of problems (1)-(5) and (18) are coincide, and the corresponding eigenfunctions of (1)-(5) are the first components of the eigenelements of the operator $\mathcal{L}$ given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_0$.

For further investigation, we need to use some definitions and facts. Let A be closed linear operator in a Hilbert space E. A regular value λ of A is a complex number such that the resolvent operator $(A - \lambda I)^{-1}$ exists, is defined on a dense set and is bounded. The resolvent set $\rho(A)$ of A consist of all regular values of A. The set $\mathbb{C} - \rho(A)$ is called a spectrum of A and is denoted by $\sigma(A)$. If the resolvent operator $(A - \lambda I)^{-1}$ does not exist then the value λ is called an eigenvalue of A. The algebraic multiplicity of the eigenvalue λ is the dimension of the linear subspace

$$K_{\lambda_0} := \bigcup_{n=1}^{\infty} \{f \in D(A^n), (A - \lambda_0 I)^n f = 0\}.$$

Let $\mathbb{G}$ be any subset of complex plane $\mathbb{C}$ and $r > 0$ be any real number. By $N(r, \mathbb{G}, A)$ we shall denote the number of eigenvalues of A belonging to $\mathbb{G}$, which are smaller than r and are counted according to their algebraic multiplicity, i.e.

$$N(r, G, A) := \sum_{n \in \{k : \lambda_k \in G,\ |\lambda| > r\}} 1.$$

Definition 5.5. Let A_1 be any closed linear operator having at least one regular point. A linear (in general, unbounded) operator A_2 is said to be A_1-compact if $D(A_2) \supseteq D(A_1)$ and if for some regular point $\lambda_0 \in \rho(A_1)$ the operator $A_2 R(\lambda_0, A_1) = A_2(A_1 - \lambda_0 I)^{-1}$ is compact (see, for example [6]).

Theorem 5.6. *Let S be self-adjoint operator in a Hilbert space the spectrum of which is discrete, $\mathcal{A}$ be S-compact operator and $£ = S + \mathcal{A}$. Then if S has a precisely numerable many positive eigenvalues and*

$$N(r(1+\varepsilon), R^+, S) \sim N(r, R^+, S), \text{ as } r \to \infty,\ \varepsilon \to 0$$

then for any α $(0 < \alpha < \frac{\pi}{2})$

$$N(r, \mathbb{G}_\alpha, £) \sim N(r, R^+, S), \text{ as } r \to \infty$$

where $R^+ = (0, \infty)$, $\mathbb{G}_\alpha$ is the angle as in the previous section and $f(\lambda) \sim g(\lambda)$ as $r \to \infty$ is the abbreviation for

$$\lim_{r \longrightarrow \infty} \frac{f(r)}{g(r)} = 1.$$

Proof. The proof of this theorem follows immediately from the results of [6]. □

Lemma 5.7. *Let the operator $\mathcal{A}$ be $\mathcal{L}_0$-compact in the Hilbert space $\mathcal{H}$. Then the spectrum of $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_0$ is discrete and consist of precisely denumerable many eigenvalues. For any arbitrary small $\alpha > 0$ all eigenvalues of $\mathcal{L}$ with the possible exception of a finite number lie in the sector $\psi_\alpha = \{\lambda \in \mathbb{C} : |arg\lambda| < \alpha\}$ of angular 2α and for the sequence of eigenvalues $(\lambda_{n,\alpha}), n \geq 0$, belonging to the sector ψ_α, which, when listen according to nondecreasing modulus and repeated according to algebraic multiplicity, satisfies the following asymptotic formula:*

$$|\lambda_{n,\alpha}| = \frac{n^2}{4} + o(n^2), \quad n \longrightarrow \infty\ . \tag{19}$$

Proof. Let $\lambda_1(\mathcal{L}_0) \leq \lambda_2(\mathcal{L}_0) \leq ...$ be the sequence of eigenvalues of $\mathcal{L}_0$ which counted with their algebraic multiplicity. By Lemma 5.3 there are real numbers m_1, m_2, such that

$$m_1 n + \frac{n^2}{4} \leq \lambda_n(\mathcal{L}_0) \leq m_2 n + \frac{n^2}{4}$$

for all $n = 1, 2, ...$ From this relation it follows that

$$N(r,\ R^+, \mathcal{L}_0) = 1 + \sqrt{r} + O(\frac{1}{\sqrt{r}}) \text{ as } r \to \infty.$$

Since for arbitrary $\varepsilon > 0$

$$\sqrt{r+\varepsilon} = \sqrt{r} + O(\frac{1}{\sqrt{r}}) \ \ as \ \ r \to \infty$$

and

$$\frac{1}{\sqrt{r+\varepsilon}} = \frac{1}{\sqrt{r}} + O(\frac{1}{\sqrt{r^{\frac{3}{2}}}}) \ \ as \ \ r \to \infty,$$

we have that

$$N(r(1+\varepsilon),\ R^+, \mathcal{L}_0) - N(r,\ R^+, \mathcal{L}_0) = O(\frac{1}{\sqrt{r}}) \ \ as \ \ r \to \infty.$$

Consequently,

$$N(r(1+\varepsilon),\ R^+,\ \mathcal{L}_0) \sim N(r,\ R^+,\ \mathcal{L}_0) \ \text{ as } \ r \to \infty\ .$$

Then by virtue of the Theorem 5.6 we have

$$N(r,\ \alpha,\ \mathcal{L}_0 + A_0) \sim N(r,\ R^+,\ \mathcal{L}_0) \ \ as \ \ r \to \infty\ .$$

Thus, we get

$$N(r,\ \psi_\alpha,\ \mathcal{L}_0 + A_0) = N(r,\ R^+,\ \mathcal{L}_0) + o(N(r,\ R^+,\ \mathcal{L}_0)) \ \ as \ \ r \to \infty\ . \tag{20}$$

Here, as usual, the expression $f(r) = o(g(r))$, *as* $r \to \infty$ means that $\lim\limits_{r \longrightarrow \infty} \frac{f(r)}{g(r)} = 0$. Now, writing (20) for $r = |\lambda_{n,\alpha}|$ yields the desired formula (19). The proof is complete. □

Theorem 5.8. *Under condition of previous Lemma the spectrum* $\sigma(\mathcal{L})$ *of the operator* $\mathcal{L}$ *is discrete and consist of denumerable many eigenvalues* $(\lambda_n(\mathcal{L}))$ *(is several non- real) which, when arranged in decreasing modulus and counted to their algebraic multiplicity, has the following asymptotic representations*

$$Re\lambda_n(\mathcal{L}) = \frac{\pi^2 n^2}{4} + o(n^2) \ \ and \ \ Im\lambda_n(\mathcal{L}) = o(n^2) \ as \ n \longrightarrow \infty. \tag{21}$$

Proof. We know that the number of eigenvalues of the operator $\mathcal{L}$ for which $|arg\lambda| > \alpha$ is finite. Taking in view this fact and using Lemma 5.7 yield

$$|\lambda_{n,\alpha}(\mathcal{L})| = \frac{n^2}{4} + o(n^2) \ \ as \ n \longrightarrow \infty. \tag{22}$$

By virtue of Theorem 4.2, there is a natural number n_α such that the inequalities

$$Re\lambda_n(\mathcal{L}) > |\lambda_n(\mathcal{L})| \cos\alpha$$

and

$$|Im\lambda_n(\mathcal{L})| < |\lambda_n(\mathcal{L})| \sin\alpha$$

are hold for all $n \geq n_\alpha$, from which it follows easily that

$$Re\lambda_n(\mathcal{L}) \sim |\lambda_n(\mathcal{L})|$$

and

$$|Im\lambda_n(\mathcal{L})| = o(|\lambda_n(\mathcal{L})|)$$

as $n \longrightarrow \infty$. Together with (22), this shows that the asymptotic formulas (21) are true. The proof is complete. □

The main result of this section is the the following theorem.

Theorem 5.9. *Let the operator $\mathcal{A}$ acted compactly from $W_2^2(-\pi,0) \oplus W_2^2(0,\pi)$ into $L_2(-\pi,0) \oplus L_2(0,\pi)$. Then, the spectrum of BVTP (1)-(5) is discrete and consist of precisely denumerable many eigenvalues λ_n, $n = 1,2,...$ (is several non-real) which, when listed according to decreasing real parts and repeated according to algebraic multiplicity has the following asymptotic representation:*

$$\lambda_n = \frac{\pi^2 n^2}{4} + o(n^2) \ \ as \ \ n \longrightarrow \infty.$$

i.e.

$$\lim_{n\to\infty} \frac{\mid \lambda_n - \frac{\pi^2 n^2}{4} \mid}{n^2} = 0.$$

Proof. By virtue of the Theorem 4.4 the resolvent operator $R(\lambda, \mathcal{L}) = (\lambda I - \mathcal{L})^{-1}$ maps the Hilbert space $\mathcal{H}$ continuously into the $\mathcal{H}_2$. At the other hand, the operator $\mathcal{A}_0$, defined by (17) is compact from $\mathcal{H}_2$ to $\mathcal{H}_0$, by assumption on $\mathcal{L}_0$. Consequently the operator $\mathcal{A}_0(\lambda I - \mathcal{L})^{-1}$ is compact in the Hilbert space $\mathcal{H}$, and so $\mathcal{A}_0$ is $\mathcal{L}$-compact operator. Now, to complete the proof it is enough to apply the Theorem 5.8. □

Remark 5.10. Note that the operator $\mathcal{A}$ satisfying the conditions of this theorem may be non-self-adjoint and/or unbounded in the Hilbert space $L_2[-\pi,0) \oplus L_2(0,\pi] \equiv L_2[-\pi,\pi]$.

6. Conclusion

In this paper we have discussed new type of discontinuous Sturm-Liouville problem (1)-(5) involving an abstract linear operator in equation (1). The pure differential part of this problem is not self-adjoint in the usual Hilbert space $L_2[-\pi,\pi]$. For operator treatment in appropriate Hilbert space we have defined an alternative inner product (6) in terms of transmission conditions (4)-(5). We want to emphasize that the spectral properties of our problem (1)-(5) is essentially different from the spectral properties of classical Sturm-Liouville problem. For instance, it is well-know that the eigenvalues of classical Sturm-Liouville problem are real and the second asymptotic term in asymptotic expansion of eigenvalues has the form $O(n)$. But the eigenvalues of our problem (1)-(5) may be also non-real complex numbers and the second asymptotic term appears in more weak form as $o(n^2)$. Moreover, we have proved such non-usual results as topological isomorphism and coercive solvability for corresponding non-homogeneous problem (9).

References

[1] K. Aydemir, O.Sh. Mukhtarov, Green's Function Method for Self-Adjoint Realization of Boundary-Value Problems with Interior Singularities, Abstr. Appl. Anal. 2013 (2013), Article ID 503267, 7 pages.

[2] B.P. Allahverdiev, E. Bairamov, E. Ugurlu, Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions, J. Math. Anal. Appl. 401 (2013) 388–396.

[3] B.P. Allahverdiev, A dissipative singular Sturm-Liouville problem with a spectral parameter in the boundary condition, J. Math. Anal. Appl. 316 (2006) 510–524.

[4] E. Bairamov, E. Uğurlu, The determinants of dissipative Sturm-Liouville operators with transmission conditions, Math. Comput. Modelling 53 (2011) 805–813.

[5] C.T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edin. 77A (1977) 293–308.

[6] I.C. Gohberg, M.G. Krein, Introduction to The Theory of Linear Non-Selfadjoint Operators, Translation of Mathematical Monographs, vol. 18, Amer. Math. Soc., Providence, Rhode Island, 1969.

[7] N.S. Imanbaev, M.A. Sadybekov, On spectral properties of a periodic problem with an integral perturbation of the boundary condition, Eurasian Math. J. 4:3 (2013) 53–62.

[8] N.S. Imanbaev, M.A. Sadybekov, Characteristic determinant of the spectral problem for the ordinary differential operator with the boundary load, AIP Conference Proceedings ICAAM 2014, 1611 (2014) 261–265.

[9] M. Kandemir, O.Sh. Mukhtarov, A method on solving irregular boundary value problems with transmission conditions, Kuwait J. Sci. Engin. 36(2A) (2009) 79–98.

[10] M. Kandemir, Irregular boundary value problems for elliptic differential-operator equations with discontinuous coefficients and transmission conditions, Kuwait J. Sci. Engin. 39 (1A)(2010) 71–97.

[11] M. Kandemir, O. Mukhtarov, Asymptotic behavior of eigenvalues of the functional-many point boundary-value problem with discontinuous coefficients, Bull. Pure Appl. Sci. 19 (2000) 373–384.

[12] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New-York, 1966.
[13] N. Kerimov, Z.S. Aliev, Basis properties of a spectral problem with a spectral parameter in the boundary condition , Mat. Sb. 197/10 (2006) 65–86 (in Russian).
[14] A.V. Likov, Yu.A. Mikhailov, The Theory of Heat and Mass Transfer, Qosenergaizdat, 1963 (in Russian).
[15] K.R. Mamedov, F.A. Cetinkaya, Inverse problem for a class of Sturm-Liouville operator with spectral parameter in boundary condition, Boundary Value Problems (2013) 2013:183.
[16] K.R. Mamedov, F. A. Cetinkaya, Boundary value problem for a Sturm-Liouville operator with piecewise continuous coefficient, Hacettepe J. Math. Stat. 44 (2015) 867–874.
[17] O.Sh. Mukhtarov, M. Kadakal Some spectral properties of one Sturm-Liouville type problem with discontinuous weight, Sib. Math. J. 46 (2005) 681–694.
[18] O.Sh. Mukhtarov, H. Olğar, K. Aydemir, Resolvent operator and spectrum of new type boundary value problems, Filomat 29 (2015) 1671–1680.
[19] O.Sh. Mukhtarov, S. Yakubov, Problems for ordinary differential equations with transmission conditions, Appl. Anal. 81 (2002) 1033–1064.
[20] M.A. Sadybekov, N.S. Imanbaev, On the basis property of root functions of a periodic problem with an integral perturbation of the boundary condition, Differential Eq. 48 (2012) 896–900.
[21] E.C. Titchmarsh, Eigenfunctions Expansion Associated with Second Order Differential Equations I, (second ed.), Oxford Univ. Press, London, 1962.
[22] A.N. Tikhonov, A.A. Samarskii, Equations of Mathematical Physics, Pergamon, Oxford and New York, 1963.
[23] N.N. Voitovich, B.Z. Katsenelbaum, A.N. Sivov, Generalized Method of Eigen-vibration in the theory of Differaction, Nakua, Mockow, 1997 (in Russian).
[24] S.Y. Yakubov, Completeness of Root Functions of Regular Differential Operators, Longman, Scientific and Technical, Newyork, 1994.
[25] A. Zettl, Adjoint and self-adjoint boundary value problems with interface conditions, Siam J. Appl. Math. 16 (1968) 851–859.