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Difference Schemes for the Semilinear Integral-Differential Equation of the Hyperbolic Type

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Abstract. In this paper, the initial-value problem for integral-differential equation of the hyperbolic type in a Hilbert space H is considered. The unique solvability of this problem is established. The first order and the second order of accuracy difference schemes approximately solving this problem are presented. The convergence estimates for the solutions of these difference schemes are obtained. Theoretical results are supported by numerical example.

1. Introduction

Hyperbolic equations have wide range of applications in fluid dynamics, theory of elasticity, vibration theories, electromagnetics, electrodynamics, hydrodynamics, wave propagation, etc. [16, 17]. There is a great deal of work in constructing and analysing difference schemes for numerical solutions of hyperbolic differential equations [1, 3–12]. In [2], linear integral-differential equations of the hyperbolic type with two dependent limits have been studied. Various difference schemes for numerical solutions of these kind of equations were the subject of previous studies [2, 13–15].

In this paper, we consider the initial value problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + Au(t) = \displaystyle\int_{-t}^{t} B(\rho)u(\rho)d\rho + f(t,u(t)), \quad -1 \le t \le 1, \\ u(0) = u_0, \quad u'(0) = u_0' \end{cases} \tag{1}$$

for semilinear integral-differential equation in a Hilbert space H with unbounded linear operators A and $B(t)$ in H with dense domain $D(A) \subset D(B(t))$. We assume that A^{-1} and $B(t)$ commute and satisfy

$$\left\| B(t)A^{-1} \right\|_{H \to H} \le M, \quad -1 \le t \le 1. \tag{2}$$

Various initial-boundary value problems for the integral-differential equation of hyperbolic type with two dependent limits can be reduced to the initial value problem (1) in a Hilbert space H.

A function $u(t)$ is called a *solution* of the problem (1) if the following conditions are satisfied:

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i) $u(t)$ is twice continuously differentiable on $[-1,1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.

iii) $u(t)$ satisfies the equations and the initial conditions (1).

A solution of problem (1) defined in this manner will from now on be referred to as a solution of problem (1) in the space $C(H) = C([-1,1],H)$ of all continuous functions $\varphi(t)$ defined on $[-1,1]$ with values in H equipped with the norm

$$\left\|\varphi\right\|_{C(H)} = \max_{-1\le t\le 1} \|\varphi(t)\|_H. \tag{3}$$

We consider the problem (1) under the assumption that A is a positive definite self-adjoint operator with $A \ge \delta I$, where $\delta > \delta_0 > 0$.

In the present paper, the unique solvability of the initial-value problem (1) is obtained. The first order and the second order of accuracy difference schemes approximately solving the initial-value problem (1) are presented. The convergence estimates for the solutions of these difference schemes are obtained. Numerical illustrations for the simple test problem are provided.

2. Unique Solvability of Problem

Theorem 2.1. *Suppose that $u_0 \in D(A)$, $u_0' \in D(A^{1/2})$, f is the continuous function on $[-1,1]\times H$, and there is an $L > 0$ such that f satisfies the Lipschitz condition*

$$\left\|A^{-1/2}\left(f(t,u) - f(t,v)\right)\right\|_H \le L\,\|u-v\|_H\,, \quad -1 \le t \le 1 \tag{4}$$

for all $u, v \in H$. Then, there is a unique solution of the initial value problem (1).

Proof. The proof of the existence and uniqueness of the solution of problem (1) is based on the following formula

$$u(t) = c(t)u_0 + s(t)u_0' + \int_0^t s(t-\tau)f\big(\tau,u(\tau)\big)d\tau + \int_{-t}^t A^{-1}\big(I - c(|t|-|\tau|)\big)B(\tau)u(\tau)d\tau$$

and the fixed point theorem. It is easy to see that the operator

$$Fu(t) = c(t)u_0 + s(t)u_0' + \int_0^t s(t-\tau)f\big(\tau,u(\tau)\big)d\tau + \int_{-t}^t A^{-1}\big(I - c(|t|-|\tau|)\big)B(\tau)u(\tau)d\tau$$

maps $C\big([-1,1],H\big)$ into $C\big([-1,1],H\big)$. By using a special value of λ in the norm

$$\|v\|_{C^*([-1,1],H)} = \max_{-1\le t\le 1} e^{-\lambda|t|}\,\|v(t)\|_H\,, \tag{5}$$

we can prove that F is the contraction operator on $C^*([-1,1],H)$. Indeed, applying (4), the triangle inequality and the following estimates

$$\|c(t)\|_H \le 1, \quad \left\|A^{1/2}s(t)\right\|_H \le 1, \quad -1 \le t \le 1,$$

we have

$$
\begin{aligned}
e^{-\lambda|t|}\left\|Fu(t)-Fv(t)\right\|_H &= e^{-\lambda|t|}\Bigg\| \int_0^t A^{1/2}s(t-\tau)A^{-1/2}\Big(f\big(\tau,u(\tau)\big)-f\big(\tau,v(\tau)\big)\Big)d\tau \\
&\quad + \int_{-t}^{t}\Big(I-c(|t|-|\tau|)\Big)B(\tau)A^{-1}\Big(u(\tau)-v(\tau)\Big)d\tau\Bigg\|_H \\
&\le e^{-\lambda|t|}\left[L\int_0^{|t|}\left\|u(\tau)-v(\tau)\right\|_H d\tau + 2M\int_{-|t|}^{|t|}\left\|u(\tau)-v(\tau)\right\|_H d\tau\right] \\
&= Le^{-\lambda|t|}\int_0^{|t|} e^{\lambda\tau}e^{-\lambda\tau}\left\|u(\tau)-v(\tau)\right\|_H d\tau \\
&\quad +2Me^{-\lambda|t|}\left[\int_0^{|t|} e^{\lambda\tau}e^{-\lambda\tau}\left\|u(\tau)-v(\tau)\right\|_H d\tau + \int_{-|t|}^{0} e^{-\lambda\tau}e^{\lambda\tau}\left\|u(\tau)-v(\tau)\right\|_H d\tau\right] \\
&\le e^{-\lambda|t|}\left[L\int_0^{|t|} e^{\lambda\tau}d\tau + 2M\int_0^{|t|} e^{\lambda\tau}d\tau + 2M\int_{-|t|}^{0} e^{-\lambda\tau}d\tau\right]\left\|u-v\right\|_{C^*([-1,1],H)} \\
&= e^{-\lambda|t|}\frac{e^{\lambda|t|}-1}{\lambda}\Big(L+4M\Big)\left\|u-v\right\|_{C^*([-1,1],H)} = \Big(L+4M\Big)\frac{1-e^{-\lambda|t|}}{\lambda}\left\|u-v\right\|_{C^*([-1,1],H)} \\
&\le \Big(L+4M\Big)\frac{1-e^{-\lambda}}{\lambda}\left\|u-v\right\|_{C^*([-1,1],H)}
\end{aligned}
$$

for any $t \in [-1,1]$. So,

$$
\left\|Fu-Fv\right\|_{C^*([-1,1],H)} \le \alpha_\lambda \left\|u-v\right\|_{C^*([-1,1],H)},
$$

where $\alpha_\lambda = \Big(L+4M\Big)\frac{1-e^{-\lambda}}{\lambda} \to 0$ when $\lambda \to \infty$. So, F is the contraction operator on $C^*([-1,1],H)$. Finally, we note that the norms (3) and (5) are equivalent in $C\Big([-1,1],H\Big)$, which completes the proof of this theorem. □

3. First Order of Accuracy Difference Scheme

We construct the first order of accuracy difference scheme

$$
\begin{cases}
\dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_{k+1} = \displaystyle\sum_{j=-k+1}^{k} B_j u_j \tau + f(t_k,u_k), \quad k=1,\dots,N-1, \\
\dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_{k-1} = -\displaystyle\sum_{j=k+1}^{-k} B_j u_j \tau + f(t_k,u_k), \quad k=-N+1,\dots,-1, \\
\tau = \dfrac{1}{N}, \quad t_k = k\tau, \quad k=-N,\dots,N, \quad B_k = \dfrac{1}{\tau}\displaystyle\int_{t_{k-1}}^{t_k} B(\rho)d\rho, \quad k=-N+1,\dots,N, \\
u_0 = u(0), \quad (I+\tau^2 A)\dfrac{u_1-u_0}{\tau} = (I+\tau^2 A)\dfrac{u_0-u_{-1}}{\tau} = u_0'
\end{cases}
\tag{6}
$$

for approximate solutions of problem (1).

Theorem 3.1. *Let $u(t)$ be a solution of (1). Assume that the requirements of the Theorem 2.1 are satisfied and $A^{-1/2}u'''(t)$, $u''(t)$, $A^{1/2}u'(t)$ are continuous functions on $[-1,1]$. Then the difference scheme (6) has a unique solution and the following convergence estimate holds*

$$\max_{-N\leq k\leq N}\|u_k-u(t_k)\|_H\leq M^*\tau, \tag{7}$$

where M^* does not depend on τ.

Proof. Subtracting (1) from (6), we obtain the system of difference equations

$$\begin{cases}\dfrac{z_{k+1}-2z_k+z_{k-1}}{\tau^2}+Az_{k+1}=\displaystyle\sum_{j=-k+1}^{k}B_jz_j\tau+\phi_k, \quad k=1,\ldots,N-1,\\ \dfrac{z_{k+1}-2z_k+z_{k-1}}{\tau^2}+Az_{k-1}=-\displaystyle\sum_{j=k+1}^{-k}B_jz_j\tau+\phi_k, \quad k=-N+1,\ldots,-1,\\ z_0=0, \quad (I+\tau^2A)\dfrac{z_1-z_0}{\tau}=\alpha, \quad (I+\tau^2A)\dfrac{z_0-z_{-1}}{\tau}=\beta,\end{cases}$$

where $z_k=u_k-u(t_k)$ is the convergence error of the difference scheme (6) and

$$\begin{aligned}\phi_k&=-\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-Au(t_{k+1})+\sum_{j=-k+1}^{k}B_ju(t_j)\tau+f(t_k,u_k)-f(t_k,u(t_k))\\&\quad+\frac{d^2u(t_k)}{dt^2}+Au(t_k)-\int_{-t_k}^{t_k}B(\rho)u(\rho)d\rho, \quad k=1,\ldots,N-1,\\ \phi_k&=-\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-Au(t_{k-1})-\sum_{j=k+1}^{-k}B_ju(t_j)\tau+f(t_k,u_k)-f(t_k,u(t_k))\\&\quad+\frac{d^2u(t_k)}{dt^2}+Au(t_k)-\int_{-t_k}^{t_k}B(\rho)u(\rho)d\rho, \quad k=-N+1,\ldots,-1,\\ \alpha&=u_0'-(I+\tau^2A)\frac{u(\tau)-u(0)}{\tau}, \quad \beta=u_0'-(I+\tau^2A)\frac{u(0)-u(-\tau)}{\tau}.\end{aligned} \tag{8}$$

By [10], the first order of accuracy difference scheme

$$\begin{cases}\dfrac{z_{k+1}-2z_k+z_{k-1}}{\tau^2}+Az_{k+1}=\psi_k, \quad k=1,\ldots,N-1,\\ z_0=0, \quad \left(I+\tau^2A\right)\dfrac{z_1-z_0}{\tau}=\alpha\end{cases}$$

has the solution

$$\begin{aligned}&z_1=\left(I+\tau^2A\right)^{-1}\tau\alpha,\\ &z_k=\left(R-\tilde{R}\right)^{-1}\left(R^k-\tilde{R}^k\right)z_1+\sum_{j=1}^{k-1}\left(R-\tilde{R}\right)^{-1}\tilde{R}R\left(R^{k-j}-\tilde{R}^{k-j}\right)\tau^2\psi_j, \quad k=2,\ldots,N,\end{aligned} \tag{9}$$

where $R=\left(I+i\tau A^{1/2}\right)^{-1}$ and $\tilde{R}=\left(I-i\tau A^{1/2}\right)^{-1}$. Using $\tilde{R}R=\left(I+\tau^2A\right)^{-1}$ and $\left(R-\tilde{R}\right)^{-1}=\dfrac{iA^{-1/2}}{2\tau}\left(\tilde{R}R\right)^{-1}$, we have

$$\begin{aligned}z_k&=\left(R-\tilde{R}\right)^{-1}\left(R^k-\tilde{R}^k\right)\tau\tilde{R}R\alpha+\sum_{j=1}^{k-1}\left(R-\tilde{R}\right)^{-1}\tilde{R}R\left(R^{k-j}-\tilde{R}^{k-j}\right)\tau^2\psi_j\\&=\frac{iA^{-1/2}}{2}\left(R^k-\tilde{R}^k\right)\alpha+\sum_{j=1}^{k-1}\frac{i\tau A^{-1/2}}{2}\left(R^{k-j}-\tilde{R}^{k-j}\right)\psi_j, \quad k=2,\ldots,N.\end{aligned}$$

By putting $\psi_k = \sum_{j=-k+1}^{k} B_j z_j \tau + \phi_k$ for $k = 1, \ldots, N-1$, we obtain

$$\begin{aligned} z_k &= \frac{iA^{-1/2}}{2}\left(R^k - \tilde{R}^k\right)\alpha + \sum_{j=1}^{k-1} \frac{i\tau A^{-1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right)\phi_j \\ &\quad + \sum_{j=1}^{k-1} \frac{i\tau A^{-1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right) \sum_{s=-j+1}^{j} B_s z_s \tau, \quad k = 2, \ldots, N. \end{aligned} \tag{10}$$

Since $i\tau A^{1/2}R = I - R$ and $-i\tau A^{1/2}\tilde{R} = I - \tilde{R}$, we have

$$\sum_{j=s}^{k-1} \frac{i\tau A^{1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right) = \sum_{j=s}^{k-1} \frac{R^{k-j-1} - R^{k-j}}{2} + \sum_{j=s}^{k-1} \frac{\tilde{R}^{k-j-1} - \tilde{R}^{k-j}}{2} = I - \frac{R^{k-s} + \tilde{R}^{k-s}}{2}$$

and in the similar way

$$\sum_{j=-s+1}^{k-1} \frac{i\tau A^{1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right) = I - \frac{R^{k+s-1} + \tilde{R}^{k+s-1}}{2}.$$

Then,

$$\begin{aligned} &\sum_{j=1}^{k-1} \frac{i\tau A^{-1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right) \sum_{s=-j+1}^{j} B_s z_s \tau \\ &= \sum_{s=1}^{k-1} \sum_{j=s}^{k-1} \frac{i\tau A^{1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right) B_s A^{-1} z_s \tau + \sum_{s=-k+2}^{0} \sum_{j=-s+1}^{k-1} \frac{i\tau A^{1/2}}{2}\left(R^{k-j} - \tilde{R}^{k-j}\right) B_s A^{-1} z_s \tau \\ &= \sum_{s=1}^{k-1} \left(I - \frac{R^{k-s} + \tilde{R}^{k-s}}{2}\right) B_s A^{-1} z_s \tau + \sum_{s=-k+2}^{0} \left(I - \frac{R^{k+s-1} + \tilde{R}^{k+s-1}}{2}\right) B_s A^{-1} z_s \tau, \quad k = 2, \ldots, N. \end{aligned} \tag{11}$$

Putting (11) in (10), we obtain

$$\begin{aligned} z_k = &\frac{\tilde{R}^k - R^k}{2i} A^{-1/2}\alpha - \tau \sum_{j=1}^{k-1} \frac{R^{k-j} - \tilde{R}^{k-j}}{2i} A^{-1/2}\phi_j \\ &+ \tau \sum_{s=1}^{k-1} \left(I - \frac{R^{k-s} + \tilde{R}^{k-s}}{2}\right) B_s A^{-1} z_s + \tau \sum_{s=-k+2}^{0} \left(I - \frac{R^{k+s-1} + \tilde{R}^{k+s-1}}{2}\right) B_s A^{-1} z_s, \quad k = 2, \ldots, N. \end{aligned}$$

Then, using (2) and the following estimates

$$\|R\|_{H\to H} \le 1, \quad \left\|\tilde{R}\right\|_{H\to H} \le 1, \quad \left\|\tau A^{1/2} R\right\|_{H\to H} \le 1, \quad \left\|\tau A^{1/2} \tilde{R}\right\|_{H\to H} \le 1$$

yields

$$\|z_k\|_H \le \left\|A^{-1/2}\alpha\right\|_H + \tau \sum_{j=1}^{k-1} \left\|A^{-1/2}\phi_j\right\|_H + 2M\tau \sum_{j=-k+1}^{k-1} \left\|z_j\right\|_H, \quad k = 2, \ldots, N. \tag{12}$$

Furthermore, from (9) we have

$$\|z_1\|_H \le \left\|A^{-1/2}\alpha\right\|_H. \tag{13}$$

In a similar way, one can prove that

$$\|z_k\|_H \le \left\|A^{-1/2}\beta\right\|_H + \tau \sum_{j=k+1}^{-1} \left\|A^{-1/2}\phi_j\right\|_H + 2M\tau \sum_{j=k+1}^{-k-1} \left\|z_j\right\|_H, \quad k = -N, \dots, -2 \tag{14}$$

and

$$\|z_{-1}\|_H \le \left\|A^{-1/2}\beta\right\|_H. \tag{15}$$

From (8) by using (2) and (4), we have

$$\begin{aligned} \left\|A^{-1/2}\phi_k\right\|_H &\le \left\|A^{-1/2}\left(u''(t_k) - \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2}\right)\right\|_H + \left\|A^{1/2}\left(u(t_{k+1}) - u(t_k)\right)\right\|_H \\ &\quad + \left\|A^{-1/2}\left(f(t_k, u_k) - f(t_k, u(t_k))\right)\right\|_H + \left\|A^{-1/2}\left(\sum_{j=-k+1}^{k} B_j u(t_j)\tau - \int_{-t_k}^{t_k} B(\rho)u(\rho)d\rho\right)\right\|_H \\ &= \left\|\frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\int_{s}^{s+\tau}\int_{z}^{t_k} A^{-1/2}u'''(\xi)d\xi dz ds\right\|_H + \left\|\int_{t_k}^{t_{k+1}} A^{1/2}u'(s)ds\right\|_H \\ &\quad + \left\|A^{-1/2}\left(f(t_k, u_k) - f(t_k, u(t_k))\right)\right\|_H + \left\|\sum_{j=-k+1}^{k}\int_{t_{j-1}}^{t_j} B(\rho)A^{-1}\int_{\rho}^{t_j} A^{1/2}u'(s)dsd\rho\right\|_H \\ &\le c_1\tau + L\|z_k\|_H, \quad k = 1, \dots, N-1. \end{aligned} \tag{16}$$

In a similar way, one can prove that

$$\left\|A^{-1/2}\phi_k\right\|_H \le c_2\tau + L\|z_k\|_H, \quad k = -N+1, \dots, -1. \tag{17}$$

Furthermore, from (8) we have

$$\begin{aligned} \left\|A^{-1/2}\alpha\right\|_H &\le \left\|A^{-1/2}\left(u_0' - \frac{u(\tau - u(0)}{\tau}\right)\right\|_H + \tau\left\|A^{1/2}\left(u(\tau) - u(0)\right)\right\|_H \\ &= \left\|\frac{1}{\tau}\int_0^{\tau}\int_0^{s} A^{-1/2}u''(z)dzds\right\|_H + \tau\left\|\int_0^{\tau} A^{1/2}u'(s)ds\right\|_H \le c_3\tau. \end{aligned} \tag{18}$$

In a similar way, one can prove that

$$\left\|A^{-1/2}\beta\right\|_H \le c_4\tau. \tag{19}$$

Thus, using (16)-(19) in (12)-(15) gives us

$$\|z_k\|_H \le c\tau + \tau(L + 2M)\sum_{j=-|k|+1}^{|k|-1}\left\|z_j\right\|_H, \quad k = -N, \dots, -1, 1, \dots, N. \tag{20}$$

Finally, using (20) and the theorem about the discrete analogue of a Gronwall type integral inequality with two dependent limits [], we obtain the estimate (7). □

4. Second Order of Accuracy Difference Scheme

We construct the second order of accuracy difference schemewe consider

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\dfrac{1}{2}Au_k+\dfrac{1}{4}A(u_{k+1}+u_{k-1}) \\ \qquad =\tau\displaystyle\sum_{j=-k+1}^{k}B_{j-\frac{1}{2}}\left(\dfrac{u_j+u_{j-1}}{2}\right)+f(t_k,u_k), \quad k=1,\dots,N-1, \\ \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\dfrac{1}{2}Au_k+\dfrac{1}{4}A(u_{k+1}+u_{k-1}) \\ \qquad =-\tau\displaystyle\sum_{j=k+1}^{-k}B_{j-\frac{1}{2}}\left(\dfrac{u_j+u_{j-1}}{2}\right)+f(t_k,u_k), \quad k=-N+1,\dots,-1, \\ \tau=\dfrac{1}{N}, \quad t_k=k\tau, \quad k=-N,\dots,N, \quad B_{k-\frac{1}{2}}=\dfrac{1}{\tau}\displaystyle\int_{t_{k-1}}^{t_k}B(\rho)d\rho, \quad k=-N+1,\dots,N, \\ u_0=u(0), \quad \left(I+\tau^2A\right)\dfrac{u_1-u_0}{\tau}=\dfrac{\tau}{2}\left(f(0,u_0)-Au_0\right)+u_0', \\ \left(I+\tau^2A\right)\dfrac{u_0-u_{-1}}{\tau}=\dfrac{\tau}{2}\left(Au_0-f(0,u_0)\right)+u_0' \end{cases} \tag{21}$$

for approximate solutions of problem (1).

Theorem 4.1. *Let $u(t)$ be a solution of (1). Assume that the requirements of the Theorem 2.1 are satisfied and $A^{-1/2}u^{(4)}(t)$, $u'''(t)$, $A^{1/2}u''(t)$ are continuous functions on $[-1,1]$. Then the difference scheme (21) has a unique solution and the following convergence estimate holds*

$$\max_{-N\le k\le N}\|u_k-u(t_k)\|_H\le M^*\tau^2, \tag{22}$$

where M^* does not depend on τ.

Proof. Subtracting (1) from (21), we obtain the system of difference equations

$$\begin{cases} \dfrac{z_{k+1}-2z_k+z_{k-1}}{\tau^2}+\dfrac{1}{2}Az_k+\dfrac{1}{4}A(z_{k+1}+z_{k-1})=\tau\displaystyle\sum_{j=-k+1}^{k}B_{j-\frac{1}{2}}\left(\dfrac{z_j+z_{j-1}}{2}\right)+\phi_k, \quad k=1,\dots,N-1, \\ \dfrac{z_{k+1}-2z_k+z_{k-1}}{\tau^2}+\dfrac{1}{2}Az_k+\dfrac{1}{4}A(z_{k+1}+z_{k-1})=-\tau\displaystyle\sum_{j=k+1}^{-k}B_{j-\frac{1}{2}}\left(\dfrac{z_j+z_{j-1}}{2}\right)+\phi_k, \quad k=-N+1,\dots,-1, \\ z_0=0, \quad (I+\tau^2A)\dfrac{z_1-z_0}{\tau}=\alpha, \quad (I+\tau^2A)\dfrac{z_0-z_{-1}}{\tau}=\beta, \end{cases}$$

where $z_k=u_k-u(t_k)$ is the convergence error of the difference scheme (21) and

$$\phi_k=-\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-\frac{1}{2}Au(t_k)-\frac{1}{4}A\left(u(t_{k+1})+u(t_{k-1})\right)+\tau\sum_{j=-k+1}^{k}B_{j-\frac{1}{2}}\left(\frac{u(t_j)+u(t_{j-1})}{2}\right)$$
$$+f(t_k,u_k)-f(t_k,u(t_k))+\frac{d^2u(t_k)}{dt^2}+Au(t_k)-\int_{-t_k}^{t_k}B(\rho)u(\rho)d\rho, \quad k=1,\dots,N-1,$$
$$\phi_k=-\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-\frac{1}{2}Au(t_k)-\frac{1}{4}A\left(u(t_{k+1})+u(t_{k-1})\right)-\tau\sum_{j=k+1}^{-k}B_{j-\frac{1}{2}}\left(\frac{u(t_j)+u(t_{j-1})}{2}\right)$$
$$+f(t_k,u_k)-f(t_k,u(t_k))+\frac{d^2u(t_k)}{dt^2}+Au(t_k)-\int_{-t_k}^{t_k}B(\rho)u(\rho)d\rho, \quad k=-N+1,\dots,-1,$$
$$\alpha=\frac{\tau}{2}\left(f(0,u_0-Au_0)\right)+u_0'-(I+\tau^2A)\frac{u(\tau)-u(0)}{\tau}, \quad \beta=\frac{\tau}{2}\left(Au_0-f(0,u_0)\right)+u_0'-(I+\tau^2A)\frac{u(0)-u(-\tau)}{\tau}.$$

Now, the proof of estimate (22) is similar to the proof of estimate (7) and is based on the following results:

$$\left\|A^{-1/2}\alpha\right\|_H \le c_1\tau^2, \quad \left\|A^{-1/2}\beta\right\|_H \le c_2\tau^2, \quad \left\|A^{-1/2}\phi_k\right\|_H \le c_3\tau^2 + L\left\|z_k\right\|_H, \quad k = -N+1,\dots,-1,1,\dots,N-1.$$

□

5. Numerical Example

In this section, we shall validate our findings by numerical illustrations for simple test problem. We consider the initial-boundary value problem:

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) = \int\limits_{-t}^{t} u_{xx}(s,x)ds + \left(\frac{2}{3}t^3 + t^2 + 2\right)\sin x - \sin\left(t^2\sin x\right) + \sin\left(u(t,x)\right), \ -1 \le t \le 1, \ 0 < x < \pi, \\ u(0,x) = 0, \quad u_t(0,x) = 0, \quad 0 \le x \le \pi, \\ u(t,0) = 0, \quad u(t,\pi) = 0, \quad -1 \le t \le 1, \end{cases} \tag{23}$$

which has the exact solution $u(t,x) = t^2\sin x$.

Firstly, applying the first order of accuracy difference scheme (6) to the problem (23), we have

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} = \tau \sum\limits_{j=-k+1}^{k} \frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + \left(\frac{2}{3}t_k^3 + t_k^2 + 2\right)\sin x_n \\ \qquad - \sin\left(t_k^2\sin x_n\right) + \sin\left(u_n^k\right), \quad n = 1,\dots,M-1, \quad k = 1,\dots,N-1, \\ \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} = -\tau \sum\limits_{j=k+1}^{-k} \frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + \left(\frac{2}{3}t_k^3 + t_k^2 + 2\right)\sin x_n \\ \qquad - \sin\left(t_k^2\sin x_n\right) + \sin\left(u_n^k\right), \quad n = 1,\dots,M-1, \quad k = -N+1,\dots,-1, \\ \tau = \frac{1}{N}, \quad h = \frac{\pi}{M}, \quad t_k = k\tau, \quad k = -N,\dots,N, \quad x_n = nh, \quad n = 0,\dots,M, \\ u_n^0 = 0, \quad u_n^1 = u_n^{-1} = 0, \quad n = 0,\dots,M, \quad u_0^k = u_M^k = 0, \quad k = -N,\dots,N. \end{cases} \tag{24}$$

The first order of accuracy difference scheme (24) is implemented for different values of M and N. The errors are computed by

$$\|Error\|_\infty = \max_{\substack{-N\le k\le N \\ 0\le n\le M}} \left|u(t_k,x_n) - u_n^k\right|, \tag{25}$$

where u_n^k represents the numerical solution at (t_k, x_n). Table 1 shows the errors between the exact solution of the initial-boundary value problem (23) and the numerical solutions computed by using the first order of accuracy difference scheme (24) for different values of N and M. We observe that the scheme has the first order convergence as is it expected to be.

	$N = M = 8$	$N = M = 16$	$N = M = 32$	$N = M = 64$	$N = M = 128$
$\|\|Error\|\|_\infty$	1.65×10^{-1}	8.28×10^{-2}	4.14×10^{-2}	2.07×10^{-2}	1.04×10^{-2}

Table 1: The errors between the exact solution of the initial-boundary value problem (23) and the numerical solutions computed by using the first order of accuracy difference scheme (24) for different values of N and M.

Secondly, applying the second order of accuracy difference scheme (21) to the initial-boundary value problem (23), we get:

$$\left\{\begin{array}{l} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2}-\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2}-\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} \\ \quad =\tau\displaystyle\sum_{j=-k+1}^{k}\left(\dfrac{u_{n+1}^j-2u_n^j+u_{n-1}^j}{2h^2}+\dfrac{u_{n+1}^{j-1}-2u_n^{j-1}+u_{n-1}^{j-1}}{2h^2}\right)+\left(\dfrac{2}{3}t_k^3+t_k^2+2\right)\sin x_n \\ \quad -\sin\left(t_k^2\sin x_n\right)+\sin\left(u_n^k\right),\quad n=1,\dots,M-1,\quad k=1,\dots,N-1, \\ \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2}-\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2}-\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} \\ \quad =-\tau\displaystyle\sum_{j=k+1}^{-k}\left(\dfrac{u_{n+1}^j-2u_n^j+u_{n-1}^j}{2h^2}+\dfrac{u_{n+1}^{j-1}-2u_n^{j-1}+u_{n-1}^{j-1}}{2h^2}\right)+\left(\dfrac{2}{3}t_k^3+t_k^2+2\right)\sin x_n \\ \quad -\sin\left(t_k^2\sin x_n\right)+\sin\left(u_n^k\right),\quad n=1,\dots,M-1,\quad k=-N+1,\dots,-1, \\ \tau=\dfrac{1}{N},\quad h=\dfrac{\pi}{M},\quad t_k=k\tau,\quad k=-N,\dots,N,\quad x_n=nh,\quad n=0,\dots,M, \\ u_n^0=0,\quad u_n^1=u_n^{-1}=\tau^2\sin x_n,\quad n=0,\dots,M,\quad u_0^k=u_M^k=0,\quad k=-N,\dots,N. \end{array}\right. \tag{26}$$

The second order of accuracy difference scheme (26) is implemented for different values of M and N. The errors between the exact solution of problem (23) and the numerical solution of the difference scheme (26) are computed by (25). Table 2 shows the errors between the exact solution of problem (23) and the numerical solutions computed by using the second order of accuracy difference schemes (26) for different values of N and M. We observe that the scheme has the second order convergence as is it expected to be.

	$N=M=8$	$N=M=16$	$N=M=32$	$N=M=64$	$N=M=128$
$\|\|Error\|\|_\infty$	4.24×10^{-3}	9.34×10^{-4}	2.15×10^{-4}	5.14×10^{-5}	1.25×10^{-5}

Table 2: The errors between the exact solution of the initial-boundary value problem (23) and the numerical solutions computed by using the second order of accuracy difference scheme (26) for different values of N and M.

6. Conclusions

We considered the initial-value problem (1) for semilinear integral-differential equation in a Hilbert space H. We have proved that this problem has a unique solution. We constructed the first order and the second order of accuracy difference schemes approximately solving this problem. The convergence estimates for the solutions of these difference schemes were obtained. Numerical illustrations for the simple test problem were provided to support our theoretical results.

Finally, we note that higher order accurate difference schemes [11, 12] can be constructed and the convergence results for the solutions of these difference schemes can be established in the similar way.

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