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Beyond Cauchy and Quasi-Cauchy Sequences

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Abstract. In this paper, we investigate the concepts of downward continuity and upward continuity. A real valued function on a subset E of $\mathbb{R}$, the set of real numbers, is downward continuous if it preserves downward quasi-Cauchy sequences; and is upward continuous if it preserves upward quasi-Cauchy sequences, where a sequence (x_k) of points in $\mathbb{R}$ is called downward quasi-Cauchy if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $x_{n+1} - x_n < \varepsilon$ for $n \geq n_0$, and called upward quasi-Cauchy if for every $\varepsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that $x_n - x_{n+1} < \varepsilon$ for $n \geq n_1$. We investigate the notions of downward compactness and upward compactness and prove that downward compactness coincides with above boundedness. It turns out that not only the set of downward continuous functions, but also the set of upward continuous functions is a proper subset of the set of continuous functions.

1. Introduction

A subset E of $\mathbb{R}$, the set of real numbers, is compact if and only if any sequence of points in E has a convergent subsequence with limit in E. A subset E of $\mathbb{R}$ is bounded if and only if any sequence of points in E has a quasi-Cauchy subsequence. Boundedness coincides not only with ward compactness ([7]), but also each of the following kinds of compactness, slowly oscillating compactness ([14, Theorem 3]), statistical ward compactness ([9, Lemma 2]), lacunary statistical ward compactness ([10, Theorem 3]), N_θ-ward compactness ([2, Theorem 3.3]). Two of the results in this paper provide us with firstly, necessary and sufficient conditions for below boundedness of a subset of $\mathbb{R}$, and secondly, necessary and sufficient conditions for above boundedness of a subset of $\mathbb{R}$.

Recently, using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([39]), quasi-slowly oscillating continuity ([26]), ward continuity ([7]), statistical ward continuity ([9]), λ-statistically ward continuity ([22]), ρ-statistical ward continuity ([4]), ideal ward continuity ([17]), Abel statistical continuity ([24]), strongly lacunary-ward continuity ([18]), lacunary statistical ward continuity ([15]), and arithmetic continuity ([6, 40]). Investigation of some of these kinds of continuities lead some authors to enable interesting results related to uniform continuity of a real function in terms of sequences in the above manner ([9, Theorem 6], [39, Theorem 8], [26, Theorem 2.3], [7, Theorem 7], [1, Theorem 1], [2, Theorem 3.8], [22, Corollary 6]). Modifying the definitions of a *forward Cauchy sequence*, and a *backward Cauchy sequence* introduced in [38] (see also [29, 36]), recently, the definitions of statistically downward and upward half Cauchyness of a real sequence have been introduced in [12] (see also [3, 27, 28]).

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The purpose of this paper is to investigate the concepts of downward continuity and upward continuity, and prove interesting theorems.

2. Downward and Upward Quasi-Cauchy Sequences

Students often misunderstand the definition of Cauchy sequences when they first encounter it in an introductory real analysis course. In particular, many students fail to understand that it involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. In [1] the authors call them "quasi-Cauchy", while they were called "forward convergent to 0" sequences in [7], where a sequence (x_n) is called quasi-Cauchy if given any $\varepsilon > 0$, there exists an integer $K > 0$ such that $n \geq K$ implies that $|x_{n+1} - x_n| < \varepsilon$. A subset E of $\mathbb{R}$ is compact if and only if any sequence in E has a convergent subsequence whose limit is in E. Boundedness of a subset E of $\mathbb{R}$ coincides with that any sequence of points in E has either a Cauchy subsequence, or a quasi-Cauchy subsequence. What is the case for below boundedness and above boundedness? Upward quasi-Cauchy sequences and downward quasi-Cauchy sequences provide with the answers. Weakening the condition on the definition of a quasi-Cauchy sequence, omitting the absolute value symbol, i.e. replacing $|x_{k+1} - x_k| < \varepsilon$ with $x_{k+1} - x_k < \varepsilon$ in the definition of a quasi-Cauchy sequence given in [7], we introduce the following definition.

Definition 2.1. Suppose that (x_n) $(n = 1, 2, 3, ...)$ is a sequence of real numbers. (x_n) is called *downward quasi-Cauchy* if given any $\varepsilon > 0$ there exists an integer $K > 0$ such that $n \geq K$ implies that $x_{n+1} - x_n < \varepsilon$.

Trivially, any quasi-Cauchy sequence is downward quasi-Cauchy, but there are downward quasi-Cauchy sequences which are not quasi-Cauchy. For example, the sequence $(b_n) = (-n)$ is downward quasi-Cauchy, but not quasi-Cauchy. Thus the set of all quasi-Cauchy sequences is a proper subset of the set of downward quasi-Cauchy sequences. Therefore any Cauchy sequence is downward quasi-Cauchy, so is a convergent sequence. Any downward half Cauchy sequence is downward quasi-Cauchy, but the converse is not always true, i.e. there are downward quasi-Cauchy sequences which are not downward half Cauchy, where a sequence (x_n) is called downward half Cauchy if given any $\varepsilon > 0$ there exists an integer $K > 0$ such that $m \geq n \geq K$ implies that $x_m - x_n < \varepsilon$ ([36]). As a counterexample consider the sequence $(-\log n)$. Thus the set of all downward half Cauchy sequences is a proper subset of the set of downward quasi-Cauchy sequences. Any subsequence of a downward half Cauchy sequence is downward half Cauchy. The analogous property fails for downward quasi-Cauchy sequences. Counterexample is the sequence $(a_n) = (\sqrt{n})$ with the subsequence $(a_{n^2}) = (n)$. In [36], Palladino proved that a sequence in $\mathbb{R}$ converges if and only if it is bounded and downward half Cauchy. The situation is different for downward quasi-Cauchyness, i.e. there are non-convergent sequences which are bounded and downward quasi-Cauchy. The bounded sequence $(\cos(6\log(n + 1)))$ is downward quasi-Cauchy, but neither Cauchy, nor downward half Cauchy.

As in the case that a sequence is Cauchy if and only if every subsequence of it is quasi-Cauchy, which was proved in [1], we prove that a sequence is downward half Cauchy if and only if every subsequence of it is downward quasi-Cauchy:

Theorem 2.2. *A sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is downward half Cauchy if and only if every subsequence of $\mathbf{x}$ is downward quasi-Cauchy.*

Proof. It is clear that if $\mathbf{x} = (x_k)$ is downward half Cauchy, then any subsequence of $\mathbf{x}$ is downward half Cauchy, so is downward quasi-Cauchy. To prove the converse now suppose that (x_k) is not downward half Cauchy so that there is a positive real number ε_0 such that for every positive integer n there exist positive integers k_i and k_j satisfying $k_j > k_i > n$ and $x_{k_j} - x_{k_i} \geq \varepsilon_0$. Choose positive integers k_1 and k_2 satisfying $k_2 > k_1 > 1$ and $x_{k_2} - x_{k_1} \geq \varepsilon_0$. Then collect positive integers k_3 and k_4 satisfying $k_4 > k_3 > k_2$ and $x_{k_4} - x_{k_3} \geq \varepsilon_0$. Having inductively chosen positive integers k_n and k_{n+1} satisfying $k_{n+1} > k_n > k_{n-1}$ and $x_{k_{n+1}} - x_{k_n} \geq \varepsilon_0$ we can choose positive integers k_{n+2} and k_{n+3} satisfying $k_{n+3} > k_{n+2} > k_{n+1}$ and $x_{k_{n+3}} - x_{k_{n+2}} \geq \varepsilon_0$. Hence the subsequence (x_{k_n}) is not downward quasi-Cauchy. This completes the proof of the theorem. □

Now we introduce a definition of downward compactness of a subset of $\mathbb{R}$ by using the main idea in the definition of ward compactness.

Definition 2.3. A subset E of $\mathbb{R}$ is called *downward compact*, if any sequence of points in E has a downward quasi-Cauchy subsequence.

We note that any finite subset of $\mathbb{R}$ is downward compact and the union of two downward compact subsets of $\mathbb{R}$ is downward compact. Any subset of a downward compact set is downward compact and therefore the intersection of any family of any downward compact subsets of $\mathbb{R}$ is downward compact. A bounded subset of $\mathbb{R}$ is downward compact. The sum of two downward compact subsets of $\mathbb{R}$ is also downward compact and the product of a downward compact set with a positive real number is downward compact. A slowly oscillating compact subset of $\mathbb{R}$ is downward compact (see [14] for the definition of slowly oscillating compactness). A compact subset of $\mathbb{R}$ is also downward compact. The set of negative integers is downward compact but the set of positive integers, $\mathbb{N}$ is not downward compact. Furthermore, any above bounded subset of $\mathbb{R}$ is downward compact. These observations suggest to us the following.

Theorem 2.4. *A subset E of $\mathbb{R}$ is bounded above if and only if it is downward compact, i.e. any sequence of points in E has a downward quasi-Cauchy subsequence.*

Proof. Let E be a above bounded subset of $\mathbb{R}$. If E is also bounded below, then any sequence in E has a convergent subsequence which is also downward quasi-Cauchy. If E is unbounded below, and (x_n) is an unbounded below sequence of points in E, then for $k = 1$ we can find an x_{n_1} less than 0. For k=2 we can pick an x_{n_2} such that $x_{n_2} < -1 + x_{n_1}$. We can successively find for each $k \in \mathbb{R}$ an $x_{n_{k+1}}$ such that $x_{n_{k+1}} < -k + x_{n_k}$. Then $x_{n_{k+1}} - x_{n_k} < -k$. Therefore for every $\varepsilon > 0$ we have $x_{n_{k+1}} - x_{n_k} < -k < \varepsilon$ for $k \geq n_0$. Thus we have constructed a downward quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n).

Conversely, suppose that E is not bounded above. Pick an element x_1 of E with $x_1 > 0$. Then we can choose an element x_2 of E such that $x_2 > 1 + x_1$. Similarly we can choose an element x_3 of E such that $x_3 > 2 + x_2$. We can inductively choose x_{k+1} satisfying $x_{k+1} > k + x_k$ for each $k \in \mathbb{N}$. Then the sequence (x_n) does not have any downward quasi-Cauchy subsequence. This contradiction completes the proof. □

Now reversing the places of x_k and x_{k+1} in the definition of downward quasi-Cauchy sequence in Definition 2.1, we give the following definition.

Definition 2.5. Suppose that (x_n) $(n = 1, 2, 3, ...)$ is a sequence of real numbers. (x_n) is called *upward quasi-Cauchy* if given any $\varepsilon > 0$ there exists an integer $K > 0$ such that $n \geq K$ implies that $x_n - x_{n+1} < \varepsilon$.

A sequence is called *half quasi-Cauchy* if it is either downward quasi-Cauchy, or upward quasi-Cauchy, or both. Trivially, any quasi-Cauchy sequence is upward quasi-Cauchy, but there are upward quasi-Cauchy sequences, which are not quasi-Cauchy. For example, the sequence $(a_n) = (n)$ is upward quasi-Cauchy, but not quasi-Cauchy. Thus the set of all quasi-Cauchy sequences is a proper subset of the set of upward quasi-Cauchy sequences. The intersection of the set of upward quasi-Cauchy sequences and the set of downward quasi-Cauchy sequences is equal to the set of all quasi-Cauchy sequences. Any Cauchy sequence is upward quasi-Cauchy, so is a convergent sequence. Any upward half Cauchy sequence is upward quasi-Cauchy, but the converse is not always true, i.e. there are upward quasi-Cauchy sequences which are not upward half Cauchy, where a sequence (x_n) is called upward half Cauchy if given any $\varepsilon > 0$ there exists an integer $K > 0$ such that $m \geq n \geq K$ implies that $x_n - x_m < \varepsilon$ ([36]). As a counterexample consider the sequence $(\ln n)$. Thus the set of all upward half Cauchy sequences is a proper subset of the set of upward quasi-Cauchy sequences. Any subsequence of an upward half Cauchy sequence is upward half Cauchy. The analogous property fails for upward quasi-Cauchy sequences. A counterexample is the sequence $(a_n) = (-\sqrt{n})$ with the subsequence $(a_{n^2}) = (-n)$. In [36], Palladino proved that a sequence in $\mathbb{R}$ converges if and only if it is bounded and upward half Cauchy. The situation is different, not only for the downward quasi-Cauchyness, but also for the upward quasi-Cauchyness, i.e. there are non-convergent sequences which are bounded and upward quasi-Cauchy. The bounded sequence $(\cos(6 \log(n + 1)))$ is upward quasi-Cauchy, but not upward half Cauchy.

As in Theorem 2.2, a sequence is half downward Cauchy if and only if every subsequence of it is downward quasi-Cauchy, we prove that a sequence is upward half Cauchy if and only if every subsequence of it is upward quasi-Cauchy:

Theorem 2.6. *A sequence $\boldsymbol{x} = (x_n)$ of points in $\mathbb{R}$ is upward half Cauchy if and only if every subsequence of $\boldsymbol{x}$ is upward quasi-Cauchy.*

Proof. The proof can be obtained using a similar technique to that of Theorem 2.2, so is omitted (see also [5]). □

We now give a definition of upward compactness of a subset of $\mathbb{R}$ by using the main idea in the definition of downward compactness.

Definition 2.7. A subset E of $\mathbb{R}$ is called *upward compact* if any sequence of points in E has an upward quasi-Cauchy subsequence.

It follows that any finite subset of $\mathbb{R}$ is upward compact, the union of two upward compact subsets of $\mathbb{R}$ is upward compact and the intersection of any family of any upward compact subsets of $\mathbb{R}$ is upward compact. Any subset of an upward compact set is upward compact and any bounded subset of $\mathbb{R}$ is upward compact. The sum of two upward compact sets of $\mathbb{R}$ is also upward compact and the product of an upward compact set with a positive real number is upward compact. Any N_θ ward compact subset of $\mathbb{R}$ is upward compact (see [2], and [18] for the definition and the concepts related to N_θ ward compactness). Any compact subset of $\mathbb{R}$ is also upward compact. The set of negative integers is not upward compact, whereas the set of positive integers, $\mathbb{N}$ is upward compact. We note that if a closed subset E of $\mathbb{R}$ is upward compact, then any sequence of points in E has a Abel convergent subsequence (see [25, 42, 43]). Furthermore, any below bounded subset of $\mathbb{R}$ is upward compact. These observations suggest to us the following.

Theorem 2.8. *A subset E of $\mathbb{R}$ is bounded below if and only if it is upward compact, i.e. any sequence of points in E has an upward quasi-Cauchy subsequence.*

Proof. The proof can be obtained using a similar technique to that of Theorem 2.4, so is omitted. □

3. Downward and Upward Continuities

A real valued function f on $\mathbb{R}$ is continuous if and only if for each point ℓ in the domain, $\lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\lim_{n\to\infty} x_n = \ell$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is. This is also equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is Cauchy. These known results for continuity for real functions in terms of sequences might suggest to us introducing a new type continuity, namely, downward continuity.

Definition 3.1. A function $f : E \to \mathbb{R}$ is called *downward continuous* on a subset E of $\mathbb{R}$, if it preserves downward quasi-Cauchy sequences, i.e. the sequence $(f(x_n))$ is downward quasi-Cauchy whenever (x_n) is a downward quasi-Cauchy sequence of points in E.

It should be noted that downward continuity cannot be given by any G-continuity in the manner of [8] (see also [30, 34]). We see that the composition of two downward continuous functions is downward continuous, and for every positive real number c, cf is downward continuous, if f is a downward continuous function.

We see in the following that the sum of two downward continuous functions is downward continuous.

Theorem 3.2. *If f and g are downward continuous functions, then $f + g$ is downward continuous.*

Proof. Let f, g be two downward continuous functions on a subset E of $\mathbb{R}$. To prove that $f + g$ is downward continuous on E, take any downward quasi-Cauchy sequence (x_n) of points in E. Then $(f(x_n))$ and $(g(x_n))$ are downward quasi-Cauchy sequences. Let $\varepsilon > 0$ be given. Since $(f(x_n))$ and $(g(x_n))$ are downward quasi-Cauchy, there exist positive integers n_1 and n_2 such that
$f(x_{n+1}) - f(x_n) < \frac{\varepsilon}{2}$ for $n \geq n_1$, and $g(x_{n+1}) - g(x_n) < \frac{\varepsilon}{2}$ for $n \geq n_2$.
Write $n_0 = max\{n_1, n_2\}$. Then $n \geq n_0$ implies that
$f(x_{n+1}) - f(x_n) + g(x_{n+1}) - g(x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
This completes the proof. □

As far as the product of two functions is considered, the case is different. If f and g are bounded positive valued functions, then the product of f and g is downward continuous, whenever f and g are.

In connection with downward quasi-Cauchy sequences, and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$:

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$,
(S_ρ) $(x_n) \in S_\rho \Rightarrow (f(x_n)) \in S_\rho$,
(δ) $(x_n) \in \Delta \Rightarrow (f(x_n)) \in \Delta$,
(δ^-) $(x_n) \in \Delta^- \Rightarrow (f(x_n)) \in \Delta^-$,
$(\delta^- c)$ $(x_n) \in \Delta^- \Rightarrow (f(x_n)) \in c$,
$(c\delta^-)$ $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta^-$,

where S_ρ denotes the set of ρ-statistical convergent sequences. We see that (c) can be replaced by not only ρ-statistical continuity, but also by lacunary statistical continuity, N_θ-sequential continuity ([18]), I-sequential continuity for a nontrivial admissible ideal I ([17]), and more generally G-sequential continuity for any regular subsequential method G ([8, 13]). We see that (δ^-) is downward continuity of f. It is easy to see that $(\delta^- c)$ implies (δ^-); (δ^-) does not imply $(\delta^- c)$; (δ^-) implies $(c\delta^-)$; $(c\delta^-)$ does not imply (δ^-); $(\delta^- c)$ implies (c), and (c) does not imply $(\delta^- c)$; and (c) implies $(c\delta^-)$.

Now we give the implication of (δ^-) to (c), i.e. any downward continuous function is continuous.

Theorem 3.3. *Any downward continuous function is continuous.*

Proof. Let (x_n) be any convergent sequence with $\lim_{k\to\infty} x_k = \ell$. Then the sequence

$$(x_1, \ell, x_1, \ell, x_2, \ell, x_2, \ell, ..., x_n, \ell, x_n, \ell, ...)$$

also converges to ℓ. Thus it is downward quasi-Cauchy. Hence

$$(f(x_1), f(\ell), f(x_1), f(\ell), f(x_2), f(\ell), f(x_2), f(\ell), ..., f(x_n), f(\ell), f(x_n), f(\ell), ...)$$

is downward quasi-Cauchy. Therefore $\lim_{n\to\infty} f(x_k) = f(\ell)$. This completes the proof of the theorem. □

It is easy to give an example that the converse of the preceding theorem is not always true.

We have the following result for general sequential methods.

Corollary 3.4. *If f is downward continuous, then it is G-continuous for any regular subsequential method G.*

Proof. The proof follows from Theorem 3.3, and [8, Theorem 13] (see also [13]). □

Corollary 3.5. *If f is downward continuous, then it is I-continuous for any non-trivial admissible ideal I of $\mathbb{N}$.*

Proof. The proof follows from Theorem 3.3, and [17, Theorem 4]. □

We note that if f is downward continuous, then it is either ρ-statistically continuous, or lacunary statistically continuous or N_θ-continuous ([2, 18]).

Theorem 3.6. *A downward continuous image of any downward compact subset of $\mathbb{R}$ is downward compact.*

Proof. Let E be a subset of $\mathbb{R}$, $f : E \longrightarrow \mathbb{R}$ be a downward continuous function, and A be a downward compact subset of E. Take any sequence $\boldsymbol{y} = (y_n)$ of points in $f(A)$. Write $y_n = f(x_n)$, where $x_n \in A$ for each $n \in \mathbb{R}$, $\boldsymbol{x} = (x_n)$. Downward compactness of A implies that there is a downward quasi-Cauchy subsequence $\boldsymbol{\xi}$ of the sequence of $\boldsymbol{x}$. Write $\boldsymbol{z} = (z_k) = f(\boldsymbol{\xi}) = (f(\xi_k))$. Then $\boldsymbol{z}$ is a downward quasi-Cauchy subsequence of the sequence $f(\boldsymbol{x})$. This completes the proof of the theorem. □

We note that downward continuous image of any N_θ-sequentially compact subset of $\mathbb{R}$ is N_θ-sequentially compact, and downward continuous image of any ρ-statistically sequentially compact subset of $\mathbb{R}$ is lacunary statistically sequentially compact. Furthermore downward continuous image of any G-sequentially connected subset of $\mathbb{R}$ is G-sequentially connected (see [11, 19]).

We see in the following that the uniform limit of downward continuous functions on a subset of $\mathbb{R}$ is downward continuous.

Theorem 3.7. *If (f_n) is a sequence of downward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is downward continuous on E.*

Proof. Let $\varepsilon > 0$. Then there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. Take any downward quasi-Cauchy sequence (x_n) of points in E. As f_N is downward continuous, the sequence $(f_N(x_n))$ is a downward quasi-Cauchy sequence, so there exists a positive integer N_1, depending on ε, and greater than N such that
$f_N(x_{n+1}) - f_N(x_n) < \frac{\varepsilon}{3}$ for $n \geq N_1$.
Now for $n \geq N_1$ we have

$$\begin{aligned} f(x_{n+1}) - f(x_n) &= f(x_{n+1}) - f_N(x_{n+1}) + f_N(x_{n+1}) - f_N(x_n) + f_N(x_n) - f(x_n) \\ &\leq f(x_{n+1}) - f_N(x_{n+1}) + \frac{\varepsilon}{3} + f_N(x_n) - f(x_n) \\ &\leq |f(x_{n+1}) - f_N(x_{n+1})| + \frac{\varepsilon}{3} + |f_N(x_n) - f(x_n)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This completes the proof of the theorem. □

Definition 3.8. A function f is called *upward continuous* on a subset E of $\mathbb{R}$ if it preserves upward quasi-Cauchy sequences, i.e. the sequence $(f(x_n))$ is upward quasi-Cauchy whenever $\mathbf{x} = (x_n)$ is an upward quasi-Cauchy sequence of points in E.

We see that the sum of two upward continuous functions is upward continuous and the composite of two upward continuous functions is upward continuous.

In connection with upward quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$:
$(\delta^+)(x_n) \in \Delta^+ \Rightarrow (f(x_n)) \in \Delta^+$,
$(\delta^+ c)(x_n) \in \Delta^+ \Rightarrow (f(x_n)) \in c$,
$(c\delta^+)(x_n) \in c \Rightarrow (f(x_n)) \in \Delta^+$.
We see that (δ^+) is downward continuity of f. It is easy to see that $(\delta^+ c)$ implies (δ^+); (δ^+) does not imply $(\delta^+ c)$; (δ^+) implies $(c\delta^+)$; $(c\delta^+)$ does not imply (δ^+); $(\delta^+ c)$ implies (c), and (c) does not imply $(\delta^+ c)$.

Now we see that (δ^+) implies (c) in the following:

Theorem 3.9. *If f is upward continuous on a subset E of $\mathbb{R}$, then it is continuous on E in the ordinary sense.*

Proof. The proof can be obtained using a similar technique to that of Theorem 3.3, so is omitted (see also [5]). □

It should be noted that the converse of the preceding theorem is not always true, i.e. there are continuous functions which are not upward continuous.

We have the following result for general sequential methods.

Corollary 3.10. *If f is upward continuous, then it is G-continuous for any regular subsequential method G.*

Proof. The proof follows from Theorem 3.9, and [8, Theorem 13]. □

Corollary 3.11. *If f is upward continuous, then it is I-continuous for any non-trivial admissible ideal I of $\mathbb{N}$.*

Proof. The proof follows from Theorem 3.9, and [17, Theorem 4]. □

We note that if f is upward continuous, then it is either ρ-statistically continuous, or lacunary statistically continuous or N_θ-continuous.

Theorem 3.12. *A upward continuous image of any upward compact subset of $\mathbb{R}$ is upward compact.*

Proof. The proof can be obtained using a similar technique to that of Theorem 3.6, so is omitted (see also [5]). □

Corollary 3.13. *A upward continuous image of any G-sequentially compact subset of $\mathbb{R}$ is G-sequentially compact for a regular subsequential method G.*

We see that N_θ-sequentially compact subset of $\mathbb{R}$ is upward compact, and upward continuous image of any ρ-statistically compact subset of $\mathbb{R}$ is ρ-statistically compact (see [4]), upward continuous image of any statistically compact subset of $\mathbb{R}$ is statistically compact, and upward continuous image of any lacunary statistically compact subset of $\mathbb{R}$ is lacunary statistically compact (see [15]).

Now we prove in the following that uniform limit of upward continuous functions is upward continuous, which was placed in [5] without proof.

Theorem 3.14. *If (f_n) is a sequence of upward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is upward continuous on E.*

Proof. The proof is similar to the proof of Theorem 3.7 so is omitted. □

4. Conclusion

In this paper, the notions of downward continuity and upward continuity of a real function are introduced and investigated. In this investigation we have obtained results related to downward continuity, upward continuity, some other kinds of continuities. We also introduce and study some other continuities involving downward quasi-Cauchy sequences, upward quasi-Cauchy sequences, convergent sequences, ρ-statistical convergent sequences, lacunary λ-statistical convergent sequences of points in $\mathbb{R}$. It turns out that not only the set of downward continuous functions, but also the set of upward continuous functions is a proper subset of the set of continuous functions. We suggest to investigate downward and upward quasi-Cauchy sequences of fuzzy points in asymmetric fuzzy spaces (see [16, 33] for the definitions and related concepts in fuzzy setting). We also suggest to investigate downward and upward quasi-Cauchy double sequences (see for example [20, 31, 32, 37] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate upward quasi-Cauchy sequences of points in an asymmetric cone metric space since in a cone metric space the notion of an upward quasi-Cauchy sequence coincides with the notion of a downward quasi-Cauchy sequence, and therefore upward continuity coincides with downward continuity (see [21, 23, 35, 41]).

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