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On the Stability of the Schrödinger Equation with Time Delay

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Abstract. In the present paper, the initial value problem for the Schrödinger equation with time delay in a Hilbert space is investigated. Theorems on stability estimates for the solution of the problem are established. The applications of theorems for three types of Schrödinger problems are provided.

1. Introduction

It is known that various problems in physics lead to the Schrödinger equation. Methods of solutions of the problems for Schrödinger equation have been studied extensively by many researchers (see, e.g., [2, 6–9, 11, 14–16, 19–22], and the references given therein).

Time delay is one of the most common phenomena occurring in many engineering applications. In control theory, the process of sampled-data control is a typical example where time delay happens in the transmission from measurement to controller.

Theory and applications of delay linear and nonlinear Schrödinger equations with the delay term is an operator of lower order with respect to the operator term were widely investigated (see, e.g., [13, 17, 18, 25–27], and the references given therein).

For example, in the article [26], the boundary stabilization of a Schrödinger equation with variable coefficient where the boundary observation suffers from a fixed time delay was studied. This is a generalization of the similar work for the Schrödinger equation in [18] by using the separation principle [17] for constant coefficients. The variable coefficients make the system too complicated to estimate the solution, which relies on the estimation of the eigenvalues and eigenfunctions by asymptotic analysis. In [25], existence and uniqueness of local solutions of nonlinear Schrödinger equation with delay was investigated. In [13], the existence and upper semi-continuity of the global attractor for discrete nonlinear delay Schrödinger equation was established. The paper [27] was devoted to study of traveling waves of nonlinear Schrödinger equation with distributed delay by applying geometric singular perturbation theory, differential manifold theory and the regular perturbation analysis for a Hamiltonian system. Under the assumptions that the distributed delay kernel was strong general delay kernel and the average delay was small, the existence of solitary wave solutions was investigated by differential manifold theory. Then by utilizing the regular perturbation analysis for a Hamiltonian system, the periodic traveling wave solutions were explored. Finally, theory and applications of partial differential equations with the delay term is an operator of same order

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with respect to the other operator term were widely investigated for delay parabolic differential equations (see, e.g., [1, 3–5, 10, 12, 23], and the references given therein).

Our goal in this paper is to investigate the initial value problem for the Schrödinger equation with time delay

$$i\frac{dv(t)}{dt} + Av(t) = bAv([t]), \quad 0 < t < \infty, v(0) = \varphi \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator $A, A \geq \delta I$,where $\delta > 0$. Here φ is the given element of $D(A)$ and $[t]$ denotes the greatest-integer function and $0 \leq b \leq 1$.

Theorems on stability estimates for the solution of problem (1) are established. The application of theorems for three types of Schrödinger problems is provided. Therefore, this article has great significance for obtaining stability estimates of systems described by Schrödinger type of partial differential equations.

The paper is organized as follows. Section 1 is introduction. In Section 2, theorems on stability of problem (1) are established. In Section 3, theorems on the stability estimates for the solution of three problems for the Schrödinger equation are obtained. Finally, Section 4 is conclusion.

2. Theorems on Stability

A function $v(t)$ is called a solution of problem (1), if the following conditions are satisfied:

i. $v(t)$ is continuously differentiable function on the interval $[0, \infty)$. The derivative at the endpoint $t = 0$ is understood as the appropriate unilateral derivative.

ii. The element $v(t)$ belongs to $D(A)$ for all $t \in [0, \infty)$, and the function $Av(t)$ is continuous on the interval $[0, \infty)$.

iii. $v(t)$ satisfies the main equation and initial conditions in (1).

For the self-adjoint positive definite operator A, we have that

$$\left\|exp\{itA\}\right\|_{H\to H} \leq 1, \forall t \in Z. \tag{2}$$

Theorem 2.1. *For the solution of problem* (1) *the following estimates hold:*

$$\max_{0\leq t\leq 1} \|v(t)\|_H \leq \left\|\varphi\right\|_H, \tag{3}$$

$$\max_{n\leq t\leq n+1} \|v(t)\|_H \leq \max_{n-1\leq t\leq n} \|v(t)\|_H, \ n = 1, 2, ... \tag{4}$$

Proof. Let $t \in [0, 1]$. Then, applying (1), we get

$$e^{-iAt}\left[v'(t) - iAv(t)\right] = be^{-iAt}\left[-iAv([t])\right]. \tag{5}$$

Taking the integral with respect to s from 0 to t, we get

$$\int_0^t \left[v(s)e^{-iAs}\right]' ds = b\int_0^t e^{-iAs}(-i)Av([s])ds.$$

Then

$$v(t) - e^{iAt}v(0) = b\int_0^t e^{iA(t-s)}(-i)Av([s])ds.$$

Since $v([s]) = \varphi$ for $0 \le s \le t \le 1$, we have that

$$v(t) = e^{iAt}\varphi + bA\int_{0}^{t} e^{iA(t-s)}(-i)ds\varphi.$$

Applying formula

$$\int_{0}^{t} e^{iA(t-s)}(-i)ds = (A)^{-1}[I - e^{iAt}],$$

we get

$$v(t) = e^{iAt}\varphi + b[I - e^{iAt}]\varphi = b\varphi + (1-b)\,e^{iAt}\varphi. \tag{6}$$

Let $t \in [n-1, n], n = 1, 2, ...$ Then, applying (5) and taking the integral with respect to s from $n-1$ to t, we get

$$\int_{n-1}^{t} [v(s)e^{-iAs}]' ds = b\int_{n-1}^{t} e^{-iAs}(-i)Av([s])ds.$$

Then, using $v([s]) = v(n-1)$ for $n-1 \le s \le t \le n$, we get

$$v(t) - e^{iA(t-n+1)}v(n-1) = b\int_{n-1}^{t} e^{iA(t-s)}(-i)Av(n-1)ds.$$

Applying formula

$$\int_{n-1}^{t} e^{iA(t-s)}(-i)ds = (A)^{-1}[I - e^{iA(t-n+1)}],$$

we get

$$v(t) = e^{iA(t-n+1)}v(n-1) + b[I - e^{iA(t-n+1)}]v(n-1) = bv(n-1) + (1-b)\,e^{iA(t-n+1)}v(n-1). \tag{7}$$

So, there exists a unique solution of problem (1), and for the solution we have the formulas (6) and (7). Now, we establish the estimates (3) and (4). Using formula (6), the triangle inequality and estimate (2), we get

$$\|v(t)\|_H \le b\left\|\varphi\right\|_H + (1-b)\left\|e^{iAt}\right\|_{H\to H}\left\|\varphi\right\|_H \le \left\|\varphi\right\|_H$$

for $t \in [0, 1]$. From that estimate (3) follows. Using formula (7), the triangle inequality and estimate (2), we get

$$\|v(t)\|_H \le b\,\|v(n-1)\|_H + (1-b)\left\|e^{iA(t-n+1)}\right\|_{H\to H}\|v(n-1)\|_H \le \|v(n-1)\|_H$$

for $t \in [n-1, n], n = 1, 2, ...$ From that estimate (4) follows. Theorem 2.1 is established. □

Theorem 2.2. *Assume that $\varphi \in D(A)$. Then, for the solution of problem (1) the following estimates hold:*

$$\max_{0\le t\le 1}\|Av(t)\|_H \le \left\|A\varphi\right\|_H,$$

$$\max_{n\le t\le n+1}\left\|v'(t)\right\|_H + \max_{n\le t\le n+1}\|Av(t)\|_H \le \max_{n-1\le t\le n}\|Av(t)\|_H,\ n = 1, 2, ...$$

The proof of Theorem 2.2 follows from the scheme of the proof of Theorem 2.1, and it is based on formulas (6), (7), on estimate (2) and on commutation of A and e^{iAt}.

Note that from Theorem 2.1 and Theorem 2.2, the following stability estimates

$$\sup_{0\le t<\infty} \|v(t)\|_H \le \|\varphi\|_H, \tag{8}$$

$$\sup_{0\le t<\infty} \left\|v'(t)\right\|_H + \sup_{0\le t<\infty} \|Av(t)\|_H \le \|A\varphi\|_H \tag{9}$$

can be obtained for the solution of problem (1).

3. Applications

In this section, we consider the applications of Theorems 2.1-2.2.

First, the boundary value problem for the Schrödinger equation with time delay

$$\begin{cases} iu_t(t,x) - (a(x)u_x(t,x))_x + \delta u = b\left(-\left(a(x)u_x([t],x)\right)_x + \delta u([t],x)\right), \\ 0 < t < \infty,\ 0 < x < 1, \\ u(0,x) = \varphi(x), 0 \le x \le 1, \\ u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1), \quad 0 \le t < \infty \end{cases} \tag{10}$$

is considered. Problem (10) has a unique solution $u(t,x)$ for the smooth functions $a(x) \ge a > 0$, $x \in (0,1)$, $\delta > 0$, $a(1) = a(0)$, $\varphi(x)$ $(x \in [0,1])$ and $0 \le b \le 1$. This allows us to reduce the boundary value problem (10) to the boundary value problem (1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -(a(x)u_x)_x + \delta u \tag{11}$$

with domain

$$D(A^x) = \{u(x) : u(x), u_x(x), (a(x)u_x)_x \in L_2[0,1], u(1) = u(0), u_x(1) = u_x(0)\}.$$

Applying the symmetry property of the space operator A^x with the domain $D(A^x) \subset W_2^2[0,1]$ and estimates (8) and (9) in $H = L_2[0,1]$, we can obtain the following theorem on stability of problem (10).

Theorem 3.1. *For solutions of problem* (10) *we have following stability estimates*

$$\sup_{0\le t<\infty} \|u(t,\cdot)\|_{L_2[0,1]} \le \|\varphi\|_{L_2[0,1]},$$

$$\sup_{0\le t<\infty} \|u_t(t,\cdot)\|_{L_2[0,1]} + \sup_{0\le t<\infty} \|u(t,\cdot)\|_{W_2^2[0,1]} \le M_1 \|\varphi\|_{W_2^2[0,1]},$$

where M_1 does not depend on $\varphi(x)$. Here, $W_2^2[0,1]$ is the Sobolev space of all square integrable functions $\psi(x)$ defined on $[0,1]$ *equipped with the norm*

$$\|\psi\|_{W_2^2[0,1]} = \left\{\int_0^1 \left[\psi^2(x) + \psi_{xx}^2(x)\right] dx\right\}^{1/2}.$$

Second, let Ω be the unit open cube in the n-dimensional Euclidean space.

$\mathbb{R}^n (x = (x_1, \cdots, x_n) : 0 < x_k < 1, k = 1, \cdots, n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0, \infty) \times \Omega$, the boundary value problem for the multi-dimensional Schrödinger equation with time delay and the Dirichlet condition

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x))_{x_r} = -b \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}([t], x))_{x_r}, \\ 0 < t < \infty, x \in \Omega, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, \ x \in S, \ \ 0 \leq t < \infty \end{cases} \tag{12}$$

is considered. Here $a_r(x) \geq a > 0$, $(x \in \Omega)$, $\varphi(x)(x \in \overline{\Omega})$ are given smooth functions and $0 \leq b \leq 1$.

We consider the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \left(\int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right)^{\frac{1}{2}}.$$

Problem (12) has a unique solution $u(t,x)$ for the smooth functions $\varphi(x)$, $a_r(x)$. This allows us to reduce the problem (12) to the boundary value problem (1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} \tag{13}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, u(x) = 0, x \in S \right\}.$$

Therefore, estimates (8) and (9) in $H = L_2(\overline{\Omega})$ permit us to get the following theorem on stability of problem (12).

Theorem 3.2. *For the solutions of problem* (12), *we have following stability estimates*

$$\sup_{0 \leq t < \infty} \|u(t, \cdot)\|_{L_2(\overline{\Omega})} \leq \left\| \varphi \right\|_{L_2(\overline{\Omega})},$$

$$\sup_{0 \leq t < \infty} \|u_t(t, \cdot)\|_{L_2(\overline{\Omega})} + \sup_{0 \leq t < \infty} \|u(t, \cdot)\|_{W_2^2(\overline{\Omega})} \leq M_2 \left\| \varphi \right\|_{W_2^2(\overline{\Omega})},$$

where M_2 does not depend on $\varphi(x)$. Here and in the future, $W_2^2(\overline{\Omega})$ is the Sobolev space of all square integrable functions $\psi(x)$ defined on $\overline{\Omega}$ equipped with the norm

$$\left\| \psi \right\|_{W_2^2(\overline{\Omega})} = \left(\int \cdots \int_{x \in \overline{\Omega}} \left[\left| \psi(x) \right|^2 + \sum_{r=1}^{n} \left| \psi_{x_r x_r}(x) \right|^2 \right] dx_1 \cdots dx_n \right)^{\frac{1}{2}}.$$

The proof of Theorem 3.2 is based on estimates (8) and (9) in $H = L_2(\overline{\Omega})$ and the symmetry property of the operator A^x defined by formula (13) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.3. *For the solution of the elliptic differential problem [24]*

$$\begin{cases} A^x u(x) = \mu(x), x \in \Omega, \\ u(x) = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \left\| u_{x_r x_r} \right\|_{L_2(\overline{\Omega})} \le M_3 \|\mu\|_{L_2(\overline{\Omega})}.$$

Here M_3 does not depend on $\mu(x)$.

Third, in $[0, \infty) \times \Omega$, the boundary value problem for the multi-dimensional Schrödinger equation with time delay and Neumann boundary condition

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x))_{x_r} + \delta u(t,x) = b\left(- \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}([t], x))_{x_r} + \delta u([t], x) \right), \\ 0 < t < \infty, x \in \Omega, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial \vec{n}} = 0, \ x \in S, \ 0 \le t < \infty \end{cases} \tag{14}$$

is considered. Here, $\vec{n}$ is the normal vector to S, $a_r(x) \ge a > 0$, $(x \in \Omega)$, $\varphi(x)$ $(x \in \overline{\Omega})$ are given smooth functions and $\delta > 0, 0 \le b \le 1$.

Problem (14) has a unique solution $u(t, x)$ for the smooth functions $\varphi(x)$ and $a_r(x)$. This allows us to reduce the problem (14) to the boundary value problem (1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \delta u \tag{15}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, \frac{\partial u(x)}{\partial \vec{n}} = 0, x \in S \right\}.$$

Therefore, estimates (8) and (9) in $H = L_2(\overline{\Omega})$ permit us to get the following theorem on stability of problem (14).

Theorem 3.4. *For the solutions of problem* (14)*, we have following stability estimates*

$$\sup_{0 \le t < \infty} \|u(t, \cdot)\|_{L_2(\overline{\Omega})} \le \left\| \varphi \right\|_{L_2(\overline{\Omega})},$$

$$\sup_{0 \le t < \infty} \|u_t(t, \cdot)\|_{L_2(\overline{\Omega})} + \sup_{0 \le t < \infty} \|u(t, \cdot)\|_{W_2^2(\overline{\Omega})} \le M_4 \left\| \varphi \right\|_{W_2^2(\overline{\Omega})},$$

where M_4 does not depend on $\varphi(x)$.

The proof of Theorem 3.4 is based on estimates (8) and (9) in $H = L_2(\overline{\Omega})$ and the symmetry property of the operator A^x defined by formula (14) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.5. *For the solution of the elliptic differential problem [24]*

$$\begin{cases} A^x u(x) = \mu(x), x \in \Omega, \\ \frac{\partial u(x)}{\partial \vec{n}} = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \left\| u_{x_r x_r} \right\|_{L_2(\overline{\Omega})} \leq M_5 \|\mu\|_{L_2(\overline{\Omega})}.$$

Here M_5 does not depend on $\mu(x)$.

4. Conclusion

In the present paper, the initial value problem for the Schrödinger equation with time delay in a Hilbert space is investigated. Theorems on stability estimates for the solution of the problem are established. The applications of these theorems for three types of Schrödinger problems are provided. Moreover, applying the result of the monograph [10], the single-step difference schemes for the numerical solution of boundary value problem (1) can be constructed. Of course, such type of results of stability estimates hold for the solutions of these difference schemes. Applying this approach and method of [10], we can study the initial value problem for the Schrödinger differential equation with time delay

$$\begin{cases} i\frac{dv(t)}{dt} + Av(t) = bAv(t-w) + f(t), \ \ 0 < t < \infty, \\ \\ v(t) = \varphi(t), \ \ -w \leq t \leq 0 \end{cases}$$

in a Hilbert space H with a self-adjoint positive definite operator A. Here $\varphi(t)$ is a continuous abstract function defined on the interval $[-w, 0]$ with values in H, $f(t)$ is continuous abstract function defined on the interval $[0, \infty)$ with values in H.

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