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A Unified Class of Harmonic Functions with Varying Argument of Coefficients

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Abstract. In this paper we investigate several classes of harmonic functions with varying argument of coefficients which are defined by means of the principle of subordination between harmonic functions. Properties such as the coefficient estimates, distortion theorems, convolution properties, radii of convexity, starlikeness and the closure properties of these classes under the generalized Bernardi-Libera-Livingston integral operators are investigated.

1. Introduction and definitions

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]). Let $\mathcal{H}$ denote the family of continuous complex-valued functions that are harmonic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$.
Denote by $\mathcal{S}_{\mathcal{H}}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic and orientation preserving in the open unit disc $\mathcal{U}$ so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = g'(0) = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=2}^{\infty} b_m z^m,$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m}. \tag{2}$$

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For functions $f \in \mathcal{H}$ given by (2) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=2}^{\infty} B_m z^m}, \tag{3}$$

we denote the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=2}^{\infty} b_m B_m z^m} \quad (z \in \mathcal{U}). \tag{4}$$

Let $V_{\mathcal{H}}$ be the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [7], consisting of functions f of the form (1) in $\mathcal{S}_{\mathcal{H}}$ for which there exists a real number ξ such that

$$\eta_m + (m-1)\,\xi \equiv \pi\,(mod\,2\pi), \quad \delta_m + (m+1)\,\xi \equiv 0\,(mod\,2\pi), \quad m \geq 2, \tag{5}$$

where $\eta_m = arg\,(a_m)$ and $\delta_m = arg\,(b_m)$.

For $0 \leq \alpha < 1$ we let $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ and $\mathcal{S}^c_{\mathcal{H}}(\alpha)$, respectively, denote the subclasses of $\mathcal{S}_{\mathcal{H}}$ consisting of harmonic starlike and harmonic convex functions of order α.

A functions f of the form (2) is in $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if (see [3],[2])

$$\frac{\partial}{\partial\theta}\left(arg\, f\left(re^{i\theta}\right)\right) > \alpha, |z| = r < 1.$$

Similarly, a function f of the form (2) is in $\mathcal{S}^c_{\mathcal{H}}(\alpha)$ if and only if

$$\frac{\partial}{\partial\theta}\left(arg\,\frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right) > \alpha, |z| = r < 1.$$

We note that $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if

$$\Re\frac{\mathscr{D}_{\mathcal{H}} f(z)}{f(z)} > \alpha, |z| = r < 1$$

or

$$\left|\frac{\mathscr{D}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathscr{D}_{\mathcal{H}} f(z) + (1-\alpha) f(z)}\right| < 1, |z| = r < 1,$$

where $\mathscr{D}_{\mathcal{H}} f(z) := zh'(z) - \overline{zg'(z)}$.

For $0 \leq \alpha < 1$ if $f \in \mathcal{S}^c_{\mathcal{H}}(\alpha)$ then $\mathscr{D}_{\mathcal{H}} f(z) \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$.

Let k, A and B be real parameters such that

$$k \geq 0,\ 0 \leq B \leq 1 \text{ and } -1 \leq A < B.$$

Also let $\varphi, \phi \in \mathcal{H}$. Motivated by Dziok and Srivastava [5] ,we denote by $\mathscr{W}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)$, $0 \leq B < 1$ the class of functions $f \in \mathcal{H}$ such that

$$(\varphi * f)(z) \neq 0, z \in \mathcal{U} \setminus \{0\}$$

and

$$\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - k\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - 1\right| - \frac{1-AB}{1-B^2}\right| < \frac{B-A}{1-B^2} \quad (z \in \mathcal{U}). \tag{6}$$

If $B = 1$, then we have

$$\Re\left(\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)}\right) - k\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - 1\right| > \frac{1+A}{2} \quad (z \in \mathcal{U}). \tag{7}$$

Let us define

$$\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right) := \mathscr{W}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right) \cap V_{\mathcal{H}}.$$

We assume that f is of the form (2), φ and ϕ are the functions of the following forms:

$$\varphi(z) = z + \sum_{m=2}^{\infty} c_m z^m + \overline{\sum_{m=2}^{\infty} d_m z^m} \text{ and } \phi(z) = z + \sum_{m=2}^{\infty} e_m z^m + \overline{\sum_{m=2}^{\infty} f_m z^m} \tag{8}$$

where

$$0 \leq c_m \leq e_m \text{ and } 0 \leq d_m \leq f_m. \tag{9}$$

2. Coefficient estimates

Theorem 2.1. *Let* $0 \leq B \leq 1$, $-1 \leq A < B$ and $(\varphi * f)(z) \neq 0, z \in \mathcal{U} \setminus \{0\}$. *If*

$$\sum_{m=2}^{\infty} \left(|a_m| \alpha_m + |b_m| \beta_m\right) \leq B - A \tag{10}$$

then $f \in \mathscr{W}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)$ *where*

$$\alpha_m = (k+1)(1+B) e_m + [(B-A) - (k+1)(1+B)] c_m, \tag{11}$$

$$\beta_m = (k+1)(1+B) f_m + [(B-A) - (k+1)(1+B)] d_m. \tag{12}$$

Proof. Let $B \neq 1$.

$$\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - k\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - 1\right| - \frac{1-AB}{1-B^2}\right| \leq$$

$$\leq (k+1)\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - 1\right| + \frac{B(B-A)}{1-B^2} \leq$$

$$\leq (k+1)\frac{\sum_{m=2}^{\infty}\left\{|a_m|(e_m - c_m) + |b_m|(f_m - d_m)\right\}|z|^{m-1}}{1 - \sum_{m=2}^{\infty}\left(|a_m| c_m + |b_m| d_m\right)|z|^{m-1}} + \frac{B(B-A)}{1-B^2}.$$

Thus, by (10), we obtain (6). Consequently, $f \in \mathscr{W}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)$.

If we suppose that $B = 1$, then

$$k\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - 1\right| - \Re\left(\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - \frac{1+A}{2}\right) \leq (k+1)\left|\frac{\left(\phi * f\right)(z)}{(\varphi * f)(z)} - 1\right| \leq$$

$$\leq (k+1)\frac{\sum_{m=2}^{\infty}\left\{|a_m|(e_m-c_m)+|b_m|(f_m-d_m)\right\}|z|^{m-1}}{1-\sum_{m=2}^{\infty}(|a_m|c_m+|b_m|d_m)|z|^{m-1}},$$

which, by means of (10), leads us to (7). Therefore, $f \in \mathscr{W}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$. □

Theorem 2.2. *Let f be a function of the form* (2) *satisfying the argument property* (5). *Then $f \in \mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$ if and only if condition* (10) *holds true.*

Proof. Let f be a function of the form (2) satisfying the argument property (5) belongs to the class $\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$. Then, putting $z=re^{i\xi}$ in the conditions (6) and (7), we obtain

$$(k+1)\frac{\sum_{m=2}^{\infty}\left\{|a_m|(e_m-c_m)+|b_m|(f_m-d_m)\right\}r^{m-1}}{1-\sum_{m=2}^{\infty}(|a_m|c_m+|b_m|d_m)r^{m-1}}<\frac{B-A}{1+B}.$$

We thus find that

$$\sum_{m=2}^{\infty}\left\{|a_m|\left[(k+1)(1+B)(e_m-c_m)+(B-A)c_m\right]+\right.$$
$$\left.+|b_m|\left[(k+1)(1+B)(f_m-d_m)+(B-A)d_m\right]\right\}r^{m-1}<B-A,$$

which, upon letting $r\to 1^-$, readily yields the assertion (10). □

Corollary 1. *If a function f of the form* (2) *belongs to the class $\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$, then*

$$|a_m|\leq\frac{B-A}{\alpha_m},\quad |b_m|\leq\frac{B-A}{\beta_m},\quad (m\in\{2,3,\ldots\}) \tag{13}$$

where α_m and β_m are defined by (10).
The result is sharp and the extremal functions are

$$f_{1,m}=z-\frac{B-A}{\alpha_m}e^{i(1-m)\xi}z^m, \tag{14}$$

and

$$f_{2,m}=z+\frac{B-A}{\beta_m}e^{i(1+m)\xi}\overline{z}^m,\ m\in\{2,3,\cdots\}. \tag{15}$$

3. Distortion theorems

Theorem 3.1. *Let f be a function in the class $\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$ then*

$$r-\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2\leq\left|f(z)\right|\leq r+\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2, \tag{16}$$

where $\alpha_2\leq\alpha_m$, $\beta_2\leq\beta_m$ $(m\in\mathbb{N}\setminus\{1\})$.

Proof. Let a function f of the form (2) which belongs to the class $\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$.
Then, for $|z|=r<1$, taking the absolute value of f, we obtain

$$\left|f(z)\right| \le r+\sum_{m=2}^{\infty}\left(|a_m|+|b_m|\right)r^m \le r+\sum_{m=2}^{\infty}\left(|a_m|+|b_m|\right)r^2.$$

This implies that

$$\left|f(z)\right| \le r+\frac{1}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}\sum_{m=2}^{\infty}\frac{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}{1}\left(|a_m|+|b_m|\right)r^2$$

$$\left|f(z)\right| \le r+\frac{1}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}\sum_{m=2}^{\infty}\left(\alpha_m|a_m|+\beta_m|b_m|\right)r^2 \le$$
$$\le r+\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2.$$

So

$$\left|f(z)\right| \le r+\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2$$

and analogously

$$\left|f(z)\right| \ge r-\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2.$$

□

4. Radii of starlikeness and convexity

Let $\mathcal{B}\subseteq\mathcal{H}$. We define the radius of starlikeness and the radius of convexity of the class $\mathcal{B}$:

$$R_\alpha^* := \inf_{f\in\mathcal{B}}\left(\sup\left\{r\in(0,1] : f \text{ is starlike of order } \alpha \text{ in } \mathcal{U}(r)\right\}\right),$$

$$R_\alpha^c := \inf_{f\in\mathcal{B}}\left(\sup\left\{r\in(0,1] : f \text{ is convex of order } \alpha \text{ in } \mathcal{U}(r)\right\}\right),$$

$$\text{where} \quad U(r) := \left\{z\in\mathbb{C} : |z|<r\le 1\right\}.$$

Theorem 4.1. *Let $0\le\alpha<1$ and α_k and β_k be defined by* (10). *Then*

$$R_\alpha^*\left(\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)\right)=\inf_{m\ge2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\alpha_m}{m-\alpha},\frac{\beta_m}{m+\alpha}\right\}\right)^{\frac{1}{m-1}}.$$

Proof. Let $f\in\mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$ be of the form (14). Then for $|z|=r<1$, we have

$$\left|\frac{\mathscr{D}_{\mathcal{H}}f(z)-(1+\alpha)f(z)}{\mathscr{D}_{\mathcal{H}}f(z)+(1-\alpha)f(z)}\right| =$$

$$=\left|\frac{-\alpha z+\sum_{m=2}^{\infty}\left\{(m-1-\alpha)a_m z^m-(m+1+\alpha)b_m\overline{z}^m\right\}}{(2-\alpha)z+\sum_{m=2}^{\infty}\left\{(m+1-\alpha)a_m z^m-(m-1+\alpha)b_m\overline{z}^m\right\}}\right|$$

$$\leq \frac{2\alpha + \sum_{m=1}^{\infty} \{(m-1-\alpha)|a_m| + (m+1+\alpha)|b_m|\} r^{m-1}}{(4-2\alpha) - \sum_{m=1}^{\infty} \{(m+1-\alpha)|a_m| + (m-1+\alpha)|b_m|\} r^{m-1}}.$$

We note that f is starlike of order α in $\mathcal{U}(r)$ if and only if (see also [1], [4], [6])

$$\left|\frac{\mathcal{D}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathcal{D}_{\mathcal{H}} f(z) + (1-\alpha) f(z)}\right| < 1, z \in U(r)$$

or

$$\sum_{m=2}^{\infty} \left(\frac{m-\alpha}{1-\alpha}|a_m| + \frac{m+\alpha}{1-\alpha}|b_m|\right) r^{m-1} \leq 1. \tag{17}$$

Also, we have

$$\sum_{m=2}^{\infty} \left(\frac{\alpha_m}{B-A}|a_m| + \frac{\beta_m}{B-A}|b_m|\right) \leq 1.$$

The condition (17) is true if

$$\frac{m-\alpha}{1-\alpha} r^{m-1} \leq \frac{\alpha_m}{B-A} \text{ and } \frac{m+\alpha}{1-\alpha} r^{m-1} \leq \frac{\beta_m}{B-A} \quad (m = 2, 3, \ldots)$$

or if

$$r \leq \left(\frac{1-\alpha}{B-A} min \left\{\frac{\alpha_m}{m-\alpha}, \frac{\beta_m}{m+\alpha}\right\}\right)^{\frac{1}{m-1}}, (m = 2, 3, \ldots).$$

So, the function f is starlike of order α in the disk $\mathcal{U}(r^*)$ where

$$r^* := \inf_{m \geq 2} \left(\frac{1-\alpha}{B-A} min \left\{\frac{\alpha_m}{m-\alpha}, \frac{\beta_m}{m+\alpha}\right\}\right)^{\frac{1}{m-1}}.$$

□

Similarly, we get:

Theorem 4.2. *Let* $0 \leq \alpha < 1$ *and* α_k *and* β_k *be defined by* (10). *Then*

$$R_\alpha^c \left(\mathcal{W}\mathcal{V}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)\right) = \inf_{m \geq 2} \left(\frac{1-\alpha}{B-A} min \left\{\frac{\alpha_m}{m(m-\alpha)}, \frac{\beta_m}{m(m+\alpha)}\right\}\right)^{\frac{1}{m-1}}.$$

5. Convolution properties

Theorem 5.1. *If* $f, F \in \mathcal{W}\mathcal{V}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)$ *and* $|a_m|, |b_m|, |A_m|, |B_m| \in [0, 1]$ *then* $f * F \in \mathcal{W}\mathcal{V}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)$.

Proof. Let the functions f and g of the forms:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \text{ and } F(z) = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=2}^{\infty} B_m z^m}, \quad (z \in \mathcal{U})$$

belong to the class $\mathcal{WV}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$. Then from Theorem 2.1 we have

$$\sum_{m=2}^{\infty}\left(|a_m|\,\alpha_m+|b_m|\,\beta_m\right)\leq B-A$$

and

$$\sum_{m=2}^{\infty}\left(|A_m|\,\alpha_m+|B_m|\,\beta_m\right)\leq B-A.$$

Thus, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{m=2}^{\infty}\frac{1}{B-A}\sqrt{\left(|a_m|\,\alpha_m+|b_m|\,\beta_m\right)\left(|A_m|\,\alpha_m+|B_m|\,\beta_m\right)}\leq 1.$$

In order to proove that $f*F\in\mathcal{WV}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$ we need to show that

$$\sum_{m=2}^{\infty}\left(|a_mA_m|\,\alpha_m+|b_mB_m|\,\beta_m\right)\leq B-A.$$

But by using again Cauchy-Schwarz inequality we have

$$\left(|a_mA_m|\,\alpha_m+|b_mB_m|\,\beta_m\right)^2\leq\left(|a_m|^2\,\alpha_m+|b_m|^2\,\beta_m\right)\left(|A_m|^2\,\alpha_m+|B_m|^2\,\beta_m\right)\leq$$
$$\leq\left(|a_m|\,\alpha_m+|b_m|\,\beta_m\right)\left(|A_m|\,\alpha_m+|B_m|\,\beta_m\right)$$

or, equivalently

$$|a_mA_m|\,\alpha_m+|b_mB_m|\,\beta_m\leq\sqrt{\left(|a_m|\,\alpha_m+|b_m|\,\beta_m\right)\left(|A_m|\,\alpha_m+|B_m|\,\beta_m\right)}.$$

□

6. Integral properties

Now, we will examine the closure properties of the class $\mathcal{WV}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$ under the generalized Bernardi-Libera -Livingston integral operator $\mathcal{L}_c(f)$, $(c>-1)$ which is defined by $\mathcal{L}_c(f)=\mathcal{L}_c(h)+\overline{\mathcal{L}_c(g)}$ where

$$\mathcal{L}_c(h)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt\quad\text{and}\quad\mathcal{L}_c(g)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt$$

(see [8]).

Theorem 6.1. *Let* $f\in\mathcal{WV}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$. *Then* $\mathcal{L}_c(f)\in\mathcal{WV}_{\mathcal{H}}\left(\phi,\varphi;A^*,B;k\right)$ *where* $A^*=min\left\{A_1^*,A_2^*\right\}>A$,

$$A_1^*=B-\frac{(B-A)(k+1)(1+B)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m+(k+1)(1+B)(c+m)(e_m-c_m)},$$

$$A_2^*=B-\frac{(B-A)(k+1)(1+B)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m+(k+1)(1+B)(c+m)(f_m-d_m)}.$$

Proof. Since $f \in \mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi, \varphi; A, B; k\right)$ we have

$$\frac{\sum_{m=2}^{\infty}\left(|a_m|\,\alpha_m + |b_m|\,\beta_m\right)}{B-A} \leq 1$$

where α_m and β_m are given in Theorem 2.1 by (11) and (12). We know from Theorem 2.1 that $\mathcal{L}_c(f) \in \mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi, \varphi; A^*, B; k\right)$ if and only if

$$\frac{\sum_{m=2}^{\infty}\left(|a_m|\,\alpha'_m \frac{c+1}{c+m} + |b_m|\,\beta'_m \frac{c+1}{c+m}\right)}{B-A^*} \leq 1 \tag{18}$$

where

$$\alpha'_m = (k+1)(1+B)\,e_m + \left[(B-A^*) - (k+1)(1+B)\right] c_m,$$
$$\beta'_m = (k+1)(1+B)\,f_m + \left[(B-A^*) - (k+1)(1+B)\right] d_m.$$

We note that the inequalities

$$\frac{\sum_{m=2}^{\infty}\left(|a_m|\,\alpha'_m \frac{c+1}{c+m} + |b_m|\,\beta'_m \frac{c+1}{c+m}\right)}{B-A^*} \leq \frac{\sum_{m=2}^{\infty}\left(|a_m|\,\alpha_m + |b_m|\,\beta_m\right)}{B-A} \tag{19}$$

imply (18), so it is sufficient to determine A^* such that

$$\frac{\alpha'_m \frac{c+1}{c+m}}{B-A^*} \leq \frac{\alpha_m}{B-A} \tag{20}$$

and

$$\frac{\beta'_m \frac{c+1}{c+m}}{B-A^*} \leq \frac{\beta_m}{B-A} \tag{21}$$

hold true. (20) is equivalent to

$$A_1^* = A^* < B - \frac{(B-A)(k+1)(1+B)(c+1)(e_m - c_m)}{(B-A)(m-1)\,c_m + (k+1)(1+B)(c+m)(e_m - c_m)}. \tag{22}$$

(21) is equivalent to

$$A_2^* = A^* < B - \frac{(B-A)(k+1)(1+B)(c+1)(f_m - d_m)}{(B-A)(m-1)\,d_m + (k+1)(1+B)(c+m)(f_m - d_m)} \tag{23}$$

From (22) and (23) we choose the smaller one.
Let us consider the functions $E_1, E_2 : [2;\infty) \to \mathbb{R}$,

$$E_1(x) = B - \frac{(B-A)(k+1)(1+B)(c+1)(e_m - c_m)}{(B-A)(x-1)\,c_m + (k+1)(1+B)(c+x)(e_m - c_m)}$$

and

$$E_2(x) = B - \frac{(B-A)(k+1)(1+B)(c+1)(f_m - d_m)}{(B-A)(x-1)\,d_m + (k+1)(1+B)(c+x)(f_m - d_m)}$$

then their derivatives are:

$$E_1'(x) = \frac{(B-A)(k+1)(1+B)(c+1)(e_m-c_m)\left[(B-A)c_m+(k+1)(1+B)(e_m-c_m)\right]}{\left[(B-A)(x-1)c_m+(k+1)(1+B)(c+x)(e_m-c_m)\right]^2} > 0$$

and

$$E_2'(x) = \frac{(B-A)(k+1)(1+B)(c+1)(f_m-d_m)\left[(B-A)d_m+(k+1)(1+B)(f_m-d_m)\right]}{\left[(B-A)(x-1)d_m+(k+1)(1+B)(c+x)(f_m-d_m)\right]^2} > 0.$$

$E_1, E_2(x)$ are increasing functions. In our case we need $A_1^* \le E_1(m)$ and $A_2^* \le E_2(m), \forall m \ge 2$ and for this reason we choose

$$A_1^* = E_1(2) = B - \frac{(B-A)(k+1)(1+B)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m+(k+1)(1+B)(c+m)(e_m-c_m)}$$

and

$$A_2^* = E_2(2) = B - \frac{(B-A)(k+1)(1+B)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m+(k+1)(1+B)(c+m)(f_m-d_m)}.$$

We must check $A_1^* > A, A_2^* > A$ that are equivalent to

$$(B-A)(m-1)c_m+(k+1)(1+B)(m-1)(e_m-c_m) > 0$$

and

$$(B-A)(m-1)d_m+(k+1)(1+B)(m-1)(f_m-d_m) > 0$$

which are true. □

Theorem 6.2. *Let* $f \in \mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B;k\right)$. *Then* $\mathcal{L}_c(f) \in \mathscr{W}\mathscr{V}_{\mathcal{H}}\left(\phi,\varphi;A,B^*;k\right)$ *where* $B^* = min\left\{B_1^*, B_2^*\right\} < B$,
$B_1^* = A + \frac{(B-A)(k+1)(1+A)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m+(k+1)(e_m-c_m)\left[(1+B)(c+m)-(c+1)(B-A)\right]}$,
$B_2^* = A + \frac{(B-A)(k+1)(1+A)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m+(k+1)(f_m-d_m)\left[(1+B)(c+m)-(c+1)(B-A)\right]}$.

Proof. The proof is similar to the demonstration for Theorem 6.1. □

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