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On Distributivity Between Aggregation Operators with Annihilator and Mayor's Aggregation Operators

Dragan Jočić[a], Ivana Štajner-Papuga[b]

[a]*Novi Sad School of Business*
[b]*Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad*

Abstract.
The issue of distributivity for different classes of aggregation operators is a topic that is being currently investigated by a number of researchers. The focus of this paper is on characterization of pairs of aggregation operators that are satisfying distributivity law where one of them is a commutative, associative aggregation operator with annihilator and the other one is a Mayor's aggregation operator. The results presented here extend and upgrade some known research, e.g., results concerning distributivity between semi-uninorms and Mayor's aggregation operators.

1. Introduction

Recently, the problem of distributivity for different classes of aggregation operators have been intensively studied by a number of researchers. Aggregation operators are highly interesting research topic due to their applicability in various fields, from mathematics and natural sciences to economics and social sciences (see [7, 9, 12]). Therefore, the characterization of pairs of aggregation operators that are satisfying the distributivity law has captivated a high level of attention. This topic has roots in [1] and can go into two direction. The first one considers distributivity on the whole domain, and results for t-norms and t-conorms can be found in [7], for quasi-arithmetic means in [2, 27], for uninorms and nullnorms in [3, 6, 17, 18, 23, 26, 28], for semi-t-operators and uninorms in [4, 5, 21], for Mayor's aggregation operators in [2, 10, 22], etc. The second direction consists of the problem of distributivity on the restricted domain i.e., it considers the restricted (conditional) distributivity [12–15, 24, 25]. The significance of this topic follows not only from the theoretical point of view, but also because from its applicability in the integration theory [25] and in the utility theory [9, 11, 16].

This paper deals with distributivity equations on the whole domain, involving aggregation operators defined in the sense of G. Mayor [20], and some classes of commutative, associative aggregation operators with annihilator studied in [19]. T. Calvo in [2] considered distributivity between Mayor's aggregation operators and t-norms and t-conorms. Jočić, Štajner-Papuga in [10] focused on semi-nullnorms and semi-uninorms, while, in [22], Qin and Wang focused on semi-t-operators. Since commutative, associative

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Email addresses: dragan.jocic@vps.ns.ac.rs (Dragan Jočić), ivana.stajner-papuga@dmi.uns.ac.rs (Ivana Štajner-Papuga)

aggregation operators with annihilator present a generalization of uninorms and nullnorms (see [19]) the research presented here extends the one presented in [10].

This paper is organized as follows. Some preliminary notions concerning aggregation operators, Mayor's aggregation operators, uninorms, T-uninorms, S-uninorms, bi-uninorms and distributivity equations are given in the Section 2. Results on distributivity between T-uninorms and Mayor's aggregation operators are given in the third section. Distributivity between S-uninorms, bi-uninorms, and Mayor's aggregation operators is investigated in the fourth and the fifth section, respectively. Some concluding remarks are given in the sixth section.

2. Preliminaries

An overview of some concepts that are necessary for the research that follows is given in this section (see [3, 8, 9, 12, 19, 20]).

2.1. Aggregation operators

The starting point of this research is the notion of an aggregation operator in $[0,1]^n$.

Definition 2.1. *([9]) An aggregation operator in* $[0,1]^n$ *is a function* $A^{(n)} : [0,1]^n \to [0,1]$ *that is nondecreasing in each variable and that fulfills the following boundary conditions*

$$A^{(n)}(0,\ldots,0) = 0 \quad \text{and} \quad A^{(n)}(1,\ldots,1) = 1.$$

Since the focus of this paper is on binary aggregation operators, further the simple notation A will be used instead of $A^{(2)}$. The sequel of this section contains an overview of classes of aggregation operators that are essential for the presented research. The focus is on certain commutative and associative operators with a neutral element, then with an annihilator and on Mayor's operators.

2.1.1. Commutative and associative aggregation operators with neutral element

Definition 2.2. *([29]) A uninorm* $U : [0,1]^2 \to [0,1]$ *is a binary aggregation operator that is commutative, associative, and for which there exists a neutral element* $e \in [0,1]$*, i.e.,* $U(x,e) = x$ *for all* $x \in [0,1]$.

For $e = 1$ the uninorm U becomes a t-norm denoted by T, and for $e = 0$, U is a t-conorm denoted by S. For any uninorm we have $U(0,1) \in \{0,1\}$. If $U(0,1) = 0$, the uninorm in question is the conjunctive uninorm, and if $U(0,1) = 1$, it is the disjunctive uninorm. Additionally, if both functions $U(x,0)$ and $U(x,1)$ are continuous (except perhaps at the point e) the following characterized is obtained (see [8]).

Theorem 2.3. *([8]) Let U be a uninorm with a neutral element* $e \in (0,1)$ *such that both functions* $U(x,1)$ *and* $U(x,0)$ *are continuous except at the point* $x = e$.

(i) If $U(0,1) = 0$*, then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{1}$$

where T is a t-norm, and S is a t-conorm.

(ii) If $U(0,1) = 1$*, then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{2}$$

where T is a t-norm, and S is a t-conorm.

A t-norm T from (1) (and (2)) is called the underlying t-norm of U and a t-conorm S is called the underlying t-conorm of U. Generally, the class of all uninorms of the form (1) is denoted by $\mathcal{U}_{\min}$, while the class of all uninorms of the form (2) is denoted by $\mathcal{U}_{\max}$. More on this subject can be found in [8, 9, 29].

Remark 2.4. The first class of uninorms were considered by Yager and Rybalov (see [29]). They introduced the idempotent uninorms U_e^{min} and U_e^{max} of the following form

$$U_e^{min} = \begin{cases} \max & \text{on } [e,1]^2, \\ \min & \text{otherwise,} \end{cases} \tag{3}$$

and

$$U_e^{max} = \begin{cases} \min & \text{on } [0,e]^2, \\ \max & \text{otherwise.} \end{cases} \tag{4}$$

Uninorms given by (3) and (4) are the only idempotent uninorms from classes $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$. Consequently, the only idempotent t-norm and t-conorm are operators minimum and maximum, respectively.

2.1.2. Commutative and associative aggregation operators with annihilator

Another class of aggregation operators that is necessary for the presented research is a subclass of aggregation operator with an annihilator (absorbing element). Generally, an element $a \in [0,1]$ is an annihilator for aggregation operator A if $A(a,x) = A(x,a) = a$ for all $x \in [0,1]$. The class of commutative aggregation operators with an annihilator a, generally known as a-CAOA, was studied in [19] and of the special interest for this research is its associative case. For an arbitrary binary operator $A : [0,1]^2 \to [0,1]$, and a fixed element $c \in [0,1]$, by A_c is denoted the section $A_c : [0,1] \to [0,1]$ given by $A_c(x) = A(c,x)$. As it can be seen from the following results from [19], the continuity of A_0 and A_1 plays a crucial role in classification of associative a-CAOA.

Definition 2.5. *([19]) Let $A : [0,1]^2 \to [0,1]$ be an associative a-CAOA.*

- *A is called a S-uninorm if:*
  - *A_0 is continuous and A_1 is not;*
  - *there exists $e \in (0,1)$ such that e is idempotent, A_e is continuous and $A_e(1) = 1$.*
- *A is called a T-uninorm if:*
  - *A_1 is continuous and A_0 is not;*
  - *there exists $e \in (0,1)$ such that e is idempotent, A_e is continuous and $A_e(0) = 0$.*
- *A is called a bi-uninorm if:*
  - *A_0 and A_1 are not continuous;*
  - *there exist idempotent elements $e_0, e_1 \in (0,1)$ such that A_{e_0} and A_{e_1} are continuous and $A_{e_0}(0) = 0$ and $A_{e_1}(1) = 1$.*
- *A is called a nullnorm if:*
  - *A_0 and A_1 are continuous.*

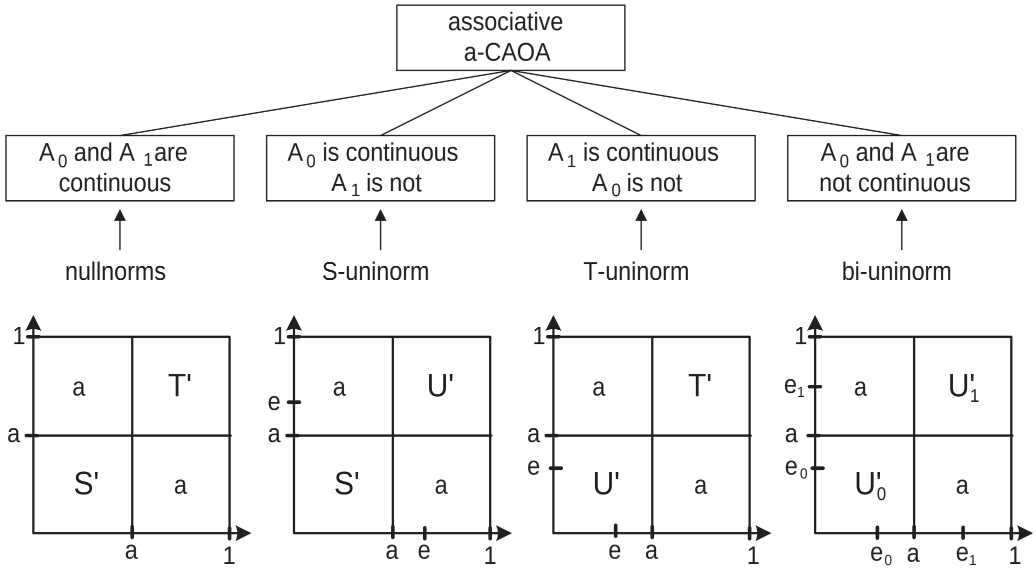

Figure 1

Remark 2.6. *Distributivity between semi-nullnorm (or nullnorm) and Mayor's operators was investigated in [10], therefore that type of associative a-CAOA will not be considered further in this paper.*

As it can be seen from the following overview of results from [19], the form of associative a-CAOA is closely related to uninorms, t-norms and t-conorms.

Theorem 2.7. *([19]) Let $A : [0,1]^2 \to [0,1]$ be a binary operator. The following statements are equivalent:*

(i) A is a S-uninorm.

(ii) There exists $a \in [0,1)$, a t-conorm S' and a conjunctive uninorm U' with neutral element $e' \in (0,1)$ such that A is given by

$$A(x,y) = \begin{cases} aS'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)U'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a]\times[a,1] \cup [a,1]\times[0,a] \end{cases} \tag{5}$$

(iii) There exists $a \in [0,1)$, a t-conorm S and a conjunctive uninorm U with neutral element $e \in (0,1)$ such that $U(x,a) \le a$ for all $x \in [0,1]$, $U \le S$ and $A = med(a, U, S)$.

Remark 2.8. *Let $A : [0,1]^2 \to [0,1]$ be a S-uninorm.*

- *For $a = 0$ the observed S-uninorm becomes a conjunctive uninormi.e., $A = U'$.*
- *$a \ne 1$ in order to ensure discontinuity of A_1, and since $A_e(1) = 1$, there holds $a < e$.*
- *If $U' \in \mathcal{U}_{\min}$, then the accepted convention is that A is a S-uninorm in $\mathcal{U}_{\min}$.*

Applying the previous theorem on some well-known operators, the following example of S-uninorm can be constructed.

Example 2.9. *Binary Operator $A : [0,1]^2 \to [0,1]$ given by*

$$A(x,y) = \begin{cases} a & \text{if } (x,y) \in [0,a]\times[a,1] \cup [a,1]\times[0,a], \\ \max(x,y) & \text{if } (x,y) \in [0,a]^2 \cup [e,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{6}$$

is an idempotent S-uninorm in $U_{\min}$ with annihilator a, obtained by (5) for $S' = \max$ and $U' = U_e^{\min}$.

Theorem 2.10. *([19]) Let $A : [0,1]^2 \to [0,1]$ be a binary operator. The following statements are equivalent:*

(i) A is a T-uninorm.

(ii) There exists $a \in (0,1]$, a t-norm T' and a disjunctive uninorm U' with neutral element $e' \in (0,1)$ such that A is given by

$$A(x,y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)T'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \end{cases} \tag{7}$$

(iii) There exists $a \in (0,1]$, a t-norm T and a disjunctive uninorm U with neutral element $e \in (0,1)$ such that $U(x,a) \geq a$ for all $x \in [0,1]$, $T \leq U$ and $A = med(a, T, U)$.

Remark 2.11. *Let $A : [0,1]^2 \to [0,1]$ be a T-uninorm.*

- *For $a = 1$ the observed T-uninorm becomes a disjunctive uninormi.e., $A = U'$.*
- *$a \neq 0$ in order to ensure the discontinuity of A_0, and since $A_e(0) = 0$, there holds $e < a$.*
- *If $U' \in U_{\max}$, then the accepted convention is that A is a T-uninorm in $U_{\max}$.*

Again, by applying the previous theorem on some well-known operators, an interesting example of T-uninorm can be constructed.

Example 2.12. *Binary Operator $A : [0,1]^2 \to [0,1]$ given by*

$$A(x,y) = \begin{cases} a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a], \\ \min(x,y) & \text{if } (x,y) \in [0,e]^2 \cup [a,1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{8}$$

is an idempotent T-uninorm in $U_{\max}$ with annihilator a, obtained by (7) for $T' = \min$ and $U' = U_e^{\max}$.

Theorem 2.13. *([19]) Let $A : [0,1]^2 \to [0,1]$ be a binary operator. The following statements are equivalent:*

(i) A is a bi-uninorm.

(ii) There exists $a \in (0,1)$, a disjunctive uninorm U_0' and a conjunctive uninorm U_1' with neutral elements $e_0', e_1' \in (0,1)$, respectively, such that A is given by

$$A(x,y) = \begin{cases} aU_0'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)U_1'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \end{cases} \tag{9}$$

(iii) There exists $a \in (0,1)$, a disjunctive uninorm U_0 and conjunctive uninorm U_1 with neutral elements $e_0, e_1 \in (0,1)$, respectively, such that $U_1(x,a) \leq a \leq U_0(x,a)$ for all $x \in [0,1]$, $U_1 \leq U_0$ and $A = med(a, U_1, U_0)$.

Remark 2.14. *Let $A : [0,1]^2 \to [0,1]$ be a bi-uninorm.*

- *$0 < a < 1$ in order to ensure discontinuity of A_0 and A_1, and since $A_{e_0}(0) = 0$ and $A_{e_1}(1) = 1$, there holds $e_0 < a < e_1$.*
- *If $U_0' \in U_{\max}$ and $U_1' \in U_{\min}$, then the accepted convention is that A is a bi-uninorm in $U_{\max} \cup U_{\min}$.*

2.1.3. Mayor's aggregation operators

Aggregation operators introduced by G. Mayor in [20] that, for the sake of simplicity, will be refereed to as *the GM aggregation operators* are given by the following definition.

Definition 2.15. *([20]) A GM aggregation operator* $F : [0,1]^2 \to [0,1]$ *is a commutative binary aggregation operator that satisfy the following boundary conditions for all* $x \in [0,1]$ *:*

$$F(x,0) = F(0,1)x \quad \text{and} \quad F(x,1) = (1 - F(0,1))x + F(0,1).$$

The following properties of the GM aggregation operators are essential for the further characterizations.

Theorem 2.16. *([20]) Let* $F : [0,1]^2 \to [0,1]$ *be a GM aggregation operator. Then the follwing holds:*

(i) F is associative if and only if F is a t-norm or t-conorm;

(ii) $F = \min$ *or* $F = \max$ *if and only if* $F(0,1) = 0$ *or* $F(0,1) = 1$ *and* $F(x,x) = x$ *for all* $x \in [0,1]$*;*

(iii) F is idempotent if and only if $\min \le F \le \max$ *.*

2.2. Distributivity equations

Finally, functional equations that are called left and right distributivity laws ([1], p. 318) are given by the following definition.

Definition 2.17. *Let* $F, G : [0,1]^2 \to [0,1]$ *be two operators. F is distributive over G, if the following two laws hold:*

(LD) F is left distributive over G i.e.,

$$F(x, G(y,z)) = G(F(x,y), F(x,z)) \quad \text{for all } x,y,z \in [0,1]$$

and

(RD) F is right distributive over G i.e.,

$$F(G(y,z), x) = G(F(y,x), F(z,x)) \quad \text{for all } x,y,z \in [0,1]$$

Of course, for a commutative F, laws (LD) and (RD) coincide.

The following two lemmas provide some additional information on distributivity law that are needed for the further research.

Lemma 2.18. *([3]) Let* $X \neq \emptyset$ *and* $F : X^2 \to X$ *have neutral element e in a subset* $Y \subset X$ *(i.e.* $\forall_{x \in Y} F(e,x) = F(x,e) = x$*). If operator F is left or right distributive over operator* $G : X^2 \to X$ *fulfilling* $G(e,e) = e$*, then G is idempotent in Y.*

Lemma 2.19. *([3]) Every increasing function* $F : [0,1]^2 \to [0,1]$ *is distributive over* $\max$ *and* $\min$ *.*

3. Distributivity between GM aggregation operators and T-uninorms

Let F be a GM aggregation operator such that $k = F(0,1)$, and let G be a T-uninorm with annihilator $0 < a \le 1$ in $U_{\max}$. Now, two problems can be distinguished: distributivity of F over G and distributivity of G over F.

3.1. Distributivity of F over G

Through this section the distributivity of GM aggregation operator over T-uninorm is being considered. Here two subcases can be distinguished $a = 1$ and $a < 1$.

3.1.1. Case $a = 1$

Let $a = 1$. In this case operator G is a uninorm U and the obtained result extends the Theorem 23 from [10].

Theorem 3.1. *Let F be a GM aggregation operator and let G be a uninorm from the class $\mathcal{U}_{\max}$. F is distributive over U if and only if U is an idempotent uninorm, i.e., $U = U_e^{max}$ and F is given by*

$$F = \begin{cases} A & \text{on } [0,e]^2, \\ B & \text{on } [e,1]^2, \\ \max & \text{otherwise}, \end{cases} \tag{10}$$

where $A : [0,e]^2 \to [0,e]$ is a commutative aggregation operator with neutral element 0 and $B : [e,1]^2 \to [e,1]$ is a commutative aggregation operator with neutral element e.

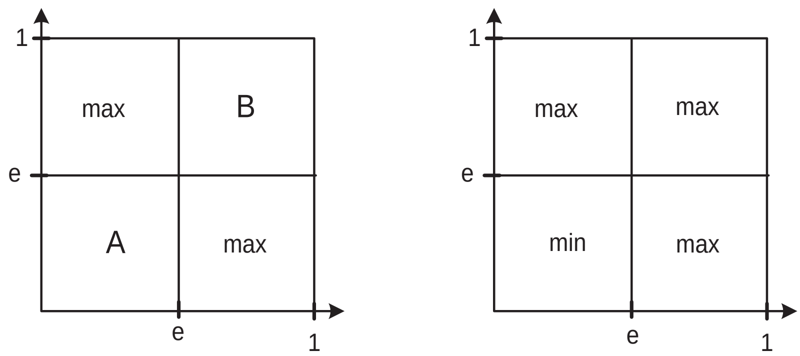

Figure 2. Distributive pair of operators from Theorem 3.1.

Proof. ($\Rightarrow$) As in the Theorem 23 from [10] there can prove that k has to be from $\{0,1\}$ and that $U = U_e^{max}$. Now, the next step is to show that k can not be 0. Let us assume the opposite, i.e., that $k = 0$. From definition of GM aggregation operator follow that $F(x,1) = x$ and $F(x,0) = 0$ for all $x \in [0,1]$. For $0 < x < e,\ y = 0,\ z = 1$ assumed distributivity gives the following contradiction:

$$x = F(x,1) = F(x,U(0,1)) = U(F(x,0),F(x,1)) = U(0,x) = 0.$$

Therefore, $k = 1$, i.e., $F(x,0) = x$ and $F(x,1) = 1$ for all $x \in [0,1]$.

Next, there should be shown that $F(e,e) = e$: for $x = z = e,\ y = 0$ distributivity law insures

$$e = F(e,0) = F(e,U(0,e)) = U(F(e,0),F(e,e)) = U(e,F(e,e)) = F(e,e).$$

Now, for $x \le e$, there holds $e = F(e,0) \le F(x,e) \le F(e,e) = e$, i.e., $F(x,e) = e$ for $x \le e$. On the other hand, for $x \ge e$ the following holds:

$$x = F(x,0) = F(x,U(0,e)) = U(F(x,0),F(x,e)) = U(x,F(x,e)).$$

Since $F(x,e) \ge F(x,0) = x \ge e$, we obtain that

$$x = U(x,F(x,e)) = \max(x,F(x,e)) = F(x,e).$$

Consequently

$$F(x,e) = \begin{cases} e & \text{for } x \le e, \\ x & \text{for } x \ge e. \end{cases} \tag{11}$$

Thus, the restrictions $A = F\mid_{[0,e]^2}$ and $B = F\mid_{[e,1]^2}$ are aggregation operators with the desired properties. It is easy to show that on the remaining part of the unit square holds $F = \max$.
($\Leftarrow$) Conversely, let F be given by (10) and let $U = U_e^{max}$. On the squares $[0,e]^2$ and $[e,1]^2$ distributivity follows from Lemma 2.19. Otherwise, $U(y,z) = z$ for $y < e < z$. Let $L = F(x, U(y,z)) = F(x,z)$ and $R = U(F(x,y), F(x,z))$. Now, the following holds:

- if $x \leq e$, then $L = \max(x,z) = z$ and, since $F(x,y) \leq e < z$, there holds $R = U(F(x,y), F(x,z)) = \max(F(x,y), z) = z$;
- if $x \geq e$, then $L = F(x,z)$ and, since $F(x,z) \geq x \geq e$ and $F(x,y) = \max(x,y) = x$, there holds $R = U(F(x,y), F(x,z)) = \max(x, F(x,z)) = F(x,z)$.

For all considered cases we obtain $L = R$, that is the distributivity law holds. □

Remark 3.2. a) The previous result also holds if commutativity and associativity are left out from the definition of uninorm, i.e., a semi-uninorm from the class $N_e^{\max}$ is used (see [3]).

b) The following result concerning a semi-uninorm from the class $N_e^{\max}$ has been proved in [10](Theorem 23 [10]):

Let F be a GM aggregation operator such that $F(0,1) = k$ and let $G \in N_e^{\max}$. If F is distributive over G, then $k \in \{0,1\}$ and $G = U_e^{max}$.

Therefore, Theorem 3.1 extends and upgrades the result from [10], because it shows that k can not be 0 and it gives both necessary and sufficient condition.

3.1.2. Case $a < 1$

Now, the case $a < 1$ is being considered.

Theorem 3.3. *Let G be a T-uninorm with annihilator $a < 1$ in $U_{\max}$, and let F be a GM aggregation operator. F is distributive over G if and only if G is an idempotent T-uninorm given by (8), and F is given by*

$$F = \begin{cases} A_1 & \text{on } [0,e]^2, \\ A_2 & \text{on } [e,a]^2, \\ B & \text{on } [a,1]^2, \\ \max & \text{otherwise,} \end{cases} \tag{12}$$

where $A_1 : [0,e]^2 \to [0,e]$ is a commutative aggregation operator with neutral element 0, $A_2 : [e,a]^2 \to [e,a]$ is a commutative aggregation operator with neutral element e, and $B : [a,1]^2 \to [a,1]$ is a commutative aggregation operator with neutral element a.

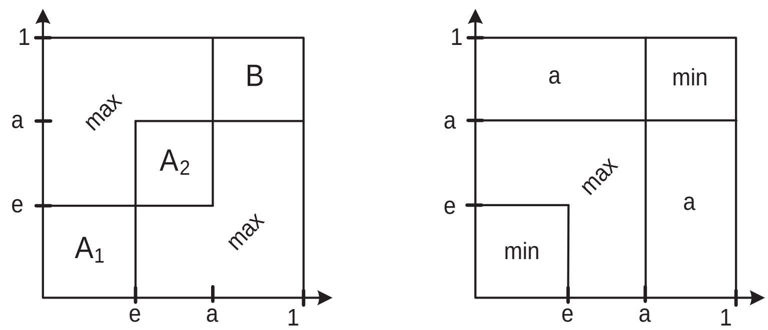

Figure 3. Distributive pair of operators from Theorem 3.3.

Proof. (⇒) If $x = 1,\ y = 0,\ z = 1$, the (LD) condition provides

$$F(1, a) = F(1, G(0, 1)) = G(F(1, 0), F(1, 1)) = G(k, 1).$$

Now, as in Theorem 3.4, there can be proved that $k \in \{0, 1\}$. Also, it is easy to show that $G(x, x) = x$ for all $x \in [0, 1]$:

- for $k = 0$, it holds $x = F(x, 1) = F(x, G(1, 1)) = G(F(x, 1), F(x, 1)) = G(x, x)$;
- for $k = 1$, it holds $x = F(x, 0) = F(x, G(0, 0)) = G(F(x, 0), F(x, 0)) = G(x, x)$.

Now, there should be shown that k can not be 0.
Let suppose the opposite, i.e., that $k = 0$. Now, since

$$F(x, a) = F(x, G(0, 1)) = G(F(x, 0), F(x, 1)) = G(0, x)$$

the following is obtained:

- for $x \geq a$, since $G(0, x) = a$, there holds $F(x, a) = a$;
- for $e < x \leq a$, since $G(0, x) = x$, there holds $F(x, a) = x$;
- for $x \leq e$, since $G(0, x) = 0$, there holds $F(x, a) = 0$ and thus $F(e, a) = 0$.

Let $z \leq e < y, x \leq a$. The (LD) condition insures

$$F(x, G(y, z)) = F(x, \max(y, z)) = F(x, y) = G(F(x, y), F(x, z)).$$

Since $F(x, z) \leq F(a, e) = 0$, the (LD) condition has the form

$$F(x, y) = G(F(x, y), 0) \text{ for all } (x, y) \in (e, a]^2.$$

Additionally for $(x, y) \in (e, a]^2$ holds $0 = F(e, e) \leq F(x, y) \leq F(a, a) = a$ and there can be concluded that restriction $F\ |_{(e,a]^2} = A_3 : (e, a]^2 \to [0, a]$ is a commutative, increasing operator with neutral element a. An example of the operator A_3 can be given by the following

$$A_3(x, y) = \begin{cases} e & \text{if } \max(x, y) < a, \\ \min(x, y) & \text{if } \max(x, y) = a. \end{cases} \tag{13}$$

Now, if $(x, y) \in (e, a)^2$, the (LD) condition yields $e = G(e, 0) = 0$ which is a contradiction.
Therefore $k = 1$ and, using the similar arguments as previous, the following can be shown

$$F(x, a) = \begin{cases} x & \text{for } x \geq a, \\ a & \text{for } x \leq a. \end{cases} \tag{14}$$

Thus $A = F\ |_{[0,a]^2}$ is a commutative aggregation operator with neutral element 0, $B = F\ |_{[a,1]^2}$ is a commutative aggregation operator with neutral element a and F is given by

$$F = \begin{cases} A & \text{on } [0, a]^2, \\ B & \text{on } [a, 1]^2, \\ \max & \text{otherwise.} \end{cases} \tag{15}$$

By applying Theorem 3.1 to the square $[0, a]^2$, the following form of A is obtained

$$A = \begin{cases} A_1 & \text{on } [0,e]^2, \\ A_2 & \text{on } [e,a]^2, \\ \max & \text{otherwise}, \end{cases} \tag{16}$$

where $A_1 : [0,e]^2 \to [0,e]$ is a commutative aggregation operator with neutral element 0, $A_2 : [e,a]^2 \to [e,a]$ is a commutative aggregation operator with neutral element e.
(⇐) Conversely, let F be given by (12) and let G be idempotent T-uninorm given by (8). On the square $[0,a]^2$ distributivity can be proved as in Theorem 3.1, and on the square $[a,1]^2$ distributivity holds from Lemma 2.19. Otherwise $G(y,z) = a$ for $y < a < z$, and $L = F(x, G(y,z)) = F(x,a)$ is given by (14). On the other hand, for the right side of the distributivity law $R = G(F(x,y), F(x,z))$ holds:

- if $x \leq a$, then $L = a$ and, since $F(x,y) \leq F(a,y) = a < z = F(x,z)$, there is $R = G(F(x,y), F(x,z)) = a$,
- if $x \geq a$, then $L = x$ and, since $F(x,z) \geq F(x,a) = x \geq a$, there is $R = G(\max(x,y), F(x,z)) = \min(x, F(x,z)) = x$.

In all considered cases the equality $L = R$ is obtained, which proves that distributivity law holds. □

3.2. Distributivity of G over F

The problem of distributivity of G over F is being addressed in this section. Since for $a = 1$ operator G is a uninorm from the class $U_{\max}$, which is the case investigated in [10], the working assumption is that $a < 1$. Therefore, the following results are "the next step" regarding the research from [10].

Theorem 3.4. *Let F be a GM aggregation operator such that the function $f(x) = F(x,x)$ is a right continuous at the point $x = e$, and let G be a T-uninorm with annihilator $a < 1$ in $U_{\max}$. G is distributive over F if and only if $F = \min$ or $F = \max$.*

Proof. (⇒) First, let us prove that $k \in \{0,1\}$. Since the (LD) condition insures

$$G(1,k) = G(1, F(0,1)) = F(G(1,0), G(1,1)) = F(a,1) = (1-k)\cdot a + k,$$

we have the following:

- if $k = a$, and since $G(1,a) = a$, there follows $(1-a)a = 0$, i.e., $a = 0$ or $a = 1$ which is a contradiction;
- if $k < a$, then $G(1,k) = a$, therefore $k(1-a) = 0$, i.e., $k = 0$;
- if $k > a$, then $G(1,k) = k$, therefore $(1-k)a = 0$, i.e., $k = 1$.

That is, k has to be either 0 or 1.
The next step is to show that F is indeed an idempotent operator, i.e., that $F(x,x) = x$ for all $x \in [0,1]$:

- if $x \geq a$, then
$$x = G(x,1) = G(x, F(1,1)) = F(G(x,1), G(x,1)) = F(x,x),$$
- if $e < x \leq a$, then
$$x = G(x,0) = G(x, F(0,0)) = F(G(x,0), G(x,0)) = F(x,x).$$

Therefore, $F(x,x) = x$ for all $x \in (e,1]$ and by right continuity of the function f at the point $x = e$ also holds $F(e,e) = e$. Now, by Lemma 2.18, F is an idempotent operator in $Y = [0,a]$. Consequently F is idempotent, and, since $k \in \{0,1\}$, from Theorem 2.16 follows that $F = min$ or $F = \max$.
(⇐) Follows from Lemma 2.19. □

If the assumption of right continuity at $x = e$ for function $f(x) = F(x,x)$ is omitted from the previous theorem, only the necessary condition remains, i.e., the following result holds.

Theorem 3.5. *Let F be a GM aggregation operator and let G be a T-uninorm with annihilator $a < 1$ in $U_{\max}$. If G is distributive over F, then $k \in \{0,1\}$ and $F(x,x) = x$ for all $x > e$.*

4. Distributivity between GM aggregation operators and *S*-uninorms

The logical next step is investigation distributivity between S-uninorms and GM aggregation operators. Therefore, in this section F is a GM aggregation operator and G is a S-uninorm with annihilator $0 \leq a < 1$ in $\mathcal{U}_{\min}$. Again, two cases can be distinguished: distributivity of F over G and distributivity of G over F.

4.1. *Distributivity of F over G*

Now, the distributivity of GM aggregation operator over S-uninorm is considered. Again, two cases can be distinguished: $a = 0$ and $a > 0$.

4.1.1. *Case* $a = 0$

Let $a = 0$. Now, operator G is a uninorm U and the obtained result extends the Theorem 25 from [10].

Theorem 4.1. *Let F be a GM aggregation operator and let U be a uninorm from the class $\mathcal{U}_{\min}$. F is distributive over U if and only if U is idempotent, i.e., $U = U_e^{min}$ and F is given by*

$$F = \begin{cases} A & on \ [0, e]^2, \\ B & on \ [e, 1]^2, \\ \min & otherwise, \end{cases} \tag{17}$$

where $A : [0, e]^2 \to [0, e]$ is a commutative aggregation operator with neutral element e and $B : [e, 1]^2 \to [e, 1]$ is a commutative aggregation operator with neutral element 1.

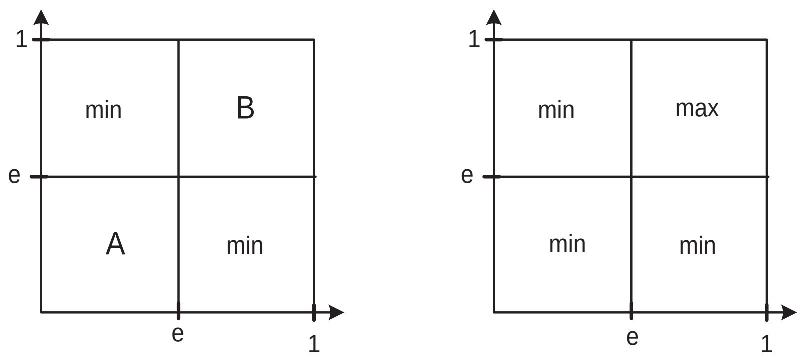

Figure 4. Distributive pair of operators from Theorem 4.1.

Proof. (⇒) Similar to the previous theorems, there can be proved that k takes value from $\{0, 1\}$ and that U is an idempotent uninorm from the class $\mathcal{U}_{\min}$. Now, there has to shown that k can not be 1. Let assume the opposite, i.e., that $k = 1$. For $e < x < 1, \ y = 0, \ z = 1$ distributivity gives a contradiction

$$x = F(x, 0) = F(x, U(0, 1)) = U(F(x, 0), F(x, 1)) = U(x, 1) = 1.$$

Therefore $k = 0$ and, similar to Theorem 3.1, the following can be proved:

$$F(x, e) = \begin{cases} x & \text{for } x \leq e, \\ e & \text{for } x \geq e. \end{cases} \tag{18}$$

Thus, the restrictions $A = F\,|_{[0,e]^2}$ and $B = F\,|_{[e,1]^2}$ are aggregation operators with the desired properties. It is easy to show that on the remaining part of the unit square holds $F = \min$.

(⇐) This direction can be proved as in Theorem 3.1. □

Remark 4.2. Theorem 4.1 also holds when instead of a uninorm $U \in \mathcal{U}_{\min}$ a semi-uninorm from the class $\mathcal{N}_e^{\min}$ is used. The following result has been proved in [10] (see Theorem 25):

> *Let F be a GM aggregation operator such that $F(0,1) = k$ and let $G \in \mathcal{N}_e^{\min}$. If F is distributive over G, then $k \in \{0,1\}$ and $G = U_e^{min}$.*

Therefore, Theorem 4.1 upgrades the result from [10], because it shows that k can not be 1 and it gives both necessary and sufficient conditions.

4.1.2. Case $a > 0$

Theorem 4.3. *Let G be a S-uninorm with an annihilator $a > 0$ in $\mathcal{U}_{\min}$ and let F be a GM aggregation operator. F is distributive over G if and only if G is the idempotent S-uninorm given by (6), and F is given by*

$$F = \begin{cases} A & on\ [0,a]^2, \\ B_1 & on\ [a,e]^2, \\ B_2 & on\ [e,1]^2, \\ \min & otherwise, \end{cases} \tag{19}$$

where $A : [0,a]^2 \to [0,a]$ is a commutative aggregation operator with neutral element a, $B_1 : [a,e]^2 \to [a,e]$ is a commutative aggregation operator with neutral element e, and $B_2 : [e,1]^2 \to [e,1]$ is a commutative aggregation operator with neutral element 1.

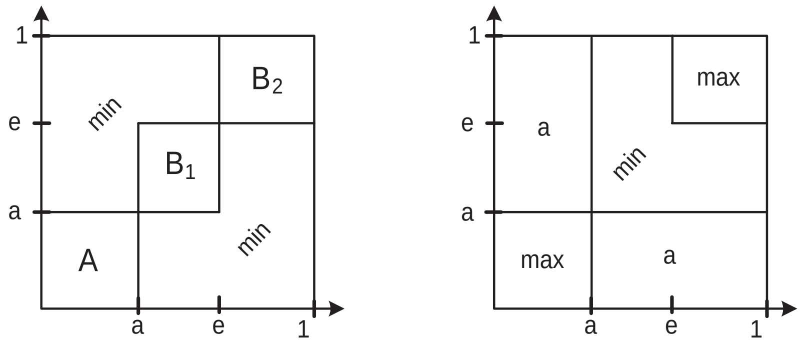

Figure 5. Distributive pair of operators from Theorem 4.3.

Proof. (⇒) As in previous theorems there can be proved that $k \in \{0,1\}$ and that G is an idempotent operator given by (6). Again, the next step is to show that k can not be 1. If the opposite is supposed, i.e., if $k = 1$, the following is obtained

$$F(x,a) = F(x,G(0,1)) = G(F(x,0),F(x,1)) = G(x,1),$$

and

- for $x \leq a$, since $G(x,1) = a$, holds $F(x,a) = a$;
- for $a \leq x < e$, since $G(1,x) = x$, holds $F(x,a) = x$;
- for $x \geq e$, since $G(1,x) = 1$, holds $F(x,a) = 1$ and thus $F(e,a) = 1$.

Let $a \leq x, y < e \leq z$. The (LD) condition insures

$$F(x,G(y,z)) = F(x,\min(y,z)) = F(x,y) = G(F(x,y),F(x,z)).$$

Since $F(x,z) \geq F(a,e) = 1$ the (LD) condition has the form

$$F(x,y) = G(F(x,y),1) \text{ for all } (x,y) \in [a,e)^2.$$

Additionally, for $(x, y) \in [a, e)^2$, $1 = F(e, e) \geq F(x, y) \geq F(a, a) = a$ and the conclusion is that restriction $F|_{[a,e)^2} = B_3 : [a, e)^2 \to [a, 1]$ is a commutative, increasing operator with neutral element a. An example of the operator B_3 is given by

$$B_3(x, y) = \begin{cases} e & \text{if } \min(x, y) > a, \\ \max(x, y) & \text{if } \min(x, y) = a. \end{cases} \tag{20}$$

Now, if $(x, y) \in (a, e)^2$, the (LD) condition insures $e = G(e, 1) = 1$ which is a contradiction. Therefore $k = 0$.

Now, using similar arguments as in Theorem 3.3, there can prove that F is given by (19).

($\Leftarrow$) The other direction can be proved as in Theorem 3.3. □

4.2. Distributivity of G over F

The focus of this section is on distributivity of a S-uninorm over a GM aggregation operator. Since the case when $a = 0$, i.e., when G is a uninorm from the class $U_{\min}$ is showed in [10], now the assumption is that $a > 0$.

Theorem 4.4. *Let F be a GM aggregation operator such that the function $f(x) = F(x, x)$ is left continuous at the point $x = e$, and let G be a S-uninorm with an annihilator $a > 0$ in $U_{\min}$. G is distributive over F if and only if $F = \min$ or $F = \max$.*

Proof. ($\Rightarrow$) Similarly as in Theorem 3.4 thee an be proved that k takes value in $\{0, 1\}$. The next step is to show that F is an idempotent operator:

- if $x \leq a$, then
$$x = G(x, 0) = G(x, F(0, 0)) = F(G(x, 0), G(x, 0)) = F(x, x);$$
- if $a \leq x < e$, then
$$x = G(x, 1) = G(x, F(1, 1)) = F(G(x, 1), G(x, 1)) = F(x, x).$$

Therefore, $F(x, x) = x$ for all $x \in [0, e)$ and from left continuity of the function f at the point $x = e$ follows $F(e, e) = e$. Again, as in Theorem 3.4, the conclusion is that $F = \min$ or $F = \max$.

($\Leftarrow$) Conversely, distributivity law holds from Lemma 2.19. □

If the assumption that function $f(x) = F(x, x)$ is a left continuous at the point $x = e$ is omitted from the previous theorem, only the necessary condition is obtained, i.e., the following result holds.

Theorem 4.5. *Let F be a GM aggregation operator and let G be a S-uninorm with an annihilator $a > 0$ in $U_{\min}$. If G is distributive over F, then $k \in \{0, 1\}$ and $F(x, x) = x$ for all $x < e$.*

5. Distributivity between GM aggregation operators and bi-uninorms

Finally, the distributivity between bi-uninorms and GM aggregation operators is considered. In this section F is a GM aggregation operator and G is a bi-uninorm with annihilator $0 < a < 1$ in $U_{\min} \cup U_{\max}$, such that e_0 is neutral element of the disjunctive uninorm U_0, and e_1 is neutral element of the conjunctive uninorm U_1.

5.1. Distributivity of F over G

Concerning distributivity of GM-aggregation operators over bi-uninorms by taking into account Theorem 3.3 and Theorem 4.3, the following negative result emerges.

Theorem 5.1. *Let G be a bi-uninorm in $U_{\min} \cup U_{\max}$ with annihilator $0 < a < 1$. There is no GM aggregation operator F distributive over G.*

Proof. Let us suppose that there exists a GM aggregation operator F distributive over G. As in previous theorems there can be proved that $k \in \{0, 1\}$ and that G is an idempotent operator. Also, as in the Theorem 3.3 can be shown that $k \neq 0$, and as in the Theorem 4.3 that $k \neq 1$. Consequently, there is no GM aggregation operator F distributive over bi-uninorm G from the class $U_{\min} \cup U_{\max}$. □

5.2. Distributivity of G over F

Theorem 5.2. *Let F be a GM aggregation operator such that the function $f(x) = F(x, x)$ is left continuous at the point $x = e_1$, and right continuous at the point $x = e_0$, and let G be a bi-uninorm. G is distributive over F if and only if $F = \min$ or $F = \max$.*

Proof. (⇒) Similar to Theorem 3.4, there can be proved that k is from $\{0, 1\}$. The next step is to show that F is an idempotent operator.

- If $e_0 < x \le a$, then
$$x = G(x, 0) = G(x, F(0, 0)) = F(G(x, 0), G(x, 0)) = F(x, x).$$
- If $a \le x < e_1$, then
$$x = G(x, 1) = G(x, F(1, 1)) = F(G(x, 1), G(x, 1)) = F(x, x).$$

Since the function f is left continuous at the point $x = e_1$ there can be obtained that $F(e_1, e_1) = e_1$, and by Lemma 2.18, $F(x, x) = x$ for all $x \in [a, 1]$. Analogously, since the function f is right continuous at the point $x = e_0$, there can be obtained that $F(e_0, e_0) = e_0$, and that $F(x, x) = x$ for all $x \in [0, a]$. Therefore, F is an idempotent operator and, according to Theorem 2.16, $F = \min$ or $F = \max$.
(⇐) The opposite direction follows from from Lemma 2.19. □

Again, the assumption that function $f(x) = F(x, x)$ is a left continuous at the point $x = e_1$ and right continuous at the point $x = e_0$ is can be omitted from the previous theorem. However, it that case, only the necessary condition is obtained.

Theorem 5.3. *Let F be a GM aggregation operator and let G be a bi-uninorm. If G is distributive over F, then $k \in \{0, 1\}$ and $F(x, x) = x$ for all $e_0 < x < e_1$.*

6. Conclusion

Distributivity law on the whole domain between GM aggregation operators and associative a-CAOA, when the underlying uninorms are from the classes $U_{\min}$ and $U_{\max}$, is considered through this paper. As it can be seen from this paper, distributivity law considerably simplifies the structure of inner operator since it is being reduced to an idempotent operator. Results from the third and fourth section of this paper complete and upgrade the corresponding ones from [10]. In the forthcoming work the focus will be on distributivity law when the underlying uninorms of associative a-CAOA are from some other classes of uninorms. Also, since restricted setting turns out to be useful for modelling behavior of some decision makers [11], the further research will also be focused on the conditional distributivity for associative a-CAOA and possible application of obtained structures to the utility theory.

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