Filomat 32:4 (2018), 1221–1231
https://doi.org/10.2298/FIL1804221M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the (b,c)–Inverse in Rings

Dijana Mosić[a], Honglin Zou[b], Jianlong Chen[c]

[a]Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
[b]School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China
[c]School of Mathematics, Southeast University, Nanjing 210096, China

Abstract. We present new characterizations for the existence of the (b,c)–inverse in a ring. The set of all (b,c)–invertible elements is described too. Necessary and sufficient conditions which ensure that the (b,c)–inverse of a given element commutes with that element are investigated. As an application of these results, we obtain new characterizations for the existence of the image-kernel (p,q)–inverse.

1. Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. The sets of all idempotents and invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\bullet$ and $\mathcal{R}^{-1}$, respectively.

An element $a \in \mathcal{R}$ is called regular if there exists $x \in \mathcal{R}$ satisfying $axa = a$. In this case, x is an inner inverse of a. The set of all inner inverses of a will be denoted by $a\{1\}$.

Let $p, q \in \mathcal{R}^\bullet$, $p \neq q$. Then $p\mathcal{R}p$ is a ring with the unit p and we can talk about invertibility of its elements. Since $p\mathcal{R}q$ does not have a unit, we will talk about invertibility of its elements only in the following sense: let $p, q \in \mathcal{R}^\bullet$, an element $a \in \mathcal{R}$ is $(-,p,q)$–invertible if there exists $a' \in q\mathcal{R}p$ such that

$$a \in p\mathcal{R}q, \quad aa' = p \quad \text{and} \quad a'a = q.$$

If the $(-,p,q)$–inverse a' of a exists, it is unique and denoted by $a^{-(p,q)}$. By $\mathcal{R}^{-(p,q)}$ will be denoted the set of all $(-,p,q)$–invertible elements of $\mathcal{R}$.

Lemma 1.1. *Let $a \in \mathcal{R}$. There exist $p, q \in \mathcal{R}^\bullet$ such that a is $(-,p,q)$–invertible if and only if a is regular.*

For $a \in \mathcal{R}$, if $xax = x$ holds for some $x \in \mathcal{R}\backslash\{0\}$, then x is an outer generalized inverse of a. The outer inverse is not unique in general, but it is unique if we fix the corresponding idempotents [3]: let $a \in \mathcal{R}$, and let $p, q \in \mathcal{R}^\bullet$. An element $x \in \mathcal{R}$ satisfying

$$xax = x, \qquad xa = p \qquad \text{and} \qquad 1 - ax = q,$$

2010 *Mathematics Subject Classification*. Primary 16B99; Secondary 16U99, 15A09
Keywords. (b,c)–inverse, outer inverse, group inverse, ring
Received: 29 December 2016; Accepted: 27 March 2018
Communicated by Dragan S. Djordjević
Research supported by the National Natural Science Foundation of China (No. 11771076). The first author is supported by the Ministry of Education and Science, Republic of Serbia, grant no. 174007. The second author is supported by China Postdoctoral Science Foundation (No. 2018M632385)
Email addresses: dijana@pmf.ni.ac.rs (Dijana Mosić), honglinzou@163.com (Honglin Zou), jlchen@seu.edu.cn (Jianlong Chen)

will be called (p,q)–outer generalized inverse of a, written $x = a^{(2)}_{p,q}$. If $a^{(2)}_{p,q}$ exists, it is unique. Note that, for $a \in \mathcal{R}$ and $p,q \in \mathcal{R}^\bullet$, $a^{(2)}_{p,q}$ exists if and only if $(1-q)a = (1-q)ap$ and there exists some $x \in \mathcal{R}$ such that $px = x$, $xq = 0$, $xap = p$ and $ax = 1-q$ [3]. If $a^{(2)}_{p,q}$ satisfies $aa^{(2)}_{p,q}a = a$, then $a^{(2)}_{p,q} = a^{(1,2)}_{p,q}$ is called a (p,q)–reflexive generalized inverse of a.

Instead of prescribing the idempotents ax and xa, we may prescribe certain kernel and image ideals related to these idempotents [6]: let $p,q \in \mathcal{R}^\bullet$, an element $x \in \mathcal{R}$ is the image-kernel (p,q)–inverse of a if

$$xax = x, \quad xa\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (1-ax)\mathcal{R} = q\mathcal{R}.$$

The image-kernel (p,q)–inverse x is unique if it exists, and it will be denoted by $a^\times_{p,q}$. We use $\mathcal{R}^\times_{p,q}$ to denote the set of all image-kernel (p,q)–invertible elements of $\mathcal{R}$.

Theorem 1.2. [8, Theorem 2.1] *Let $p,q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *$a^\times_{p,q}$ exists,*

(ii) *there exists some $x \in \mathcal{R}$ such that*

$$x = px, \quad xap = p, \quad xq = 0, \quad 1-q = (1-q)ax.$$

Observe that element x in the part (ii) of Theorem 1.2 satisfies $x = a^\times_{p,q}$. The image-kernel (p,q)-inverse of Kantún-Montiel [6] coincides with the (p,q,l)-outer generalized inverse of Cao and Xue [2].

Drazin [4] introduced the following class of outer generalized inverses: let $b,c \in \mathcal{R}$, an element $a \in \mathcal{R}$ is (b,c)–invertible if there exists $y \in \mathcal{R}$ such that

$$y \in (b\mathcal{R}y) \cap (y\mathcal{R}c), \quad yab = b \quad \text{and} \quad cay = c.$$

The (b,c)–inverse y of a satisfies $yay = y$, it is unique (if exists) and denoted by $a^{\|(b,c)}$ [4]. We will use $\mathcal{R}^{\|(b,c)}$ to denote the set of all (b,c)–invertible elements of $\mathcal{R}$.

Lemma 1.3. [9] *Let $a,b,c \in \mathcal{R}$. If a has a (b,c)–inverse, then b, c and cab are regular.*

The special type of outer inverse is a group inverse. An element $a \in \mathcal{R}$ is group invertible if there is $a^\# \in \mathcal{R}$ such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\# \quad \text{and} \quad aa^\# = a^\#a.$$

The group inverse $a^\#$ of a is uniquely determined by these equations. Denote by $\mathcal{R}^\#$ the set of all group invertible elements of $\mathcal{R}$. The spectral idempotent of $a \in \mathcal{R}^\#$ is the element $a^\pi = 1 - aa^\#$.

In this paper, we investigate some properties of the (b,c)–inverse in a ring. Precisely, some new equivalent conditions for the existence of the (b,c)–inverse are presented. We fully characterize the set of all (b,c)–invertible elements. Also, several characterizations for the (b,c)–inverse of a given element to commute with that element are given. We consider too the (b,c)–inverse of a given element which is an inner inverse of that element. As an application of our results, we get new characterizations for the existence of the image-kernel (p,q)–inverse in a ring.

2. The (b,c)–inverse in rings

In this section, we give new characterizations of the existence of the (b,c)–inverse in a ring.

Theorem 2.1. *Let $a,b,c \in \mathcal{R}$. Then*

(a) *a is (b,c)–invertible if and only if b, c are regular and, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$, one of the following equivalent statements holds:*

(i) *$cabb^-$ is $(bb^-, 1-cc^-)$–reflexive generalized invertible,*

(ii) *$cabb^-$ is $(-, cc^-, bb^-)$–invertible.*

(b) *a is (b, c)–invertible if and only if b, c are regular and, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$, one of the following equivalent statements holds:*

(i) *c^-cab is $(b^-b, 1 - c^-c)$–reflexive generalized invertible,*

(ii) *c^-cab is $(-, c^-c, b^-b)$–invertible.*

In addition, if one of the previous statements holds, then

$$a^{\|(b,c)} = (cabb^-)^{(1,2)}_{bb^-,1-cc^-}c = b(c^-cab)^{(1,2)}_{b^-b,1-c^-c},$$

$$(cabb^-)^{(1,2)}_{bb^-,1-cc^-} = a^{\|(b,c)}c^- = (cabb^-)^{-(cc^-,bb^-)},$$

$$(c^-cab)^{(1,2)}_{b^-b,1-c^-c} = b^-a^{\|(b,c)} = (c^-cab)^{-(c^-c,b^-b)}.$$

Proof. (a) Suppose that a is (b, c)–invertible and y is the (b, c)–inverse of a. Then $y = bty = ysc$, for some $t, s \in \mathcal{R}$, $yab = b$, $cay = c$ and, by Lemma 1.3, b, c are regular. For $b^- \in b\{1\}$ and $c^- \in c\{1\}$, notice that $cabb^-$ is $(bb^-, 1 - cc^-)$–reflexive generalized invertible and $(cabb^-)^{(1,2)}_{bb^-,1-cc^-} = yc^-$:

$$yc^-cabb^- = yscc^-cabb^- = yabb^- = bb^-,$$

$$cabb^-yc^- = cabb^-btyc^- = cayc^- = cc^-,$$

$$yc^-cabb^-yc^- = bb^-yc^- = yc^-,$$

$$cabb^-yc^-cabb^- = cc^-cabb^- = cabb^-.$$

So, the condition (i) is satisfied. Since $cabb^- = cc^-cabb^- \in cc^-\mathcal{R}bb^-$ and $yc^- = bb^-btyscc^- \in bb^-\mathcal{R}cc^-$, we deduce that (ii) holds and $(cabb^-)^{-(cc^-,bb^-)} = yc^-$.

Let b, c be regular, $b^- \in b\{1\}$ and $c^- \in c\{1\}$. If the statement (i) holds, that is, $cabb^-$ is $(bb^-, 1-cc^-)$–reflexive generalized invertible and $(cabb^-)^{(1,2)}_{bb^-,1-cc^-} = x$, then we verify that $y = xc$ is the (b, c)–inverse of a:

$$y = xc = bb^-xc = bb^-y \in b\mathcal{R}y$$

$$y = xc = xcc^-c = yc^-c \in y\mathcal{R}c,$$

$$yab = xcab = xcabb^-b = bb^-b = b,$$

$$cay = caxc = cabb^-xc = cc^-c = c.$$

In the same way, by condition (ii), we conclude that a is (b, c)–invertible.

Similarly, we check that (b) is satisfied. □

As a consequence of Theorem 2.1, we obtain the next results. The first of them recovers [1, Theorem 4.1].

Corollary 2.2. *Let $a, b, c \in \mathcal{R}$. Suppose that b, c are regular, $b^- \in b\{1\}$ and $c^- \in c\{1\}$.*

(a) *If $bb^- = cc^-$, then the following statements are equivalent:*

(i) *a is (b, c)–invertible,*

(ii) *$cabb^- \in \mathcal{R}^\#$ and $(cabb^-)^\pi = 1 - bb^-$,*

(iii) *$cabb^- \in (bb^-\mathcal{R}bb^-)^{-1}$.*

(b) *If $c^-c = b^-b$, then the following statements are equivalent:*

(i) *a is (b, c)–invertible,*

(ii) $c^-cab \in \mathcal{R}^\#$ *and* $(c^-cab)^\pi = 1 - c^-c$,

(iii) $c^-cab \in (c^-c\mathcal{R}c^-c)^{-1}$.

Corollary 2.3. *Let* $a, b, c \in \mathcal{R}$. *Then* a *is* (b, c)*–invertible if and only if* b, c *are regular and, for* $b^- \in b\{1\}$ *and* $c^- \in c\{1\}$, *one of the following statements holds:*

(i) abb^- *is* (b, c)*–invertible,*

(ii) c^-ca *is* (b, c)*–invertible,*

(iii) c^-cabb^- *is* (b, c)*–invertible.*

In addition, if one of the previous statements holds, then

$$a^{\|(b,c)} = (abb^-)^{\|(b,c)} = (c^-ca)^{\|(b,c)} = (c^-cabb^-)^{\|(b,c)}.$$

Applying Corollary 2.3, we prove the following result.

Corollary 2.4. *Let* $a, b, c \in \mathcal{R}$. *If* a *is* (b, c)*–invertible and* $x, y \in \mathcal{R}$, *then the following statements hold for* $b^- \in b\{1\}$ *and* $c^- \in c\{1\}$:

(i) $a + x(1 - bb^-)$ *is* (b, c)*–invertible,*

(ii) $a + (1 - c^-c)y$ *is* (b, c)*–invertible,*

(iii) $a + x(1 - bb^-) + (1 - c^-c)y$ *is* (b, c)*–invertible.*

In addition,

$$\begin{aligned} a^{\|(b,c)} &= (a + x(1 - bb^-))^{\|(b,c)} = (a + (1 - c^-c)y)^{\|(b,c)} \\ &= (a + x(1 - bb^-) + (1 - c^-c)y)^{\|(b,c)}. \end{aligned}$$

Proof. Since a is (b, c)–invertible, by Corollary 2.3, we deduce that $abb^- = (a+x(1-bb^-))bb^-$ is (b, c)–invertible. The part (ii) follows similarly. Using (i) and (ii), we get that (iii) holds. □

More characterizations for the existence of the (b, c)–inverse are presented in the next result.

Theorem 2.5. *Let* $a, b, c \in \mathcal{R}$. *Then* a *is* (b, c)*–invertible if and only if* b, c *are regular and, for* $b^- \in b\{1\}$ *and* $c^- \in c\{1\}$, *one of the following equivalent statements holds:*

(i) a *is* (bb^-, c^-c)*–invertible,*

(ii) a *is image-kernel* $(bb^-, 1 - c^-c)$*–invertible.*

In addition, if one of the previous statements holds, then

$$a^{\|(b,c)} = a^{\|(bb^-,c^-c)} = a^\times_{bb^-,1-c^-c}.$$

Proof. Let a be (b, c)–invertible and $y = a^{\|(b,c)}$. Since $y = bty = ysc$, for some $t, s \in \mathcal{R}$, $yab = b$, $cay = c$ and b, c are regular, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$, we obtain

$$y = bb^-bty = yscc^-c, \quad yabb^- = bb^-, \quad c^-cay = c^-c, \tag{1}$$

i.e. a is (bb^-, c^-c)–invertible and $y = a^{\|(bb^-,c^-c)}$. Hence, the statement (i) is satisfied.

By part (i), we have that $y = a^{\|(bb^-,c^-c)}$ satisfies (1). Thus,

$$bb^-y = y, \quad yabb^- = bb^-, \quad y(1 - c^-c) = 0, \quad c^-cay = c^-c. \tag{2}$$

So, by Theorem 1.2(ii), we observe that (ii) holds, that is, a is image-kernel $(bb^-, 1 - c^-c)$–invertible and $a^\times_{(bb^-,1-c^-c)} = y$.

Suppose that b, c are regular and (ii) holds, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$. Set $y = a^\times_{(bb^-,1-c^-c)}$. Using (2), we have that a is (b, c)–invertible and $y = a^{\|(b,c)}$. □

Now, we fully describe the set $\mathcal{R}^{\|(b,c)}$. The following result recovers [1, Theorem 5.1].

Theorem 2.6. *Let $b, c \in \mathcal{R}$ be regular, $b^- \in b\{1\}$ and $c^- \in c\{1\}$.*

(i) *Then*

$$\mathcal{R}^{\|(b,c)} = c^-\mathcal{R}^{-(cc^-,bb^-)} + (1-c^-c)\mathcal{R}bb^- + \mathcal{R}(1-bb^-).$$

In addition, for $x, y \in \mathcal{R}$ and $u \in \mathcal{R}^{-(cc^-,bb^-)}$,

$$(c^-u)^{\|(b,c)} = (c^-u + (1-c^-c)xbb^- + y(1-bb^-))^{\|(b,c)} = u^{-(cc^-,bb^-)}c.$$

(ii) *Also,*

$$\mathcal{R}^{\|(b,c)} = \mathcal{R}^{-(c^-c,b^-b)}b^- + c^-c\mathcal{R}(1-b^-b) + (1-c^-c)\mathcal{R}.$$

In addition, for $x, y \in \mathcal{R}$ and $v \in \mathcal{R}^{-(c^-c,b^-b)}$,

$$(vb^-)^{\|(b,c)} = (vb^- + c^-cx(1-bb^-) + (1-c^-c)y)^{\|(b,c)} = bv^{-(c^-c,b^-b)}.$$

Proof. (i) If $a \in \mathcal{R}^{\|(b,c)}$, then

$$a = c^-cabb^- + (1-c^-c)abb^- + a(1-bb^-).$$

By Theorem 2.1, we have that $cabb^- \in \mathcal{R}^{-(cc^-,bb^-)}$ and so $a \in c^-\mathcal{R}^{-(cc^-,bb^-)} + (1-c^-c)\mathcal{R}bb^- + \mathcal{R}(1-bb^-)$.

Conversely, assume that $u \in \mathcal{R}^{-(cc^-,bb^-)}$ and $a = c^-u$. Since $cabb^- = cc^-ubb^- = u \in \mathcal{R}^{-(cc^-,bb^-)}$, by Theorem 2.1, we conclude that $a \in \mathcal{R}^{\|(b,c)}$ and $a^{\|(b,c)} = u^{-(cc^-,bb^-)}$. Using Corollary 2.4, notice that $a + (1-c^-c)xbb^- + y(1-bb^-) \in \mathcal{R}^{\|(b,c)}$ and $a^{\|(b,c)} = (a + (1-c^-c)xbb^- + y(1-bb^-))^{\|(b,c)}$.

(ii) In the same manner as (i), we verify this part. □

Necessary and sufficient conditions which involve the corresponding outer inverses of products ab, ca or cab, for the existence and representation of $a^{\|(b,c)}$ are given too.

Theorem 2.7. *Let $a, b, c \in \mathcal{R}$. Then*

(i) *a is (b,c)–invertible if and only if b is regular and, for $b^- \in b\{1\}$, (ab) is (b^-b, c)–invertible. Moreover,*

$$(ab)^{\|(b^-b,c)} = b^-a^{\|(b,c)} \qquad \text{and} \qquad a^{\|(b,c)} = b(ab)^{\|(b^-b,c)}.$$

(ii) *a is (b,c)–invertible if and only if c is regular and, for $c^- \in c\{1\}$, (ca) is (b, cc^-)–invertible. Moreover,*

$$(ca)^{\|(b,cc^-)} = a^{\|(b,c)}c^- \qquad \text{and} \qquad a^{\|(b,c)} = (ca)^{\|(b,cc^-)}c.$$

(iii) *a is (b,c)–invertible if and only if b, c are regular and, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$, (cab) is (b^-b, cc^-)–invertible. Moreover,*

$$(cab)^{\|(b^-b,cc^-)} = b^-a^{\|(b,c)}c^- \qquad \text{and} \qquad a^{\|(b,c)} = b(cab)^{\|(b^-b,cc^-)}c.$$

Proof. (i) ⇒: Because $a^{\|(b,c)} = bta^{\|(b,c)} = a^{\|(b,c)}sc$, for some $t, s \in \mathcal{R}$, then

$$b^-a^{\|(b,c)} = b^-bta^{\|(b,c)} = b^-btbb^-a^{\|(b,c)} \in b^-b\mathcal{R}b^-a^{\|(b,c)},$$

$$b^-a^{\|(b,c)} = b^-a^{\|(b,c)}sc \in b^-a^{\|(b,c)}\mathcal{R}c,$$

$$b^-a^{\|(b,c)}abb^-b = b^-a^{\|(b,c)}ab = b^-b,$$

$$cabb^-a^{\|(b,c)} = caa^{\|(b,c)} = c,$$

that is, $(ab)^{\|(b^-b,c)} = b^-a^{\|(b,c)}$.

$\Leftarrow$: Since $(ab)^{\|(b^-b,c)} = b^-bt_1(ab)^{\|(b^-b,c)} = (ab)^{\|(b^-b,c)}s_1c$, for some $t_1, s_1 \in \mathcal{R}$, $b^-b = (ab)^{\|(b^-b,c)}abb^-b = (ab)^{\|(b^-b,c)}ab$ and $cab(ab)^{\|(b^-b,c)} = c$, we get

$$b(ab)^{\|(b^-b,c)} = bb^-bt_1(ab)^{\|(b^-b,c)} = bt_1b^-b(ab)^{\|(b^-b,c)} \in b\mathcal{R}b(ab)^{\|(b^-b,c)},$$

$$b(ab)^{\|(b^-b,c)} = b(ab)^{\|(b^-b,c)}s_1c \in b(ab)^{\|(b^-b,c)}\mathcal{R}c,$$

$$b(ab)^{\|(b^-b,c)}ab = bb^-b = b,$$

$$cab(ab)^{\|(b^-b,c)} = c.$$

Hence, $a^{\|(b,c)} = b(ab)^{\|(b^-b,c)}$.

Similarly as (i), we prove parts (ii) and (iii). □

Now, we will see that a is (b,c)–invertible if and only if au^{-1} is (ub,uc)–invertible (or $u^{-1}a$ is (bu,cu)–invertible).

Theorem 2.8. *Let $a,b,c \in \mathcal{R}$ and $u \in \mathcal{R}^{-1}$. Then the following statement are equivalent:*

(i) *a is (b,c)–invertible,*

(ii) *au^{-1} is (ub,uc)–invertible,*

(iii) *$u^{-1}a$ is (bu,cu)–invertible.*

In addition, if any of statements (i)–(iii) *holds, then*

$$a^{\|(b,c)} = u^{-1}(au^{-1})^{\|(ub,uc)} = (u^{-1}a)^{\|(bu,cu)}u^{-1},$$

$$(au^{-1})^{\|(ub,uc)} = ua^{\|(b,c)} \quad \text{and} \quad (u^{-1}a)^{\|(bu,cu)} = a^{\|(b,c)}u.$$

Proof. (i) $\Leftrightarrow$ (ii): Observe that a is (b,c)–invertible if and only if there exists $y \in \mathcal{R}$ such that $y = bty = ysc$, for some $t,s \in \mathcal{R}$, $yab = b$ and $cay = c$ if and only if there exists $y \in \mathcal{R}$ such that $uy = (ub)tu^{-1}(uy) = (uy)su^{-1}(uc)$, for some $t,s \in \mathcal{R}$, $uyau^{-1}ub = ub$ and $ucau^{-1}uy = uc$ which is equivalent to au^{-1} is (ub,uc)–invertible.

(i) $\Leftrightarrow$ (iii): It follows as (i) $\Leftrightarrow$ (ii). □

In the cases that d is (b,b)–invertible and/or e is (c,c)–invertible, we characterize (b,c)–invertible of a by (b,c)–invertible of abd, eca or $ecabd$.

Theorem 2.9. *Let $a,b,c,d,e \in \mathcal{R}$.*

(i) *If d is (b,b)–invertible, then a is (b,c)–invertible if and only if abd is (b,c)–invertible. Moreover, for $b^- \in b\{1\}$,*

$$(abd)^{\|(b,c)} = d^{\|(b,b)}b^-a^{\|(b,c)} \quad \text{and} \quad a^{\|(b,c)} = bd(abd)^{\|(b,c)}.$$

(ii) *If e is (c,c)–invertible, a is (b,c)–invertible if and only if eca is (b,c)–invertible. Moreover, for $c^- \in c\{1\}$,*

$$(eca)^{\|(b,c)} = a^{\|(b,c)}c^-e^{\|(c,c)} \quad \text{and} \quad a^{\|(b,c)} = (eca)^{\|(b,c)}ec.$$

(iii) *If d is (b,b)–invertible and e is (c,c)–invertible, then a is (b,c)–invertible if and only if $ecabd$ is (b,c)–invertible. Moreover, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$,*

$$(ecabd)^{\|(b,c)} = d^{\|(b,b)}b^-a^{\|(b,c)}c^-e^{\|(c,c)} \quad \text{and} \quad a^{\|(b,c)} = bd(ecabd)^{\|(b,c)}ec.$$

Proof. (i) Assume that d is (b,b)–invertible and a is (b,c)–invertible. For $b^- \in b\{1\}$ and $c^- \in c\{1\}$, by

$$d^{\|(b,b)}b^-a^{\|(b,c)} = bb^-d^{\|(b,b)}b^-a^{\|(b,c)} \in b\mathcal{R}d^{\|(b,b)}b^-a^{\|(b,c)},$$

$$d^{\|(b,b)}b^-a^{\|(b,c)} = d^{\|(b,b)}b^-a^{\|(b,c)}c^-c \in d^{\|(b,b)}b^-a^{\|(b,c)}\mathcal{R}c,$$

$$d^{\|(b,b)}b^-a^{\|(b,c)}abdb = d^{\|(b,b)}b^-bdb = d^{\|(b,b)}db = b,$$

$$cabdd^{\|(b,b)}b^-a^{\|(b,c)} = cabb^-a^{\|(b,c)} = caa^{\|(b,c)} = c,$$

we deduce that abd is (b,c)–invertible and $(abd)^{\|(b,c)} = d^{\|(b,b)}b^-a^{\|(b,c)}$.

Conversely, let d be (b,b)–invertible and abd be (b,c)–invertible. Since, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$,

$$bd(abd)^{\|(b,c)} = bb^-bd(abd)^{\|(b,c)} \in b\mathcal{R}bd(abd)^{\|(b,c)},$$

$$bd(abd)^{\|(b,c)} = bd(abd)^{\|(b,c)}c^-c \in bd(abd)^{\|(b,c)}\mathcal{R}c,$$

$$\begin{aligned} bd(abd)^{\|(b,c)}ab &= bd(abd)^{\|(b,c)}abdd^{\|(b,b)} = bd((abd)^{\|(b,c)}abdb)b^-d^{\|(b,b)} \\ &= bdbb^-d^{\|(b,b)} = bdd^{\|(b,b)} = b, \end{aligned}$$

$$cabd(abd)^{\|(b,c)} = c,$$

then a is (b,c)–invertible and $a^{\|(b,c)} = bd(abd)^{\|(b,c)}$.

We can prove parts (ii) and (iii) in the same manner. □

Remark that the condition d is (b,b)–invertible in Theorem 2.9 can be replaced with d is Mary invertible along b. For details about the Mary inverse, see [7]. Notice that Theorem 2.9 recovers [10, Theorem 3.7].

In the following theorem, we investigate when the equality $aa^{\|(b,c)} = a^{\|(b,c)}a$ is satisfied. If $a^{\|(b,c)}$ satisfies $aa^{\|(b,c)} = a^{\|(b,c)}a$, then $a^{\|(b,c)} \in \mathcal{R}^{\#}$ and $(a^{\|(b,c)})^{\#} = a^2a^{\|(b,c)}$.

Theorem 2.10. *Let $a, b, c \in \mathcal{R}$. If a is (b,c)–invertible, then the following statements are equivalent:*

(i) $aa^{\|(b,c)} = a^{\|(b,c)}a$,

(ii) *there exist* $c^{-(cc^-,aa^{\|(b,c)})}$ *and* $b^{-(a^{\|(b,c)}a,b^-b)}$ *such that* $c^{-(cc^-,aa^{\|(b,c)})} = a^{\|(b,c)}ac^-$ *and* $b^{-(a^{\|(b,c)}a,b^-b)} = b^-aa^{\|(b,c)}$*, for* $b^- \in b\{1\}$ *and* $c^- \in c\{1\}$,

(iii) *there exist* $c^{(1,2)}_{aa^{\|(b,c)},1-cc^-}$ *and* $b^{(1,2)}_{b^-b,1-a^{\|(b,c)}a}$ *such that* $c^{(1,2)}_{aa^{\|(b,c)},1-cc^-} = a^{\|(b,c)}ac^-$ *and* $b^{(1,2)}_{b^-b,1-a^{\|(b,c)}a} = b^-aa^{\|(b,c)}$*, for* $b^- \in b\{1\}$ *and* $c^- \in c\{1\}$.

Proof. (i) ⇒ (ii): Set $x = a^{\|(b,c)}ac^-$, for $c^- \in c\{1\}$. The equality $aa^{\|(b,c)} = a^{\|(b,c)}a$ implies

$$c = cc^-c = cc^-caa^{\|(b,c)} \in cc^-\mathcal{R}aa^{\|(b,c)},$$

$$x = a^{\|(b,c)}ac^- = aa^{\|(b,c)}c^-cc^- \in aa^{\|(b,c)}\mathcal{R}cc^-,$$

$$cx = ca^{\|(b,c)}ac^- = caa^{\|(b,c)}c^- = cc^-,$$

$$xc = a^{\|(b,c)}ac^-c = aa^{\|(b,c)}c^-c = aa^{\|(b,c)}.$$

Thus, there exists $c^{-(cc^-,aa^{\|(b,c)})} = x$. Similarly, we check that $b^{-(a^{\|(b,c)}a,b^-b)}$ exists and $b^{-(a^{\|(b,c)}a,b^-b)} = b^-aa^{\|(b,c)}$, for $b^- \in b\{1\}$.

(ii) ⇒ (i): If $c^{-(cc^-,aa^{\|(b,c)})} = a^{\|(b,c)}ac^-$ and $b^{-(a^{\|(b,c)}a,b^-b)} = b^-aa^{\|(b,c)}$, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$, then

$$aa^{\|(b,c)} = a^{\|(b,c)}ac^-c = bb^-a^{\|(b,c)}ac^-c = bb^-aa^{\|(b,c)} = a^{\|(b,c)}a.$$

(i) ⇔ (iii): In the similar way as (i) ⇔ (ii). □

By Theorem 2.10, we obtain the next result.

Corollary 2.11. *Let $a, b, c \in \mathcal{R}$. If a is (b, c)–invertible, $cc^- = aa^{\|(b,c)}$ and $b^-b = a^{\|(b,c)}a$, for $b^- \in b\{1\}$ and $c^- \in c\{1\}$, then the following statements are equivalent:*

(i) $aa^{\|(b,c)} = a^{\|(b,c)}a$,

(ii) *there exist $c^\#$ and $b^\#$ such that* $c^\# = a^{\|(b,c)}ac^-$, $c^\pi = 1 - cc^-$, $b^\# = b^-aa^{\|(b,c)}$ *and* $b^\pi = 1 - b^-b$.

Now, we study equivalent conditions for the (b, c)–inverse $a^{\|(b,c)}$ to be an inner inverse of a.

Theorem 2.12. *Let $a, b, c \in \mathcal{R}$. If a is (b, c)–invertible, then the following statements are equivalent:*

(i) $aa^{\|(b,c)}a = a$,

(ii) $\mathcal{R} = b\mathcal{R} \oplus a^\circ$,

(iii) $\mathcal{R} = \mathcal{R}c \oplus {}^\circ a$.

Proof. Recall that $aa^{\|(b,c)}a = a \Leftrightarrow \mathcal{R} = a^{\|(b,c)}\mathcal{R} \oplus a^\circ \Leftrightarrow \mathcal{R} = \mathcal{R}a^{\|(b,c)} \oplus {}^\circ a$. The rest follows by $a^{\|(b,c)}\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}a^{\|(b,c)} = \mathcal{R}c$. □

Theorem 2.13. *Let $a, b, c \in R$. Then the following statements are equivalent:*

(i) *a is (b, c)–invertible, and* $aa^{\|(b,c)}a = a$,

(ii) $a \in abR$, $a \in Rca$, $b \in Rab$ *and* $c \in caR$.

Proof. (i) ⇒ (ii): This follows by the definition of (b, c)–inverse.

(ii) ⇒ (i): From the hypotheses, we have that

$$a = abt_1 = t_2ca, b = t_3ab \text{ and } c = cat_4.$$

Then $b = t_3t_2cab \in Rcab$ and $c = cabt_1t_4 \in cabR$, which imply a is (b, c)–invertible by [4, Theorem 2.2]. Also, $aa^{\|(b,c)}a = aa^{\|(b,c)}abt_1 = abt_1 = a$. □

Theorem 2.14. *Let $a, b, c \in \mathcal{R}$. If a is (b, c)–invertible, $aa^{\|(b,c)}a = a$, $b^- \in b\{1\}$ and $c^- \in c\{1\}$, then $a^{\|(b,c)} = (c^-cabb^-)^{(1,2)}_{bb^-,1-c^-c}$. In addition, if $bb^- = c^-c$, then $c^-cabb^- \in \mathcal{R}^\#$ and $a^{\|(b,c)} = (c^-cabb^-)^\#$.*

Proof. Since

$$a^{\|(b,c)}c^-cabb^-a^{\|(b,c)} = a^{\|(b,c)}aa^{\|(b,c)} = a^{\|(b,c)},$$
$$c^-cabb^-a^{\|(b,c)}c^-cabb^- = c^-caa^{\|(b,c)}abb^- = c^-cabb^-,$$
$$a^{\|(b,c)}c^-cabb^- = a^{\|(b,c)}abb^- = bb^-,$$
$$c^-cabb^-a^{\|(b,c)} = c^-caa^{\|(b,c)} = c^-c,$$

we deduce that $(c^-cabb^-)^{(1,2)}_{bb^-,1-c^-c} = a^{\|(b,c)}$. □

One new representation for $a^{\|(b,c)}$ is given now.

Theorem 2.15. *Let $a, b, c \in \mathcal{R}$. If a is (b, c)–invertible and $x \in (cab)\{1\}$, then $a^{\|(b,c)} = bxc$.*

Proof. By Lemma 1.3, b, c and cab are regular. Let $x \in (cab)\{1\}$, $b^- \in b\{1\}$, $c^- \in c\{1\}$ and $y = bxc$. Then $y = bxc = bb^-bxc = bb^-y \in b\mathcal{R}y$ and $y = bxc = bxcc^-c = yc^-c \in y\mathcal{R}c$. Since $cabxcab = cab$, then $abxcab - ab \in c^\circ = (a^{\|(b,c)})^\circ$. So, $a^{\|(b,c)}abxcab = a^{\|(b,c)}ab$, i.e. $yab = bxcab = b$. Also, by $cabxca - ca \in {}^\circ b = {}^\circ(a^{\|(b,c)})$, we get $cabxcaa^{\|(b,c)} = caa^{\|(b,c)}$, that is, $cay = cabxc = c$. Therefore, $y = a^{\|(b,c)}$. □

Next, we consider the reverse order law for the (b, c)-inverse.

Theorem 2.16. *Let $a, b, c, d \in \mathcal{R}$ be such that $ab = ba$ and $ac = ca$. If both a and d are (b, c)–invertible, then ad is (b, c)–invertible and $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.*

Proof. Let $y = d^{\|(b,c)}a^{\|(b,c)}$. Then we obtain that

$$y = bb^{-}d^{\|(b,c)}a^{\|(b,c)} \in bRy \quad\text{and}\quad y = d^{\|(b,c)}a^{\|(b,c)}c^{-}c \in yRc.$$

From the conditions $ab = ba$ and $ac = ca$, it follows that $aa^{\|(b,c)} = a^{\|(b,c)}a$ by [5, Corollary 2.4(i)]. Then

$$\begin{aligned} y(ad)b &= d^{\|(b,c)}a^{\|(b,c)}adb = d^{\|(b,c)}aa^{\|(b,c)}db \\ &= d^{\|(b,c)}c^{-}(caa^{\|(b,c)})db = d^{\|(b,c)}db = b \end{aligned}$$

and

$$c(ad)y = cadd^{\|(b,c)}a^{\|(b,c)} = acdd^{\|(b,c)}a^{\|(b,c)} = aca^{\|(b,c)} = c.$$

This completes the proof of the theorem. □

3. The image-kernel (p, q)–inverse in rings

In this section, as an application of results proved in Section 2, we obtain new characterizations for the existence of the image-kernel (p, q)–inverse in rings.

Applying Theorem 2.5, notice that $a \in \mathcal{R}$ is (p, q)–invertible if and only if a is image-kernel $(p, 1 - q)$–invertible in the case that $p, q \in \mathcal{R}^{\bullet}$.

Corollary 3.1. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$. Then the following statements are equivalent:*

(i) *a is (p, q)–invertible,*

(ii) *a is image-kernel $(p, 1 - q)$–invertible.*

Moreover, if one of the previous statements holds, then $a^{\|(p,q)} = a^{\times}_{p,1-q}$.

By Corollary 3.1 and Theorem 2.1, we get next equivalent conditions for the existence of the image-kernel (p, q)–inverse.

Corollary 3.2. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$. Then the following statements are equivalent:*

(i) *a is image-kernel (p, q)–invertible,*

(ii) *$(1 - q)ap$ is (p, q)–reflexive generalized invertible,*

(iii) *$(1 - q)ap$ is $(-, 1 - q, p)$–invertible.*

In addition, if one of the previous statements holds, then

$$a^{\times}_{p,q} = ((1 - q)ap)^{(1,2)}_{p,q}(1 - q) = p((1 - q)ap)^{(1,2)}_{p,q},$$

$$((1 - q)ap)^{(1,2)}_{p,q} = a^{\times}_{p,q}(1 - q) = pa^{\times}_{p,q} = ((1 - q)ap)^{-(1-q,p)}.$$

Using Corollary 3.2, notice that the following results hold.

Corollary 3.3. *Let $a \in \mathcal{R}$ and $p \in \mathcal{R}^{\bullet}$. Then the following statements are equivalent:*

(i) *a is image-kernel $(p, 1 - p)$–invertible,*

(ii) *$pap \in \mathcal{R}^{\#}$ and $(pap)^{\pi} = 1 - p$,*

(iii) *$pap \in (p\mathcal{R}p)^{-1}$.*

Corollary 3.4. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. Then the following statements are equivalent:*

(i) *a is $(p, 1-q)$–reflexive generalized invertible,*

(ii) *a is $(-, q, p)$–invertible.*

Corollary 3.5. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. Then the following statements are equivalent:*

(i) *a is image-kernel (p, q)–invertible,*

(i) *ap is image-kernel (p, q)–invertible,*

(ii) *$(1-q)a$ is image-kernel (p, q)–invertible,*

(iii) *$(1-q)ap$ is image-kernel (p, q)–invertible.*

In addition, if one of the previous statements holds, then

$$a^\times_{p,q} = (ap)^\times_{p,q} = ((1-q)a)^\times_{p,q} = ((1-q)ap)^\times_{p,q}.$$

Corollary 3.6. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. If a is image-kernel (p, q)–invertible and $x, y \in \mathcal{R}$, then the following statements hold:*

(i) *$a + x(1-p)$ is image-kernel (p, q)–invertible,*

(ii) *$a + qy$ is image-kernel (p, q)–invertible,*

(iii) *$a + x(1-p) + qy$ is image-kernel (p, q)–invertible.*

The set $\mathcal{R}^\times_{p,q}$ is fully described now.

Theorem 3.7. *Let $p, q \in \mathcal{R}^\bullet$.*

(i) *Then*

$$\mathcal{R}^\times_{p,q} = \mathcal{R}^{-(1-q,p)} + q\mathcal{R}p + \mathcal{R}(1-p).$$

(ii) *Also,*

$$\mathcal{R}^\times_{p,q} = \mathcal{R}^{-(1-q,p)} + (1-q)\mathcal{R}(1-p) + q\mathcal{R}.$$

We can get the next result as Theorem 2.9.

Corollary 3.8. *Let $a, d, e \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$.*

(i) *If d is image-kernel $(p, 1-p)$–invertible, then a is image-kernel (p, q)–invertible if and only if apd is image-kernel (p, q)–invertible. Moreover,*

$$(apd)^\times_{p,q} = d^\times_{p,1-p} a^\times_{p,q} \quad \text{and} \quad a^\times_{p,q} = pd(apd)^\times_{p,q}.$$

(ii) *If e is image-kernel $(1-q, q)$–invertible, a is image-kernel (p, q)–invertible if and only if $e(1-q)a$ is image-kernel (p, q)–invertible. Moreover,*

$$(e(1-q)a)^\times_{p,q} = a^\times_{p,q} e^\times_{1-q,q} \quad \text{and} \quad a^\times_{p,q} = (e(1-q)a)^\times_{p,q} e(1-q).$$

(iii) *If d is image-kernel $(p, 1-p)$–invertible and e is image-kernel $(1-q, q)$–invertible, then a is image-kernel (p, q)–invertible if and only if $e(1-q)apd$ is image-kernel (p, q)–invertible. Moreover,*

$$(e(1-q)apd)^\times_{p,q} = d^\times_{p,1-p} a^\times_{p,q} e^\times_{1-q,q} \quad \text{and} \quad a^\times_{p,q} = pd(e(1-q)apd)^\times_{p,q} e(1-q).$$

As a consequence of Theorem 2.15, we have the following representation of $a^\times_{p,q}$.

Corollary 3.9. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. If a is image-kernel (p, q)–invertible and $x \in ((1-q)ap)\{1\}$, then $a^\times_{p,q} = px(1-q)$.*

References

[1] J. Benítez, E. Boasso, The inverse along an element in rings, Electronic Journal of Linear Algebra 31 (2016) 572–592.
[2] J. Cao, Y. Xue, The characterizations and representations for the generalized inverses with prescribed idempotents in Banach algebra, Filomat 27(5) (2013) 851–863.
[3] D.S. Djordjevic, Y. Wei, Outer generalized inverses in rings, Comm. Algebra 33 (2005) 3051–3060.
[4] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909–1923.
[5] M.P. Drazin, Commuting properties of generalized inverses, Linear Multilinear Algebra 61(12) (2013) 1675–1681.
[6] G. Kantún-Montiel, Outer generalized inverses with prescribed ideals, Linear Multilinear Algebra 62(9) (2014) 1187–1196.
[7] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434(8) (2011) 1836–1844.
[8] D. Mosić, D.S. Djordjević, G. Kantún-Montiel, Image-kernel (p, q)-inverses in rings, Electronic J. Linear 27 (2014) 272–283.
[9] L. Wang, N. Castro-González, J. L. Chen, Characterizations of outer generalized inverses, Canad. Math. Bull. 60 (2017) 861–871.
[10] H. Zhu, J. L. Chen, P. Patrício, Reverse order law for the inverse along an element, Linear Multilinear Algebra 65(1) (2017) 166–177.