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Soft Union Interior Ideals, Quasi-Ideals and Generalized Bi-Ideals of Rings

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Abstract. In this paper, soft union interior ideals, quasi-ideals and generalized bi-ideals of rings are defined and their properties are obtained and the interrelations of them are given. Moreover regular, regular duo, intra-regular and strongly regular rings are characterized in terms of these soft union ideals. This paper is a following study of [19].

1. Introduction

Probability theory, fuzzy set theory, rough set theory, vague set theory and the interval mathematics are useful approaches to describe uncertainty. However, each of these theories has its inherent difficulties. Molodtsov [13] proposed a completely new approach for modeling vagueness and uncertainty, which is called soft set theory. Since then, many related concepts about soft set operations, have undergone tremendous studies. Maji et al. [12] presented some definitions on soft sets and Ali et al. [3] introduced several operations of soft sets and Sezgin and Atagün [14] studied on soft set operations, as well. However, soft set theory have found its wide-ranging applications in the mean of algebraic structures such as groups [2, 15], semirings [8], rings [1], BCK/BCI-algebras [9–11], BL-algebras [22], near-rings [16] and soft substructures and union soft substructures [4, 17].

In [19], Sezgin Sezer made a new approach to the classical ring theory via soft set theory with the concept of soft union rings. Soft union rings, soft union left (right, two-sided) ideals, bi-ideals and soft union semiprime ideals of rings are defined, their basic properties are obtained and regular, regular duo, intra-regular and strongly regular rings are characterized by the properties of these soft union ideals in [19].

This paper is a following study of [19]. In this paper, soft union interior ideals, quasi-ideals, generalized bi-ideals of rings are defined, their basic properties with respect to soft set operations and soft int-uni product defined in [19] are obtained and the interrelations of them are investigated. Furthermore, regular, regular duo, intra-regular and strongly regular rings are characterized by the properties of these soft union ideals.

2. Preliminaries

In this section, we recall some basic notions relevant to rings and soft sets. Throughout this paper, R denotes a ring. A nonempty subgroup A of R is called a *right ideal* of R if $AR \subseteq A$ and is called a *left ideal* of

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R if $RA \subseteq A$. By *two-sided ideal* (or simply *ideal*), we mean a subset of R, which is both a left and right ideal of R. An additive subgroup $(B, +)$ of R is called a *bi-ideal* of R if $BRB \subseteq B$. An additive subgroup $(I, +)$ of R is called an *interior ideal* of R if $RIR \subseteq X$. An additive subgroup $(Q, +)$ of R is called a *quasi ideal* of R if $QR \cap RQ \subseteq Q$. A subset P of a ring R is called *semiprime* if $\forall a \in R$, $a^2 \in P$ implies that $a \in P$. A *semilattice* is a structure $S = (S, .)$, where "." is an infix binary operation, called the *semilattice operation*, such that "." is associative, commutative and idempotent. From now on, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

Definition 2.1. *([6, 13]) A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear to see that a soft set is a parametrized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2. *[6] Let $f_A, f_B \in R(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Definition 2.3. *[6] Let f_A, $f_B \in R(U)$. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A \tilde{\cup} B}$, where $f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.*

Definition 2.4. *[6] Let f_A, $f_B \in R(U)$. Then, intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A \tilde{\cap} B}$, where $f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.*

Definition 2.5. *[6] Let f_A, $f_B \in R(U)$. Then, $\wedge$-product of f_A and f_B, denoted by $f_A \wedge f_B$, is defined as $f_A \wedge f_B = f_{A \wedge B}$, where $f_{A \wedge B}(x, y) = f_A(x) \cap f_B(y)$ for all $(x, y) \in E \times E$.*

Definition 2.6. *[7] Let f_A and f_B be soft sets over the common universe U and Ψ be a function from A to B. Then, soft anti image of f_A under Ψ, denoted by $\Psi^\star(f_A)$, is a soft set over U by*

$$(\Psi^\star(f_A))(b) = \begin{cases} \bigcap\{f_A(a) \mid a \in A \text{ and } \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $b \in B$. And soft pre-image (or soft inverse image) of f_B under Ψ, denoted by $\Psi^{-1}(f_B)$, is a soft set over U by $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$ for all $a \in A$.

Definition 2.7. *[18] Let f_A be a soft set over U and $\alpha \subseteq U$. Then, lower α-inclusion of f_A, denoted by $\mathcal{L}(f_A; \alpha)$, is defined as*

$$\mathcal{L}(f_A : \alpha) = \{x \in A \mid f_A(x) \supseteq \alpha\}.$$

Definition 2.8. *[19] Let f_R and g_R be soft sets over the common universe U. Then, soft intersection-union product $f_R \diamond g_R$ is defined by*

$$(f_R \diamond g_R)(x) = \bigcap_{x = \sum_{i=1}^{m} a_i b_i} (f_R(a_i) \cup g_R(b_i))$$

if $x = \sum_{i=1}^{m} a_i b_i$ and $a_i b_i \neq 0$ for all $1 \leq i \leq m$. Otherwise, define

$$(f_R \diamond g_R)(x) = U.$$

Here note that if R is a division ring and the multiplicative identity element of R is 1_R, then $x = x \cdot 1_R = 1_R \cdot x$, and so $(f_R \diamond g_R)(x) \neq U$ for all $x \in R$.

For the sake of brevity, soft intersection-union product is abbreviated by soft int-uni product in what follows.

Theorem 2.9. *[19] Let $f_R, g_R, h_R \in R(U)$. Then,*

i) $(f_R \diamond g_R) \diamond h_R = f_R \diamond (g_R \diamond h_R)$.

ii) $f_R \diamond g_R \neq g_R \diamond f_R$*, generally. However, if R is commutative, then* $f_R \diamond g_R = g_R \diamond f_R$

iii) $f_R \diamond (g_R \widetilde{\cap} h_R) = (f_R \diamond g_R) \widetilde{\cap} (f_R \diamond h_R)$ *and* $(f_R \widetilde{\cap} g_R) \diamond h_R = (f_R \diamond h_R) \widetilde{\cap} (g_R \diamond h_R)$.

iv) $f_R \diamond (g_R \widetilde{\cup} h_R) = (f_R \diamond g_R) \widetilde{\cup} (f_R \diamond h_R)$ *and* $(f_R \widetilde{\cup} g_R) \diamond h_R = (f_R \diamond h_R) \widetilde{\cup} (g_R \diamond h_R)$.

v) If $f_R \tilde{\subseteq} g_R$*, then* $f_R \diamond h_R \tilde{\subseteq} g_R \diamond h_R$ *and* $h_R \diamond f_R \tilde{\subseteq} h_R \diamond g_R$.

vi) If $t_R, l_R \in S(U)$ *such that* $t_R \tilde{\subseteq} f_R$ *and* $l_R \tilde{\subseteq} g_R$*, then* $t_R \diamond l_R \tilde{\subseteq} f_R \diamond g_R$.

Definition 2.10. *[19] Let X be a subset of S. We denote by $\mathcal{S}_{X^c}$ the soft characteristic function of the complement X and define as*

$$\mathcal{S}_{X^c}(x) = \begin{cases} \emptyset, & \text{if } x \in X, \\ U, & \text{if } x \in S \setminus X \end{cases}$$

Theorem 2.11. *[19] Let X and Y be nonempty subsets of a ring R. Then, the following properties hold:*

i) If $Y \subseteq X$*, then* $\mathcal{S}_{X^c} \tilde{\subseteq} \mathcal{S}_{Y^c}$.

ii) $\mathcal{S}_{X^c} \widetilde{\cap} \mathcal{S}_{Y^c} = \mathcal{S}_{X^c \cap Y^c}$, $\mathcal{S}_{X^c} \widetilde{\cup} \mathcal{S}_{Y^c} = \mathcal{S}_{X^c \cup Y^c}$.

Definition 2.12. *[21] A soft set f_R over U is called a soft union ring of R, if*

i. $f_R(x + y) \subseteq f_R(x) \cup f_R(y)$

ii. $f_R(x) \subseteq f_R(-x)$

iii. $f_R(xy) \subseteq f_R(x) \cup f_R(y)$

for all $x, y \in R$.

Definition 2.13. *[21] A soft set f_R over U is called a soft union left (right) ideal of R over U if*

i. $f_R(x - y) \subseteq f_R(x) \cup f_R(y)$

ii. $f_R(xy) \subseteq f_R(y)$ $(f_R(xy) \subseteq f_R(x))$

for all $x, y \in R$. A soft set over U is called a soft union two-sided ideal (soft union ideal) of R if it is both soft union left and soft union right ideal of R over U.

Definition 2.14. *[19] An SU-ring f_R over U is called a soft union bi-ideal of R over U if*

$$f_R(xyz) \subseteq f_R(x) \cup f_R(z)$$

for all $x, y, z \in R$.

For the sake of brevity, soft union ring, soft union right (left, two-sided) ideal and soft union bi-ideal are abbreviated by SU-ring, SU-right (left, two sided) ideal and SU-bi-ideal, respectively.

It is easy to see that if $f_R(x) = \emptyset$ for all $x \in R$, then f_R is an SU-ring (right ideal, left ideal, ideal, bi-ideal) of R over U. We denote such a kind of SU-ring (right ideal, left ideal, ideal, bi-ideal) by $\widetilde{\theta}$ [19].

Lemma 2.15. *Let f_R be any SU-ring over U. Then, we have the followings:*

i) $\widetilde{\theta} \diamond \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$. *(If R is regular, then* $\widetilde{\theta} \diamond \widetilde{\theta} = \widetilde{\theta}$*).*

ii) $f_R \diamond \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$ *and* $\widetilde{\theta} \diamond f_R \widetilde{\supseteq} \widetilde{\theta}$.

iii) $f_R \widetilde{\cup} \widetilde{\theta} = f_R$ *and* $f_R \widetilde{\cap} \widetilde{\theta} = \widetilde{\theta}$.

Theorem 2.16. *[19] Let X be a nonempty subset of a ring R. Then, X is a subring (left, right, two-sided ideal, bi-ideal) of R if and only if $\mathcal{S}_{X^c}$ is an SU-ring (left, right, two-sided ideal, bi-ideal) of R.*

Proposition 2.17. *[19] Let f_R be a soft set over U and $f_R(x - y) \subseteq f_R(x) \cup f_R(y)$ for all $x, y \in R$. Then, we have the followings:*

i) f_R *is an SU-ring over U if and only if* $f_R \diamond f_R \widetilde{\supseteq} f_R$.

ii) f_R *is an SU-left (right) ideal of R over U if and only if* $\widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$ $(f_R \diamond \widetilde{\theta} \widetilde{\supseteq} f_R)$

iii) f_R *is an SU-bi-ideal of R over U if and only if* $f_R \diamond f_R \widetilde{\supseteq} f_R$ *and* $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$.

Theorem 2.18. *[19] For a ring R the following conditions are equivalent:*

1) R is regular.

2) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ *for every SU-right ideal f_R of R over U and SU-left ideal g_R of R over U.*

3. Soft union interior ideals of rings

In this section, soft union interior ideals of rings is defined and their basic properties with respect to soft operations and soft int-uni product are studied.

Definition 3.1. *Let f_R be an SU-ring over U. Then, f_R is called a soft union interior ideal of R, if*

$$f_R(xay) \subseteq f_R(a)$$

for all $x, y, a \in R$.

Corollary 3.2. *Let* $a = \sum_{i=1}^{m} x_i y_i z_i$ *and f_R be an SU-interior ideal over U. Then,* $f_R(a) = f_R(\sum_{i=1}^{m} x_i y_i z_i) \subseteq f_R(y_i)$ *for all* $1 \leq i \leq m$.

For the sake of brevity, soft union interior ideal is abbreviated by *SU*-interior ideal in what follows.

Example 3.3. *Consider the ring $R = \mathbb{Z}_6$ and let $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ be the universal set and f_R be soft set over U such that*

$$f_R(0) = \{x\},\ f_R(1) = \{e, x, y\},\ f_R(2) = \{e, y\},\ f_R(3) = \{e, x, yx\},\ f_R(4) = \{e, y\},\ f_R(5) = \{e, x, y\}.$$

Then, one can easily show that f_R is an SU-interior ideal over U.

Now, let $U = S_3$ be the symmetric group. If we construct a soft set g_R over U such that

$$g_R(0) = \{(1), (12), (13)\},\ g_R(1) = \{(1)\},\ g_R(2) = \{(1), (12)\}, g_R(3) = \{(1)\}$$

then,

$$g_R(2 \cdot 2 \cdot 3) = g_R(0) \nsubseteq g_R(2)$$

hence, g_R is not an SU-interior ideal over U.

It is easy to see that if $f_R(x) = \emptyset$ for all $x \in R$, then f_R is an SU-interior ideal over U. We denote such a kind of SU-interior ideal by $\widetilde{\theta}$. It is obvious that $\widetilde{\theta} = \mathcal{S}_{R^c}$, i.e. $\widetilde{\theta}(x) = \emptyset$ for all $x \in R$.

Theorem 3.4. *Let f_R be a soft over U. Then, f_R is an SU-bi-ideal of R over U if and only if $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$, $f_R \diamond f_R \widetilde{\supseteq} f_R$ and $\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta} \widetilde{\supseteq} f_R$.*

Proof. First assume that f_R is an SU-interior-ideal of R over U. Since f_R is an SU-ring over U, by Theorem 3.4 we have $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R \diamond f_R \widetilde{\supseteq} f_R$. Let $x \in R$. In the case, when $(\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta})(a) = u$, then it is obvious that

$$(\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta})(a) \supseteq f_R(a), \text{ thus } \widetilde{\theta} \diamond f_R \diamond \widetilde{\theta} \widetilde{\supseteq} f_R.$$

Otherwise, we have

$$\begin{aligned}
(\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta})(x) &= ((\widetilde{\theta} \diamond f_R) \diamond \widetilde{\theta})(x) \\
&= \{ \bigcap_{x=\sum_{i=1}^{m} a_i b_i} (\widetilde{\theta} \diamond f_R)(a_i) \cup \widetilde{\theta}(b_i)\} \\
&= \bigcap_{x=\sum_{i=1}^{m} a_i b_i} \{(\bigcap_{a_i=\sum_{i=1}^{m_i} a_{i_k} b_{i_k}} (\widetilde{\theta}(a_{i_k}) \cup f_R(b_{i_k}))) \cup \widetilde{\theta}(b_i)\} \\
&= \bigcap_{x=\sum_{i=1}^{m'} a_i b_i c_i} f_R(b_i) \\
&\subseteq \bigcap_{x=\sum_{i=1}^{m'} a_i b_i c_i} f_R(\sum_{i=1}^{m'} a_i b_i c_i) \\
&= f_R(x)
\end{aligned}$$

Thus, $\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta} \widetilde{\supseteq} f_R$. Here, note that if $x \neq \sum_{i=1}^{m} a_i b_i$, then $(\widetilde{\theta} \diamond f_R)(x) = U$, and so $(\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta})(x) = U \supseteq f_R(x)$.

Conversely, assume that $\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta} \widetilde{\supseteq} f_R$. Let x, a, y be any element of R. Then, we have:

$$\begin{aligned}
f_R(xay) &\subseteq (\widetilde{\theta} \diamond f_R \diamond \widetilde{\theta})(xay) \\
&= \bigcap_{xay=\sum_{i=1}^{m} x_i y_i} \{(\widetilde{\theta} \diamond f_R)(x_i) \cup \widetilde{\theta}(y_i)\}
\end{aligned} \tag{1}$$

$$
\begin{aligned}
&\subseteq (\widetilde{\theta} \diamond f_R)(xa) \cup \widetilde{\theta}(y)\\
&= (\widetilde{\theta} \diamond f_R)(xa) \cup \emptyset\\
&= \bigcap_{xa=\sum_{i=1}^{m} n_i k_i} \{\widetilde{\theta}(n_i) \cup f_R(k_i)\}\\
&\subseteq \widetilde{\theta}(x) \cup f_R(a)\\
&= f_R(a)
\end{aligned}
$$

Hence, f_R is an SU-interior ideal over U. This completes the proof. □

Corollary 3.5. *Let f_R be a soft set. Then the following conditions are equivalent:*

1) $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$.

2) $f_R(\sum_{i=1}^{m} x_i y_i z_i) \subseteq f_R(y_i)$ for all $1 \leq i \leq m$.

Theorem 3.6. *A non-empty subset I of a ring R is an interior ideal of R if and only if the soft subset f_R defined by*

$$f_R(x) = \begin{cases} \alpha, & \text{if } x \in R \setminus I,\\ \beta, & \text{if } x \in I \end{cases}$$

is an SU-interior ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Suppose I is an interior ideal of R and $x, y, a, b \in R$. If $a, b \in I$, then $a - b \in I$. Hence, $f_R(a - b) = f_R(a) = f_R(b) = \beta$ and so, $f_R(a - b) \subseteq f_S(a) \cup f_S(b)$. If $a, b \notin I$, then $a - b \in I$ or $a - b \notin I$. In any case, $f_R(a - b) \subseteq f_R(a) \cup f_R(b) = \alpha$. Now, let $a \in I$, then $xay \in I$. Hence, $f_R(xay) = f_R(a) = \beta$. If $a \notin I$, then $xay \in I$ or $xay \notin I$. In any case, $f_R(xay) \subseteq f_R(a) = \alpha$. Thus, f_R is an SU-interior ideal of S.

Conversely assume that f_R is an SU-interior ideal of R. Let $a, b \in I$ and $x, y \in R$. Then, $f_R(a - b) \subseteq f_R(a) \cup f_R(b) = \beta$. This implies that $f_R(a - b) = \beta$. Hence, $a - b \in I$. Now, $f_R(xay) \subseteq f_R(a) = \beta$. This implies that $f_R(xay) = \beta$. Hence, $xay \in I$ and so I is an interior ideal of R. □

Theorem 3.7. *Let X be a nonempty subset of a ring R. Then, X is an interior ideal of R if and only if $\mathcal{S}_{X^c}$ is an SU-interior ideal of R.*

Proof. Since

$$\mathcal{S}_{X^c}(x) = \begin{cases} U, & \text{if } x \in R \setminus X,\\ \emptyset, & \text{if } x \in X \end{cases}$$

and $U \supseteq \emptyset$, the rest of the proof follows from Theorem 3.6. □

It is obvious that every two-sided ideal of R is an interior ideal of R. Moreover, we have the following:

Proposition 3.8. *Let f_R be a soft set over U. Then, if f_R is an SU-ideal of R over U, f_R is an SU-interior ideal of R over U.*

Proof. Let f_R be an SU-ideal of R over U and $x, y \in R$. Then,

$$f_R(xyz) = f_R((xy)z) \subseteq f_R(xy) \subseteq f_R(y).$$

Hence, f_R is an SU-interior ideal of R over U. □

The following theorem shows that the converse of Proposition 3.8 holds for a regular ring.

Theorem 3.9. *Let f_R be a soft set over U, where R is a regular ring. Then, the following conditions are equivalent:*

1) f_R is an SU-ideal of R over U.

2) f_R is an SU-interior ideal of R over U.

Proof. By Proposition 3.8, it suffices to prove that (2) implies (1). Assume that (2) holds. Let a, b be any elements of R. Then, since R is regular, there exist elements x and y in R such that

$$a = axa \text{ and } b = byb.$$

Then, since f_R is an interior ideal of R, we have

$$f_R(ab) = f_R((axa)b) = f_R((ax)a(b)) \subseteq f_R(a),$$

and

$$f_R(ab) = f_R(a(byb)) = f_R((a)b(yb)) \subseteq f_R(b).$$

This means that f_R is an SU-ideal of R. Thus, (2) implies (1). □

Proposition 3.10. *Let R be a division ring and f_R be a soft set over U. Then, f_R is an SU-ideal of R if and only if f_R is an SU-interior ideal of R.*

Proof. The necessity is clear by Proposition 3.8. Now let us show the sufficiency. For $x, y \in R$, $f_R(xy) = f_R(xye) \subseteq f_R(y)$ and $f_R(xy) = f_R(exy) \subseteq f_R(x)$. Thus, f_R is an SU-ideal of R. □

It is known that a ring R is called *left (right) simple* if it contains no proper left (right) ideal of R and is called *simple* if it contains no proper ideal.

Definition 3.11. *[19] A ring R is called soft left (right) union simple if every SU-left (right) ideal of R is a constant function and is called soft union simple if every SU-ideal of R is a constant function.*

Theorem 3.12. *[19] For a ring R, the following conditions are equivalent:*

1) R is simple.

2) R is soft union simple.

Theorem 3.13. *For a regular ring R, the following conditions are equivalent:*

1) R is simple.

2) R is soft union simple.

3) Every SU-interior ideal of R is constant function.

Proof. The equivalence of (1) and (2) follows from Theorem 3.12. Assume that (2) holds. Let f_R be any SU-interior ideal of R and a and b be any element of R. Then, since R is simple, it follows that there exist elements x and y in R such that

$$a = xby.$$

Then, since f_R is an SU-interior ideal of R, we have

$$f_R(a) = f_R(xby) \subseteq f_R(b).$$

One can similarly show that $f_R(b) \subseteq f_R(a)$. Thus, $f_R(a) = f_R(b)$. Since a and b be any elements of R, f_R is a constant function and so (2) implies (3). Since every SU-interior ideal of R is an SU-ideal of R by the regularity of R, (3) implies (2). □

Definition 3.14. *[19] A soft set f_R over U is called soft union semiprime if for all $a \in R$,*

$$f_R(a) \subseteq f_R(a^2).$$

Proposition 3.15. *Let f_R be a soft union semiprime SU-interior ideal of a ring R. Then, $f_R(a^n) \subseteq f_R(a^{n+1})$ for all positive integers n.*

Proof. Let n be any positive integer. Then,

$$f_R(a^n) \subseteq f_R(a^{2n}) \subseteq f_R(a^{4n}) = f_R(a^{3n-2}a^{n+1}a) \subseteq f_R(a^{n+1}).$$

□

Proposition 3.16. *Let f_R and f_T be SU-interior ideals over U. Then, $f_R \wedge f_T$ is an SU-interior ideal of $R \times T$ over U.*

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_2) \in R \times T$. Then,

$$\begin{aligned}
f_{R\vee T}((x_1, y_1) - (x_2, y_2)) &= f_{R\vee T}(x_1 - x_2, y_1 - y_2) \\
&= f_R(x_1 - x_2) \cup f_T(y_1 - y_2) \\
&\subseteq (f_R(x_1) \cup f_R(x_2)) \cup (f_T(y_1) \cup f_T(y_2)) \\
&= (f_R(x_1) \cup f_T(y_1)) \cup (f_R(x_2) \cup f_T(y_2)) \\
&= f_{R\vee T}(x_1, y_1) \cup f_{R\vee T}(x_2, y_2),
\end{aligned}$$

$$\begin{aligned}
f_{R\vee T}((x_1, y_1)(x_2, y_2)) &= f_{R\vee T}(x_1x_2, y_1y_2) \\
&= f_R(x_1x_2) \cup f_T(y_1y_2) \\
&\subseteq (f_R(x_1) \cup f_R(x_2)) \cup (f_T(y_1) \cup f_T(y_2)) \\
&= (f_R(x_1) \cup f_T(y_1)) \cup (f_R(x_2) \cup f_T(y_2)) \\
&= f_{R\vee T}(x_1, y_1) \cup f_{R\vee T}(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned}
f_{S\vee T}((x_1, y_1)(x_2, y_2)(x_3, y_3)) &= f_{S\vee T}(x_1x_2x_3, y_1y_2y_3) \\
&= f_R(x_1x_2x_3) \cup f_T(y_1y_2y_3) \\
&\subseteq f_R(x_2) \cup f_T(y_2) \\
&= f_{S\vee T}(x_2, y_2)
\end{aligned}$$

Therefore, $f_R \vee f_T$ is an SU-interior ideal of $R \times T$ over U. □

Proposition 3.17. *If f_R and h_R are SU-interior ideals of R over U, then so is $f_R \widetilde{\cup} h_R$.*

Proof. Let $x, y, z \in R$. Then, we have

$$\begin{aligned}
(f_R \widetilde{\cup} h_R)(x - y) &= f_R(x - y) \cup h_R(x - y) \\
&\subseteq (f_R(x) \cup f_R(y)) \cup (h_R(x) \cup h_R(y)) \\
&= (f_R(x) \cup h_R(x)) \cup (f_R(y) \cup h_R(y)) \\
&= (f_R \widetilde{\cup} h_R)(x) \cup (f_R \widetilde{\cup} h_R)(y)
\end{aligned}$$

$$\begin{aligned}
(f_R \widetilde{\cup} h_R)(xy) &= f_R(xy) \cup h_R(xy) \\
&\subseteq (f_R(x) \cup f_R(y)) \cup (h_R(x) \cup h_R(y)) \\
&= (f_R(x) \cup h_R(x)) \cup (f_R(y) \cup h_R(y)) \\
&= (f_R \widetilde{\cup} h_R)(x) \cup (f_R \widetilde{\cup} h_R)(y)
\end{aligned}$$

and

$$\begin{aligned}
(f_R \widetilde{\cup} h_R)(xyz) &= f_R(xyz) \cup h_R(xyz) \\
&\subseteq f_R(y) \cup h_R(y) \\
&= (f_R \widetilde{\cup} h_R)(y)
\end{aligned}$$

Therefore, $f_R \widetilde{\cup} h_R$ is an SU-interior ideal of R over U. □

Proposition 3.18. *Let f_R be a soft set over U and α be a subset of U such that $\alpha \in Im(f_R)$, where $Im(f_R) = \{\alpha \subseteq U : f_R(x) = \alpha,\ for\ x \in R\}$. If f_R is an SU-interior ideal over U, then $\mathcal{L}(f_R;\alpha)$ is an interior ideal of R.*

Proof. Since $f_R(a) = \alpha$ for some $x \in R$, then $\emptyset \neq \mathcal{L}(f_R;\alpha) \subseteq R$. Let $a, b \in \mathcal{L}(f_R;\alpha)$ and $x, y \in R$, then $f_R(a) \subseteq \alpha$ and $f_R(b) \subseteq \alpha$. We need to show that $a - b \in \mathcal{L}(f_R;\alpha)$ and $xay \in \mathcal{L}(f_R;\alpha)$ for all $a, b \in \mathcal{L}(f_R;\alpha)$ and $x, y \in R$. Since f_R is an SU-interior ideal of R over U, it follows that $f_R(a - b) \subseteq f_R(a) \cup f_R(b)\alpha$ and $f_R(xay) \subseteq f_R(a) \subseteq \alpha$ implying that $a - b \in \mathcal{L}(f_R;\alpha)$ and $xay \in \mathcal{L}(f_R;\alpha)$. Thus, the proof is completed. □

Definition 3.19. *Let f_R be an SU-interior ideal over U. Then, the interior ideals $\mathcal{L}(f_R;\alpha)$ are called lower α-interior ideals of f_R.*

Proposition 3.20. *Let f_R be a soft set over U, $\mathcal{L}(f_R;\alpha)$ be lower α-interior ideals of f_R for each $\alpha \subseteq U$ and $Im(f_R)$ be an ordered set by inclusion. Then, f_R is an SU-interior ideal of R over U.*

Proof. Let $a, b \in R$ and $f_R(a) = \alpha_1$ and $f_R(b) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $a \in \mathcal{L}(f_R;\alpha_1)$ and $b \in \mathcal{L}(f_R;\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $a, b \in \mathcal{L}(f_R;\alpha_1)$ and since $\mathcal{L}(f_R;\alpha)$ is an interior ideal of R for all $\alpha \subseteq U$, it follows that $a - b \in \mathcal{L}(f_R;\alpha_1)$ and $xay \in \mathcal{L}(f_R;\alpha_1)$. Hence, $f_R(a - b) \subseteq \alpha_1 = \alpha_1 \cup \alpha_2 = f_R(a) \cup f_R(b)$, and $f_R(xay) \subseteq \alpha_1 = f_R(a)$. Thus, f_R is an SU-interior ideal of R over U. □

Proposition 3.21. *Let f_R and f_T be soft sets over U and Ψ be a ring isomorphism from R to T. If f_R is an SU-interior ideal of R over U, then $\Psi^\star(f_R)$ is an SU-interior ideal of T over U.*

Proof. Let $t_1, t_2, t_3 \in T$. Since Ψ is surjective, then there exist $r_1, r_2, r_3 \in R$ such that $\Psi(r_1) = t_1$, $\Psi(r_2) = t_2$, $\Psi(r_3) = t_3$. Then,

$$
\begin{aligned}
&(\Psi^\star(f_R))(t_1 - t_2)\\
&= \bigcap\{f_R(r) : r \in R, \Psi(r) = t_1 - t_2\}\\
&= \bigcap\{f_R(r) : r \in R, r = \Psi^{-1}(t_1 - t_2)\}\\
&= \bigcap\{f_R(r) : r \in R, r = \Psi^{-1}(\Psi(r_1 - r_2)) = r_1 - r_2\}\\
&= \bigcap\{f_R(r_1 - r_2) : r_i \in R, \Psi(r_i) = t_i, i = 1,2\}\\
&\subseteq \bigcap\{f_R(r_1) \cup f_R(r_2) : r_i \in R, \Psi(r_i) = t_i, i = 1,2\}\\
&= (\bigcap\{f_R : (r_1)r_1 \in R, \Psi(r_1) = t_1\}) \cup (\bigcap\{f_R(r_2) : r_2 \in R, \Psi(r_2) = t_2\})\\
&= (\Psi^\star(f_R))(t_1) \cup (\Psi^\star(f_R))(t_2)
\end{aligned}
$$

One can similarly show that $(\Psi^\star(f_R))(t_1t_2) \subseteq (\Psi^\star(f_R))(t_1) \cup (\Psi^\star(f_R))(t_2)$ Also

$$
\begin{aligned}
&(\Psi^\star(f_R))(t_1t_2t_3)\\
&= \bigcap\{f_R(s) : s \in R, \Psi(s) = t_1t_2t_3\}\\
&= \bigcap\{f_R(s) : s \in R, s = \Psi^{-1}(t_1t_2t_3)\}\\
&= \bigcap\{f_R(s) : s \in R, s = \Psi^{-1}(\Psi(s_1s_2s_3)) = s_1s_2s_3\}\\
&= \bigcap\{f_R(s_1s_2s_3) : s_i \in R, \Psi(s_i) = t_i, i = 1,2,3\}\\
&\subseteq (\bigcap\{f_R(s_2) : s_2 \in R, \Psi(s_2) = t_2\})\\
&= (\Psi^\star(f_R))(t_2)
\end{aligned}
$$

Hence, $\Psi^\star(f_R)$ is an SU-interior ideal of R over U. □

Proposition 3.22. *Let f_R and f_T be soft sets over U and Ψ be a ring homomorphism from R to T. If f_T is an SU-interior ideal of T over U, then $\Psi^{-1}(f_T)$ is an SU-interior ideal of R over U.*

Proof. Let $r_1, r_2, r_3 \in R$. Then,

$$
\begin{aligned}
(\Psi^{-1}(f_T))(r_1 - r_2) &= f_T(\Psi(r_1 - r_2))\\
&= f_T(\Psi(r_1)\Psi r_2))\\
&\subseteq f_T(\Psi(r_1)) \cup f_T(\Psi(r_2))\\
&= (\Psi^{-1}(f_T))(r_1) \cup (\Psi^{-1}(f_T))(r_2)
\end{aligned}
$$

One can similarly show that $(\Psi^{-1}(f_T))(r_1r_2) \subseteq (\Psi^{-1}(f_T))(r_1) \cup (\Psi^{-1}(f_T))(r_2)$. Also

$$\begin{array}{rcl} (\Psi^{-1}(f_T))(r_1r_2r_3) & = & f_T(\Psi(r_1r_2r_3)) \\ & = & f_T(\Psi(r_1)\Psi(r_2)\Psi(r_3)) \\ & \subseteq & f_T(\Psi(r_2)) \\ & = & (\Psi^{-1}(f_T))(r_2) \end{array}$$

Hence, $\Psi^{-1}(f_T)$ is an SU-interior ideal over U. □

4. Soft union quasi-ideals of rings

In this section, soft union quasi-ideals are defined and their properties as regards soft set operations, soft int-uni product and certain kinds of soft union ideals are studied.

Definition 4.1. *A soft set over U is called a soft union quasi-ideal of R over U if $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $(f_R \diamond \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} \diamond f_R)\widetilde{\supseteq} f_R$.*

For the sake of brevity, soft union quasi-ideal is abbreviated by SU-quasi-ideal in what follows.

Proposition 4.2. *Every SU-quasi ideal of R is an SU-ring of R.*

Proof. Let f_R be any SU-quasi-ideal of R. Then, $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and since $f_R \widetilde{\supseteq} \widetilde{\theta}$,

$$f_R \diamond f_R \widetilde{\supseteq} \widetilde{\theta} \diamond f_R \text{ and } f_R \diamond f_R \widetilde{\supseteq} f_R \diamond \widetilde{\theta}.$$

Hence,

$$f_R \diamond f_R \widetilde{\supseteq} (\widetilde{\theta} \diamond f_R)\widetilde{\cup}(f_R \diamond \widetilde{\theta})\widetilde{\supseteq} f_R$$

That is, f_R is an SU-ring over U by Proposition 2.17. □

Proposition 4.3. *Each one-sided SU-ideal of R is an SU-quasi-ideal of R.*

Proof. Let f_R be an SU-left ideal of R. Then, $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and since $\widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$, we have

$$(\widetilde{\theta} \diamond f_R)\widetilde{\cup}(f_R \diamond \widetilde{\theta})\widetilde{\supseteq}\widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R.$$

Thus, f_R is an SU-quasi-ideal of R. □

Proposition 4.4. *Every SU-quasi-ideal of R is an SU-bi-ideal of R.*

Proof. Let f_R be an SU-quasi-ideal of R. Then, $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$,

$$f_R \diamond f_R = (f_R \diamond f_R)\widetilde{\cup}(f_R \diamond f_R)\widetilde{\supseteq}(f_R \diamond \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} \diamond f_R)\widetilde{\supseteq} f_R$$

and

$$f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} \widetilde{\theta} \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} \widetilde{\theta} \diamond f_R \text{ and } f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R \diamond \widetilde{\theta} \diamond \widetilde{\theta} \widetilde{\supseteq} f_R \diamond \widetilde{\theta}$$

and so $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} (\widetilde{\theta} \diamond f_R)\widetilde{\cup}(f_R \diamond \widetilde{\theta})\widetilde{\supseteq} f_R$, as f_R is an SU-quasi-ideal of R. Hence,

$$f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R.$$

Thus, f_R is an SU-bi-ideal of R by Proposition 2.17. □

The following theorem shows that the converse of Proposition 4.4 holds for a regular ring. First, we have the following lemma:

Lemma 4.5. *Let f_R and h_R be soft SU-rings (left, (right) ideals, bi-ideals, interior ideals, quasi-ideals) of R over U, where R is a division ring. Then,*

$$(f_R \diamond h_R)(x-y) \subseteq (f_R \diamond h_R)(x) \cup (f_R \diamond h_R)(y)$$

for all $x, y \in R$.

Proof. Let f_R and h_R be soft sets over U and $x, y \in R$. Then,

$$\begin{aligned}
(f_R \diamond h_R)(x) \cup (f_R \diamond h_R)(y) &= \bigcap_{x=\sum_{i=1}^{m} a_i b_i} (f_R(a_i) \cup h_R(b_i)) \cup \bigcap_{y=\sum_{i=1}^{n} c_i d_i} (f_R(c_i) \cup h_R(d_i)) \\
&= \bigcap_{x=\sum_{i=1}^{m} a_i b_i} \bigcap_{y=\sum_{i=1}^{n} c_i d_i} (f_R(a_i) \cup f_R(c_i) \cup h_R(b_i) \cup h_R(d_i)) \\
&\supseteq \bigcap_{x+y=\sum_{i=1}^{k} x_i y_i} (f_R(x_i) \cup h_R(y_i)) \\
&= (f_R \diamond h_R)(x+y)
\end{aligned}$$

and

$$\begin{aligned}
(f_R \diamond h_R)(-x) &= \bigcap_{-x=\sum_{i=1}^{m} a_i b_i} (f_R(a_i) \cup h_R(b_i)) \\
&= \bigcap_{x=\sum_{i=1}^{m} (-a_i) b_i} (f_R(a_i) \cup h_R(b_i)) \\
&= \bigcap_{x=\sum_{i=1}^{m} (-a_i) b_i} (f_R(-a_i) \cup h_R(b_i)) \\
&= (f_R \diamond h_R)(x)
\end{aligned}$$

Thus, the proof is completed. □

Theorem 4.6. *Let f_R be a soft set over U, where R is a regular ring. Then, the following conditions are equivalent:*

1) f_R is an SU-quasi-ideal of R over U.

2) f_R is an SU-bi-ideal of R over U.

Proof. By Proposition 4.4, it suffices to prove that (2) implies (1). Assume that (2) holds. Let f_R be an SU-bi-ideal of R. Then, $\widetilde{\theta} \diamond f_R$ (resp. $f_R \diamond \widetilde{\theta}$) is an SU-left (resp. right) ideal of R. In fact, $(\widetilde{\theta} \diamond f_R)(x-y) \subseteq (\widetilde{\theta} \diamond f_R)(x) \cup (\widetilde{\theta} \diamond f_R)(y)$ by Lemma 4.5 and $\widetilde{\theta} \diamond (\widetilde{\theta} \diamond f_R) \widetilde{\supseteq} \widetilde{\theta} \diamond f_R$. It follows by Theorem 2.18 that

$$(f_R \diamond \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} \diamond f_R) = (f_R \diamond \widetilde{\theta}) \diamond (\widetilde{\theta} \diamond f_R) = f_R \diamond (\widetilde{\theta} \diamond \widetilde{\theta}) \diamond f_R = f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$$

since f_R is an SU-bi-ideal of R. Thus, f_R is an SU-quasi-ideal of R and (2) implies (1). □

Theorem 4.7. *A non-empty subset Q of a ring R is a quasi-ideal of R if and only if the soft subset f_R defined by*

$$f_R(x) = \begin{cases} \alpha, & \text{if } x \in R \setminus Q, \\ \beta, & \text{if } x \in Q \end{cases}$$

is an SU-quasi-ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. It is similar to Theorem 3.6. □

Theorem 4.8. *Let X be a nonempty subset of a ring R. Then, X is a quasi-ideal of R if and only if $\mathcal{S}_{X^c}$ is an SU-quasi-ideal of R over U.*

Proof. It follows from Theorem 4.7. □

Theorem 4.9. *Let f_R and g_R be any SU-quasi-ideal of R over U. Then, the soft int-uni product $f_R \diamond g_R$ is an SU-bi-ideal of R over U.*

Proof. Let f_R be an SU-quasi-ideal of R. Then, f_R is an SU-bi-ideal by Proposition 4.4. Hence, $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$. Moreover,

$$(f_R \diamond g_R)(x-y) \widetilde{\supseteq} (f_R \diamond g_R)(x) \cup (f_R \diamond g_R)(y)$$

and

$$(f_R \diamond g_R) \diamond (f_R \diamond g_R) = (f_R \diamond g_R \diamond f_R) \diamond g_R \widetilde{\supseteq} (f_R \diamond \widetilde{\theta} \diamond f_R) \diamond g_R \widetilde{\supseteq} f_R \diamond g_R$$

and

$$(f_R \diamond g_R) \diamond \widetilde{\theta} \diamond (f_R \diamond g_R) = (f_R \diamond (g_R \diamond \widetilde{\theta}) \diamond f_R) \diamond g_R \widetilde{\supseteq} (f_R \diamond (\widetilde{\theta} \diamond \widetilde{\theta}) \diamond f_R) \diamond g_R \widetilde{\supseteq} (f_R \diamond \widetilde{\theta} \diamond f_R) \diamond g_R \widetilde{\supseteq} f_R \diamond g_R.$$

Thus, it follows that $f_R \diamond g_R$ is an SU-bi-ideal of R over U. □

Corollary 4.10. *Let R be a regular ring and f_R, g_R be any SU-quasi-ideals of R over U. Then, $f_R \diamond g_R$ is an SU-quasi-ideal of R over U.*

Proof. Follows from Theorem 4.6 and Theorem 4.9. □

Proposition 4.11. *Let f_R be any SU-right ideal of R and g_R be any SU-left ideal of R. Then, $f_R \widetilde{\cup} g_R$ is an SU-quasi-ideal of R.*

Proof. Let f_R be any SU-right ideal of R and g_R be any SU-left ideal of R. Then, one can easily show that $(f_R \widetilde{\cup} g_R)(x-y) \widetilde{\supseteq} f_R(x) \cup g_R(y)$ as in the proof of Proposition 3.17. Moreover,

$$((f_R \widetilde{\cup} g_R) \diamond \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} \diamond (f_R \widetilde{\cup} g_R)) \widetilde{\supseteq} (f_R \diamond \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} \diamond g_R) \widetilde{\supseteq} f_R \widetilde{\cup} g_R.$$

□

Proposition 4.12. *Let R be a regular ring, f_R be any SU-right ideal of R and g_R be any SU-left ideal of R. Then, $f_R \diamond g_R$ is an SU-quasi-ideal of R.*

Proof. Let R be a regular ring and f_R be an SU-right ideal of R and g_R be an SU-left ideal of R. It follows by Proposition 4.11 that $f_R \widetilde{\cup} g_R$ is an SU-quasi-ideal of R. Since R is regular,

$$f_R \diamond g_R = f_R \widetilde{\cup} g_R$$

by Theorem 2.18. Thus, $f_R \diamond g_R$ is an SU-quasi-ideal of R. □

Proposition 4.13. *Let f_R and g_R be any SU-quasi-ideals of R. Then, $f_R \widetilde{\cup} g_R$ is an SU-quasi-ideal of R.*

Proof. Let f_R and g_R be any SU-quasi-ideals of R. Then, one can easily show that $(f_R\widetilde{\cup}g_R)(x-y)\widetilde{\supseteq}f_R(x)\cup g_R(y)$ as in the proof of Proposition 3.17. Also,

$$((f_R\widetilde{\cup}g_R)\diamond\widetilde{\theta})\widetilde{\cup}(\widetilde{\theta}\diamond(f_R\widetilde{\cup}g_R))\widetilde{\supseteq}(f_R\diamond\widetilde{\theta})\widetilde{\cup}(\widetilde{\theta}\diamond f_R)\widetilde{\supseteq}f_R$$

and

$$((f_R\widetilde{\cup}g_R)\diamond\widetilde{\theta})\widetilde{\cup}(\widetilde{\theta}\diamond(f_R\widetilde{\cup}g_R))\widetilde{\supseteq}(g_R\diamond\widetilde{\theta})\widetilde{\cup}(\widetilde{\theta}\diamond g_R)\widetilde{\supseteq}g_R.$$

Thus,

$$((f_R\widetilde{\cup}g_R)\diamond\widetilde{\theta})\widetilde{\cup}(\widetilde{\theta}\diamond(f_R\widetilde{\cup}g_R))\widetilde{\supseteq}f_R\widetilde{\cup}g_R.$$

□

Proposition 4.14. *Let f_R be a soft set over U and α be a subset of U such that $\alpha\in Im(f_R)$. If f_R is an SU-quasi-ideal of R over U, then $\mathcal{L}(f_R;\alpha)$ is a quasi-ideal of R.*

Proof. Since $f_R(x)=\alpha$ for some $x\in R$, then $\emptyset\neq\mathcal{L}(f_R;\alpha)\subseteq R$. Let $a\in(R\cdot\mathcal{L}(f_R;\alpha)\cup\mathcal{L}(f_R;\alpha)\cdot R)$. Then, there exist $x,y\in\mathcal{L}(f_R;\alpha)$ and $s,r\in R$ such that

$$a=sx=yr.$$

Thus, $f_R(x)\subseteq\alpha$ and $f_R(y)\subseteq\alpha$. Hence, $f_R(x-y)\subseteq f_R(x)\cup f_R(y)\subseteq\alpha$, implying that $x-y\in\mathcal{L}(f_R;\alpha)$. Moreover,

$$\begin{aligned}(\widetilde{\theta}\diamond f_R)(a) &= \{\bigcap_{a=\sum_{i=1}^{m}c_id_i}\{\widetilde{\theta}(c_i)\cup f_R(d_i)\}\\ &\subseteq \widetilde{\theta}(s)\cup f_R(x)\\ &= f_R(x)\\ &\subseteq \alpha\end{aligned}$$

and

$$\begin{aligned}(f_R\diamond\widetilde{\theta})(a) &= \{\bigcap_{a=\sum_{i=1}^{m}k_it_i}\{f_R(k_i)\diamond\widetilde{\theta}(t_i)\}\\ &\subseteq f_R(y)\cup\widetilde{\theta}(r)\\ &= f_R(y)\\ &\subseteq \alpha\end{aligned}$$

Since f_R is an SU-quasi-ideal of R, we have

$$f_R(a)\subseteq(\widetilde{\theta}\diamond f_R)(a)\cup(f_R\diamond\widetilde{\theta})(a)\subseteq\alpha,$$

thus $a\in\mathcal{L}(f_R;\alpha)$. This shows that $\mathcal{L}(f_R;\alpha)$ is a quasi-ideal of R. □

Definition 4.15. *Let f_R be an SU-quasi-ideal of R over U. Then, the quasi-ideals $\mathcal{L}(f_R;\alpha)$ are called lower α-quasi-ideals of f_R.*

Proposition 4.16. *Let f_R be any SU-quasi-ideal of a commutative ring R and a be any element of A. Then,*

$$f_R(a^n)\subseteq f_R(a^{n+1})$$

for every positive integer n.

Proof. For any positive integer n, we have

$$\begin{aligned}(f_R \diamond \widetilde{\theta})(a^{n+1}) &= \bigcap_{a^{n+1}=\sum_{i=1}^{m} x_i y_i} (f_R(x_i) \cup \widetilde{\theta}(y_i)) \\ &\subseteq f_R(a^n) \cup \widetilde{\theta}(a) \\ &= f_R(a^n).\end{aligned}$$

Similarly,

$$(\widetilde{\theta} \diamond f_R)(a^{n+1}) \subseteq f_R(a^n).$$

Thus, since f_R is an SU-quasi-ideal of R

$$\begin{aligned}f_R(a^{n+1}) &\subseteq ((f_R \diamond \widetilde{\theta})\widetilde{\cup}(\widetilde{\theta} \diamond f_R))(a^{n+1}) \\ &= (f_R \diamond \widetilde{\theta})(a^{n+1}) \cup (\widetilde{\theta} \diamond f_R))(a^{n+1}) \\ &\subseteq f_R(a^n) \cup f_R(a^n) \\ &= f_R(a^n)\end{aligned}$$

This completes the proof. □

5. Soft union generalized bi-ideals of rings

In this section, soft union generalized bi-ideals are defined and their properties as regards soft set operations and soft int-uni product are studied.

Definition 5.1. *A soft set over U is called a soft union generalized bi-ideal of R over U if $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R(xyz) \subseteq f_R(x) \cup f_R(z)$ for all $x, y, z \in R$.*

For the sake of brevity, soft union generalized bi-ideal is abbreviated by SU-generalized bi-ideal in what follows.

It is clear that every SU-bi-ideal of R is an SU-generalized bi-ideal of R, but the converse of this statement does not hold in general. The following theorem shows that the converse of this holds for a regular ring.

Proposition 5.2. *Every SU-generalized bi-ideal of a regular ring is an SU-bi-ideal of R.*

Proof. Let f_R be an SU-generalized bi-ideal of R and let a and b be any element of R. Then, since R is regular, there exists an element $x \in R$ such that $b = bxb$. Thus, we have

$$f_R(ab) = f_R(a(bxb)) = f_R(a(bx)b) \subseteq f_R(a) \cup f_R(b).$$

This implies that f_R is an SU-ring of R and so f_R is an SU-bi-ideal of R. □

Theorem 5.3. *Let f_R be a soft set over U. Then, f_R is an SU-generalized bi-ideal of R over U if and only if $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$.*

Proof. First assume that f_R is an SU-generalized bi-ideal of R over U. Then, $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$. Now, let $x \in R$. In the case, when $(f_R \diamond \widetilde{\theta} \diamond f_R)(x) = U$, then it is clear that $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$. Otherwise, we have

$$\begin{aligned}
(f_R \diamond \widetilde{\theta} \diamond f_R)(x) &= [(f_R \diamond \widetilde{\theta}) \diamond f_R](x) \\
&= \bigcap_{x=\sum_{i=1}^{m} a_i b_i} [(f_R \diamond \widetilde{\theta})(a_i) \cup f_R(b_i)] \\
&= \bigcap_{x=\sum_{i=1}^{m} a_i b_i} [(\bigcap_{a_i=\sum_{i=1}^{m_i} a_{i_k} b_{i_k}} (f_R(a_{i_k}) \cup \widetilde{\theta}(b_{i_k}) \cup f_R(b_i)] \\
&= \bigcap_{x=\sum_{i=1}^{m'} a_i b_i c_i} (f_R(a_i) \cup f_R(c_i)) \\
&\supseteq \bigcap_{x=\sum_{i=1}^{m'} a_i b_i c_i} f_R(\sum_{i=1}^{m'} a_i b_i c_i) \\
&= f_R(x)
\end{aligned}$$

Hence, $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$. Here, note that if $x \neq \sum_{i=1}^{m} a_i b_i$, then $(f_R \diamond \widetilde{\theta})(x) = U$, and so, $(f_R \diamond \widetilde{\theta} \diamond f_R)(x) = U \supseteq f_R(x)$.

For the converse, assume that $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$. Let $x, y, z \in R$. Then, since $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$, we have

$$\begin{aligned}
f_R(xyz) &\subseteq (f_R \diamond \widetilde{\theta} \diamond f_R)(xyz) \\
&= (f_R \diamond (\widetilde{\theta} \diamond f_R))(xyz) \\
&= \bigcap_{xyz=\sum_{i=1}^{m} x_i y_i} f_R(x_i) \cup (\widetilde{\theta} \diamond f_R)(y_i) \\
&\subseteq f_R(x) \cup (\widetilde{\theta} \diamond f_R)(yz) \\
&= f_R(x) \cup (\bigcap_{yz=\sum_{i=1}^{m} p_i q_i} \widetilde{\theta}(p_i) \cup f_R(q_i)) \\
&\subseteq f_R(x) \cup (\widetilde{\theta}(y) \cup f_R(z)) \\
&= f_R(x) \cup (\emptyset \cup f_R(z)) \\
&= f_R(x) \cup f_R(z)
\end{aligned}$$

Thus, f_R is an SU- generalized bi-ideal of R over U. This completes the proof. □

Corollary 5.4. *Let f_R be a soft set. Then the following conditions are equivalent:*

1) $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$.

2) $f_R(\sum_{i=1}^{m} x_i y_i z_i) \subseteq f_R(x_i) \cup f_R(z_i)$ *for all* $1 \leq i \leq m$.

Theorem 5.5. *A non-empty subset G of a ring R is a generalized bi-ideal of R if and only if the soft subset f_R defined by*

$$f_R(x) = \begin{cases} \alpha, & \text{if } x \in R \setminus G, \\ \beta, & \text{if } x \in G \end{cases}$$

is an SU-generalized bi-ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Theorem 5.6. *Let X be a nonempty subset of a ring R. Then, X is a generalized bi-ideal of R if and only if $\mathcal{S}_{X^c}$ is an SU-generalized bi-ideal of R over U.*

It is known that every left (right, two sided) ideal of a ring R is a bi-ideal of R. Moreover, we have the following:

Theorem 5.7. *Every SU-left (right, two sided) ideal of a ring R over U is an SU-generalized bi-ideal of R over U.*

Proof. Let f_R be an SU-left (right, two sided) ideal of R over U and $x, y, z \in R$. Then,

$$f_R(xyz) \subseteq f_R((xy)z) \subseteq f_R(z) \subseteq f_R(x) \cup f_R(z)$$

Thus, f_R is an SU-generalized bi-ideal of R. □

Theorem 5.8. *Let f_R be any soft subset of a ring R and g_R be any SU-bi-ideal of R over U. Then, the soft int-uni products $f_R \diamond g_R$ and $g_R \diamond f_R$ are SU-generalized bi-ideals of R over U.*

Proof. The proof is given for $f_R \diamond g_R$. One can easily show that $(f_R \diamond g_R)(x - y) \widetilde{\subseteq} (f_R \diamond g_R)(x) \cup (f_R \diamond g_R)(y)$ for all $x, y \in R$ as shown in the proof of Lemma 4.5. Moreover,

$$\begin{aligned} (f_R \diamond g_R) \diamond \widetilde{\theta} \diamond (f_R \diamond g_R) &= f_R \diamond (g_R \diamond (\widetilde{\theta} \diamond f_R) \diamond g_R) \\ &\widetilde{\supseteq} f_R \diamond (g_R \diamond \widetilde{\theta} \diamond g_R) \\ &\widetilde{\supseteq} f_R \diamond g_R \end{aligned}$$

It follows that $f_R \diamond g_R$ is an SU-generalized bi-ideal of R over U. It can be seen in a similar way that $g_R \diamond f_R$ is an SU-generalized bi-ideal of R over U. This completes the proof. □

Proposition 5.9. *Let f_R and f_T be SU-generalized bi-ideals over U. Then, $f_R \vee f_T$ is an SU-generalized bi-ideal of $R \times T$ over U.*

Proposition 5.10. *If f_R and h_R are two SU-generalized bi-ideals of R over U, then so is $f_R \widetilde{\cup} h_R$ of R over U.*

Proposition 5.11. *Let f_R be a soft set over U and α be a subset of U such that $\alpha \in Im(f_R)$. If f_R is an SU-generalized bi-ideal of R over U, then $\mathcal{L}(f_R; \alpha)$ is a generalized bi-ideal of R.*

Definition 5.12. *If f_R is an SU-generalized bi-ideal of R over U, then generalized bi-ideals $\mathcal{L}(f_R; \alpha)$ are called lower α generalized bi-ideals of f_R.*

Proposition 5.13. *Let f_R be a soft set over U, $\mathcal{L}(f_R; \alpha)$ be lower α generalized bi-ideals of f_R for each $\alpha \subseteq U$ and $Im(f_R)$ be an ordered set by inclusion. Then, f_R is an SU-generalized bi-ideal of R over U.*

Proposition 5.14. *Let f_R and f_T be soft sets over U and Ψ be a ring isomorphism from R to T. If f_R is an SU-generalized bi-ideal of R over U, then so is $\Psi(f_R)$ of T over U.*

Proposition 5.15. *Let f_R and f_T be soft sets over U and Ψ be a ring homomorphism from R to T. If f_T is an SU-generalized bi-ideal of T over U, then so is $\Psi^{-1}(f_T)$ of R over U.*

6. Regular rings

In this section, regular ring is characterized in terms of SU-interior ideals, SU-quasi-ideals and SU-generalized-bi-ideals.

Proposition 6.1. *[5] For a ring R, the following conditions are equivalent:*

1) R is regular.

2) $RL = R \cap L$ for every right ideal R and left ideal L of R.

3) $ARA = A$ for every quasi-ideal A of R.

Theorem 6.2. *For a ring R, the following conditions are equivalent:*

1) R is regular.

2) $f_R = f_R \diamond \widetilde{\theta} \diamond f_R$ for every SU-generalized bi-ideal f_R of R over U.

3) $f_R = f_R \diamond \widetilde{\theta} \diamond f_R$ for every SU-bi-ideal f_R of R over U.

4) $f_R = f_R \diamond \widetilde{\theta} \diamond f_R$ for every SU-quasi-ideal f_R of R over U.

Proof. First assume that (1) holds. Let f_R be any SU-generalized bi-ideal f_R of R over U and R be any element of R. Then, since R is regular, there exists an element $x \in R$ such that $s = sxs$. Thus, we have;

$$\begin{aligned}
(f_R \diamond \widetilde{\theta} \diamond f_R)(s) &= [(f_R \diamond \widetilde{\theta}) \diamond f_R](s) \\
&= \bigcap_{s=\sum_{i=1}^{m} a_i b_i} [(f_R \diamond \widetilde{\theta})(a_i) \cup f_R(b_i)] \\
&\subseteq (f_R \diamond \widetilde{\theta})(sx) \cup f_R(s) \\
&= \bigcap_{sx=\sum_{i=1}^{m} n_i k_i} \{(f_R(n_i) \cup \widetilde{\theta}(k_i)\} \cup f_R(s) \\
&\subseteq (f_R(s) \cup \widetilde{\theta}(x)) \cup f_R(s) \\
&= f_R(s)
\end{aligned}$$

and so, we have $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\subseteq} f_R$. Since f_R is an SU-generalized bi-ideal of R, $f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$. Thus, $f_R \diamond \widetilde{\theta} \diamond f_R = f_R$ which means that (1) implies (2).

(2) implies (3) and (3) implies (4) is obvious. Assume that (4) holds. In order to show that R is regular, we need to illustrate that $ARA = A$ for every quasi-ideal A of R. Let A be any quasi-ideal of R. Then, since

$$ARA \subseteq A(RR) \cup (RR)A \subseteq AR \cup RA \subseteq A,$$

$ARA \subseteq A$. Therefore, it is enough to show that $A \subseteq ARA$. Conversely, let $a \in A$ and $a \notin ARA$. Then, by Theorem 5.6, the soft characteristic function $\mathcal{S}_{A^c}$ of A is an SU-quasi-ideal of S. Thus, $(\mathcal{S}_{A^c})(a) = \emptyset$. Since $a \notin ASA$, this means that there do not exist $x, z \in A$ and $y \in R$ such that $a = xyz$. Since $\widetilde{\theta}$ is an SU-quasi ideal of S, we have,

$$(\mathcal{S}_{A^c} \diamond \widetilde{\theta} \diamond \mathcal{S}_{A^c})(a) = U$$

But this is a contradiction. Hence $A = ARA$. It follows by Proposition 3.13 that R is regular, so (4) implies (1). □

Theorem 6.3. *For a ring R the following conditions are equivalent:*

1) R is regular.

2) $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$ for every SU-quasi-ideal f_R of R and SU-ideal g_R of R over U.

3) $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$ for every SU-quasi-ideal f_R of R and SU-interior ideal g_R of R over U.

4) $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$ for every SU-bi-ideal f_R of R and SU-ideal g_R of R over U.

5) $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$ for every SU-bi-ideal f_R of R and SU-interior ideal g_R of R over U.

6) $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$ for every SU-generalized bi-ideal f_R of R and SU-ideal g_R of R over U.

7) $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$ for every SU-generalized bi-ideal f_R of R and SU-interior ideal g_R of R over U.

Proof. First assume that (1) holds. Let f_R be any SU-generalized bi-ideal and g_R be any SU-interior ideal of R over U. Then,

$$f_R \diamond g_R \diamond f_R \widetilde{\supseteq} f_R \diamond \widetilde{\theta} \diamond f_R \widetilde{\supseteq} f_R$$

and

$$f_R \diamond g_R \diamond f_R \widetilde{\supseteq} \widetilde{\theta} \diamond g_R \diamond \widetilde{\theta} \widetilde{\supseteq} g_R$$

so $f_R \diamond g_R \diamond f_R \widetilde{\supseteq} f_R \widetilde{\cup} g_R$. To show that $f_R \widetilde{\cup} g_R \widetilde{\supseteq} f_R \diamond g_R \diamond f_R$ holds, let s be any element of R. Since R is regular, there exists an element x in R such that

$$s = sxs \ (s = sx(sxs))$$

Since g_R is an SU-interior ideal of R, we have

$$\begin{aligned}
(f_R \diamond g_R \diamond f_R)(s) &= [f_R \diamond (g_R \diamond f_R)](s) \\
&= \bigcap_{s=\sum_{i=1}^{m} n_i t_i} [f_R(n_i) \cup (g_R \diamond f_R)(t_i)] \\
&\subseteq f_R(s) \cup (g_R \diamond f_R)(xsxs) \\
&= f_R(s) \cup \{ \bigcap_{xsxs=\sum_{i=1}^{m} y_i z_i} [g_R(y_i) \cup f_R(z_i)] \} \\
&= f_R(s) \cup (g_R(xsx) \cup f_R(s)) \\
&\subseteq f_R(s) \cup g_R(s) \cup f_R(s) \\
&\subseteq f_R(s) \cup g_R(s) \\
&= (f_R \widetilde{\cup} g_R)(s)
\end{aligned}$$

so we have $f_R \widetilde{\cup} g_R \widetilde{\subseteq} f_R \diamond g_R \diamond f_R$. Thus we obtain that $f_R \widetilde{\cup} g_R = f_R \diamond g_R \diamond f_R$, hence (1) implies (7).

It is clear that (7) implies (5), (5) implies (3), and that (3) implies (2). Also, (7) implies (6), (6) implies (4) and (4) implies (2) is obvious.

Assume that (2) holds. In order to show that R is regular, it is enough to show that $f_R = f_R \diamond \widetilde{\theta} \diamond f_R$ for all SU-quasi-ideal f_R of R over U by Theorem 6.2. Since $\widetilde{\theta}$ is an SU-ideal of R, we have

$$f_R = f_R \widetilde{\cup} \widetilde{\theta} = f_R \diamond \widetilde{\theta} \diamond f_R.$$

Thus, R is regular and (2) implies (1). This completes the proof. □

Theorem 6.4. *For a ring R the following conditions are equivalent:*

1) R is regular.

2) $f_R\widetilde{\cup}g_R \supseteq f_R \diamond g_R$ *for every SU-quasi-ideal* f_R *of R and SU-left ideal* g_R *of R over U.*

3) $f_R\widetilde{\cup}g_R \supseteq f_R \diamond g_R$*for every SU-bi-ideal* f_R *of R and SU-left ideal* g_R *of R over U.*

4) $f_R\widetilde{\cup}g_R \supseteq f_R \diamond g_R$ *for every SU-generalized bi-ideal* f_R *of R and SU-left ideal* g_R *of R over U.*

Proof. First assume that (1) holds. Let f_R be any SU-generalized bi-ideal and g_R be any SU-left ideal of R over U. Let s be any element of R. Then, since R is regular, there exists an element x in R such that $s = sxs$. Thus, we have

$$\begin{aligned}(f_R \diamond g_R)(s) &= \bigcap_{s=\sum_{i=1}^{m} a_ib_i} (f_R(a_i) \cup g_R(b_i)) \\ &\subseteq f_R(s) \cup g_R(xs) \\ &\subseteq (f_R(s) \cup g_R(s)) \\ &= (f_R\widetilde{\cup}g_R)(s)\end{aligned}$$

Thus, $f_R \diamond g_R\widetilde{\subseteq}f_R\widetilde{\cup}g_R$. Hence, we obtain that (1) implies (4).

It is clear that (4) implies (3), (3) implies (2). Assume that (2) holds. Since $f_R\widetilde{\cup}g_R\widetilde{\supseteq}f_R \diamond g_R$ always holds for every SU-right ideal of R is an SU-quasi-ideal of R, we have $f_R\widetilde{\cup}g_R = f_R \diamond g_R$ for every SU-right ideal f_R and SU-left ideal g_R of R. Thus, it follows by Theorem 2.18 that R is regular and (2) implies (1). □

Theorem 6.5. *For a ring R the following conditions are equivalent:*

1) R is regular.

2) $h_R\widetilde{\cup}f_R\widetilde{\cup}g_R\widetilde{\supseteq}h_R \diamond f_R \diamond g_R$ *for every SU-right ideal* h_R*, every SU-quasi-ideal* f_R *and every SU-left ideal* g_R *of R.*

3) $h_R\widetilde{\cup}f_R\widetilde{\cup}g_R\widetilde{\supseteq}h_R \diamond f_R \diamond g_R$ *for every SU-right ideal* h_R*, every SU-bi-ideal* f_R *and every SU-left ideal* g_R *of R.*

4) $h_R\widetilde{\cup}f_R\widetilde{\cup}g_R\widetilde{\supseteq}h_R \diamond f_R \diamond g_R$ *for every SU-right ideal* h_R*, every SU-generalized bi-ideal* f_R *and every SU left-ideal* g_R *of R.*

Proof. Assume that (1) holds. Let h_R, f_R and g_R be any SU-right ideal, SU-generalized bi-ideal and SU-left ideal of R, respectively. Let a be any element of R. Since R is regular, there exists an element x in R such that $a = axa$. Hence, we have:

$$\begin{aligned}(h_R \diamond f_R \diamond g_R)(a) &= [h_R \diamond (f_R \diamond g_R)](a) \\ &= \bigcap_{a=\sum_{i=1}^{m} y_iz_i} [h_R(y_i) \cup (f_R \diamond g_R)(z_i)] \\ &\subseteq h_R(ax) \cup (f_R \diamond g_R)(a) \\ &= h_R(ax) \cup \{ \bigcap_{a=\sum_{i=1}^{m} p_iq_i} [f_R(p_i) \cup g_R(q_i)]\} \\ &\subseteq h_R(a) \cup (f_R(a) \cup g_R(xa)) \\ &\subseteq h_R(a) \cup (f_R(a) \cup g_R(a)) \\ &= (h_R\widetilde{\cup}f_R\widetilde{\cup}g_R)(a)\end{aligned}$$

so we have $h_R \diamond f_R \diamond g_R \widetilde{\subseteq} h_R \widetilde{\cup} f_R \widetilde{\cup} g_R$. Thus, (1) implies (2).

It is clear that (4) implies (3), (3) implies (2). Assume that (2) holds. Let h_R and g_R be any SU-right ideal and SU-left ideal of R, respectively. It is obvious that

$$h_R \diamond g_R \widetilde{\supseteq} h_R \widetilde{\cup} g_R.$$

Since $\widetilde{\theta}$ itself is an SU-quasi-ideal of R, by assumption we have:

$$h_R \widetilde{\cup} g_R = h_R \widetilde{\cup} \widetilde{\theta} \widetilde{\cup} g_R \widetilde{\supseteq} h_R \diamond \widetilde{\theta} \diamond g_R \widetilde{\supseteq} h_R \diamond g_R.$$

It follows that $h_R \widetilde{\cup} g_R \widetilde{\supseteq} h_R \diamond g_R$ for every SU-right ideal h_R and SU-left ideal g_R of R. It follows by Theorem 2.18 that R is regular. Hence, (2) implies (1). This completes the proof. □

Proposition 6.6. *[19] A ring R is regular if and only if every SU-left (right, two-sided) ideal of R is idempotent.*

Proposition 6.7. *Let R be a regular ring and f_R be an SU-quasi-ideal of R. Then,*

$$(\widetilde{\theta} \diamond f_R) \widetilde{\cup} (f_R \diamond \widetilde{\theta}) = f_R.$$

Proof. Let f_R be any SU-quasi-ideal of R. Then, $(\widetilde{\theta} \diamond f_R) \widetilde{\cup} (f_R \diamond \widetilde{\theta}) \widetilde{\supseteq} f_R$. Thus, it suffices to show that $f_R \widetilde{\supseteq} (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (f_R \diamond \widetilde{\theta})$. One can easily show that $f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R)$ is an SU-left ideal of R. In fact,

$$\widetilde{\theta} \diamond (f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R)) = (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond (\widetilde{\theta} \diamond f_R)) = (\widetilde{\theta} \diamond f_R) \widetilde{\cup} ((\widetilde{\theta} \diamond \widetilde{\theta}) \diamond f_R) = (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond f_R) = \widetilde{\theta} \diamond f_R \subseteq f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R).$$

And $(f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R))(x - y) \subseteq (f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R))(x) \cup (f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R))(y)$. Since R is regular, every SU-left (right) ideal of R is idempotent by Proposition 6.6. Thus, we have

$$\begin{aligned}
f_R \widetilde{\supseteq} f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R) &= [f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R)] \diamond [f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R)] \\
&= \{[(f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R)] \diamond f_R\} \widetilde{\cup} \{[(f_R \widetilde{\cup} (\widetilde{\theta} \diamond f_R)] \diamond (\widetilde{\theta} \diamond f_R)\} \\
&= \{((f_R \diamond f_R) \widetilde{\cup} ((\widetilde{\theta} \diamond f_R) \diamond f_R)\} \widetilde{\cup} \{((f_R \diamond (\widetilde{\theta} \diamond f_R)) \widetilde{\cup} ((\widetilde{\theta} \diamond f_R) \diamond (\widetilde{\theta} \diamond f_R)\} \\
&= \{((f_R \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond (f_R \diamond f_R))\} \widetilde{\cup} \{((f_R \diamond (\widetilde{\theta} \diamond f_R)) \widetilde{\cup} ((\widetilde{\theta} \diamond f_R)^2\} \\
&\widetilde{\supseteq} ((\widetilde{\theta} \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond f_R)) \widetilde{\cup} ((\widetilde{\theta} \diamond (\widetilde{\theta} \diamond f_R)) \widetilde{\cup} (\widetilde{\theta} \diamond f_R)^2)(since f_R \diamond f_R \widetilde{\supseteq} f_R) \\
&\widetilde{\supseteq} (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (\widetilde{\theta} \diamond f_R)^2 (since (\widetilde{\theta} \diamond (\widetilde{\theta} \diamond f_R)) \widetilde{\supseteq} \widetilde{\theta} \diamond f_R) \\
&\widetilde{\supseteq} \widetilde{\theta} \diamond f_R
\end{aligned}$$

that is to say $f_R \widetilde{\supseteq} \widetilde{\theta} \diamond f_R$. Similarly, one can show that $f_R \widetilde{\supseteq} f_R \diamond \widetilde{\theta}$. Thus, $f_R \widetilde{\supseteq} (\widetilde{\theta} \diamond f_R) \widetilde{\cup} (f_R \diamond \widetilde{\theta})$ and so,

$$(\widetilde{\theta} \diamond f_R) \widetilde{\cup} (f_R \diamond \widetilde{\theta}) = f_R.$$

□

Theorem 6.8. *Let f_R be a soft set and R be a regular ring. Then, the following conditions are equivalent:*

1) f_R is an SU-quasi-ideal of R.

2) f_R may be presented in the form $f_R = g_R \diamond h_R$, where g_R is an SU-right ideal and h_R is an SU-left ideal of R.

Proof. Assume that (1) holds. Since R is regular, it follows by Theorem 6.2 that $f_R = f_R \diamond \widetilde{\theta} \diamond f_R$, where f_R is an SU-quasi-ideal of R. Thus,

$$f_R = f_R \diamond \widetilde{\theta} \diamond f_R = f_R \diamond (\widetilde{\theta} \diamond \widetilde{\theta}) \diamond f_R = (f_R \diamond \widetilde{\theta}) \diamond (\widetilde{\theta} \diamond f_R)$$

Since $f_R \diamond \widetilde{\theta}$ is an SU-right ideal of R and $\widetilde{\theta} \diamond f_R$ is an SU-right ideal of R, (1) implies (2).

Conversely, assume that $f_R = g_R \diamond h_R$, where g_R is an SU-right ideal and h_R is an SU-left ideal of R. Then, by Proposition 4.11, $g_R \diamond h_R$ is an SU-quasi-ideal of R. □

Proposition 6.9. *Let R be a regular ring and f_R be an SU-quasi-ideal of R. Then, $(f_R)^2 = (f_R)^3$.*

Proof. Let R be a regular ring and f_R be an SU-quasi-ideal of R. Then, by Corollary 4.10, $(f_R)^2$ is an SU-quasi-ideal of R and by Theorem 6.2,

$$(f_R)^2 = (f_R)^2 \diamond \widetilde{\theta} \diamond (f_R)^2 = f_R \diamond f_R \diamond \widetilde{\theta} \diamond f_R \diamond f_R = f_R \diamond (f_R \diamond \widetilde{\theta} \diamond f_R) \diamond f_R = f_R \diamond f_R \diamond f_R = (f_R)^3$$

□

7. Regular duo rings

In this section, a left (right) duo ring is characterized in terms of SU-ideals. A ring R is called *left (right) duo* if every left (right) ideal of R is a two-sided ideal of R. A ring R is *duo* if it is both left and right duo.

Definition 7.1. *A ring R is called soft left (right) duo if every SU-left (right) ideal of R is an SU-ideal of R and is called soft duo, if it is both soft left and soft right duo.*

Theorem 7.2. *[19] For a regular ring R, the following conditions are equivalent:*

1) R is duo.

2) R is soft duo.

Theorem 7.3. *[20] For a ring R, the following conditions are equivalent:*

1) R is a regular duo ring.

2) $A \cap B = AB$ for every left ideal A and every right ideal B of R.

3) $Q^2 = Q$ for every quasi-ideal of R. (That is, every quasi-ideal is idempotent.)

4) $EQE = E \cap Q$ for every ideal E and every quasi-ideal Q of R.

Theorem 7.4. *For a ring R, the following conditions are equivalent:*

1) R is a regular duo ring.

2) R is a regular soft duo ring.

3) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-bi-ideals f_R and g_R of R.

4) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-bi-ideal f_R and for all SU-quasi-ideal g_R of R.

5) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-bi-ideal f_R and and for all SU-right ideal g_R of R

6) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-quasi-ideal f_R and for all SU-bi-ideal g_R of R.

7) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-quasi-ideals f_R and g_R of R.

8) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-quasi-ideal f_R and for all SU-right ideal g_R of R.

9) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-left ideal f_R and for all SU-bi-ideal g_R of R.

10) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ for all SU-left ideal f_R and for all SU-right ideal g_R of R.

11) $f_R \diamond g_R = f_R \widetilde{\cup} g_R$ and $h_S \diamond k_S = h_S \widetilde{\cup} k_S$ for all SU-right ideals f_R and g_R of R and for all SU-left ideal h_S and k_S of R.

12) Every SU-quasi-ideal of R is idempotent.

Proof. The equivalence of (1) and (2) follows from Theorem 7.3. Assume that (2) holds. Let f_R and g_R be any SU-bi-ideals of R. Then, f_R is an SU-right ideal of R and g_R is an SU-left ideal of R. Since R is regular,

$$f_R \diamond g_R = f_R \widetilde{\cup} g_R$$

Thus, (2) implies (3). It is clear that (3) implies (4), (4) implies (5), (5) implies (8), (8) implies (11), (11) implies (3), (3) implies (6), (6) implies (7), (7) implies (8) and (6) implies (9), (9) implies (10), (10) implies (11).

Assume that (11) holds. Let A and B be any left ideal and right ideal of R, respectively. Let a be any element of $A \cap B$ and $a \notin AB$. Then, $a \in A$ and $a \in B$ and there do not exist $x \in A$ and $y \in B$ such that $a = xy$. Since $\mathcal{S}_{A^c}$ and $\mathcal{S}_{B^c}$ is an SU-left ideal and SU-right ideal of S, respectively, we have

$$\mathcal{S}_{A^c}(a) = \mathcal{S}_{B^c}(a) = \emptyset.$$

and

$$(\mathcal{S}_{A^c} \diamond \mathcal{S}_{B^c})(a) = U$$

But this is a contradiction, so $a \in AB$. Thus, $A \cap B \subseteq AB$. For the converse inclusion, let a be any element of AB and $a \notin A \cap B$. Then, there exist $y \in A$ and $z \in B$ such that $a = yz$. Thus,

$$(\mathcal{S}_{A^c} \widetilde{\cup} \mathcal{S}_{B^c})(a) = U$$

and

$$(\mathcal{S}_{A^c} \diamond \mathcal{S}_{B^c})(a) = \bigcap_{a=\sum_{i=1}^{k} m_i n_i} (\mathcal{S}_{A^c}(m_i) \cup \mathcal{S}_{B^c}(n_i)) \subseteq (\mathcal{S}_{A^c}(y) \cup \mathcal{S}_{B^c}(z)) = \emptyset.$$

Hence, $(\mathcal{S}_{A^c} \diamond \mathcal{S}_{B^c})(a) = \emptyset$. But this is a contradiction. This implies that $a \in A \cap B$ and that $AB \subseteq A \cap B$. Thus, we have $AB = A \cap B$. It follows by Theorem 7.3 that R is a regular duo ring. Thus (11) implies (1). It is clear that (7) implies (12) by taking $g_R = f_R$.

Conversely, assume that (12) holds. Let Q be any quasi-ideal of S and a be any element of Q and $a \notin QQ$. Then, $\mathcal{S}_{Q^c}$ is an SU-quasi-ideal of S. Thus, we have $\mathcal{S}_{Q^c}(a) = \emptyset$ and since there do not exist $y, z \in Q$ such that $a = yz$,

$$(\mathcal{S}_{Q^c} \diamond \mathcal{S}_{Q^c})(a) = U$$

But this is a contradiction. Hence, we have $a \in Q^2$ and $Q \subseteq Q^2$. Since the converse inclusion always holds, $Q = Q^2$. It follows by Theorem 7.3 that R is a regular duo ring and that (12) implies (1). This completes the proof. □

Theorem 7.5. *For a ring R, the following conditions are equivalent:*

1) R is a regular duo ring.

2) $f_R \diamond g_R \diamond f_R = f_R \widetilde{\cup} g_R$ for every SU-ideal f_R and every SU-bi-ideal g_R of R.

3) $f_R \diamond g_R \diamond f_R = f_R \widetilde{\cup} g_R$ for every SU-ideal f_R and every SU-quasi-ideal g_R of R.

Proof. First assume that (1) holds. Let f_R and g_R be any SU-bi-ideal and any SU-ideal of R, respectively. Then, we have

$$f_R \diamond g_R \diamond f_R \widetilde{\supseteq} (f_R \diamond \widetilde{\theta}) \diamond \widetilde{\theta} = f_R \diamond (\widetilde{\theta} \diamond \widetilde{\theta}) \widetilde{\supseteq} f_R \diamond \widetilde{\theta} \widetilde{\supseteq} f_R$$

On the other hand, since R is regular and duo, f_R is an SU-ideal of R. Hence, we have

$$f_R \diamond g_R \diamond f_R \widetilde{\supseteq} (\widetilde{\theta} \diamond g_R) \diamond \widetilde{\theta} \widetilde{\supseteq} g_R \diamond \widetilde{\theta} \widetilde{\supseteq} g_R$$

and so

$$f_R \diamond g_R \diamond f_R \widetilde{\supseteq} f_R \widetilde{\cup} g_R$$

In order to show the converse inclusion, let a be any element of R. Then, since R is regular, there exists an element x in R such that

$$a = axa = (axa)xa$$

Thus, we have

$$\begin{aligned}
(f_R \diamond g_R \diamond f_R)(a) &= [f_R \diamond (g_R \diamond f_R)](a) \\
&= \bigcap_{a=\sum_{i=1}^{n} x_i y_i} [f_R(x_i) \cup (g_R \diamond f_R)(y_i)] \\
&\subseteq f_R(ax) \cup (g_R \diamond f_R)(axa) \\
&= f_R(ax) \cup \{ \bigcap_{axa=\sum_{i=1}^{n} p_i q_i} [g_R(p_i) \diamond f_R(q_i)]\} \\
&\subseteq f_R(ax) \cup (g_R(a) \cup f_R(xa)) \\
&\subseteq f_R(a) \cup (g_R(a) \cup f_R(a)) \\
&= f_R(a) \cup g_R(a) \\
&= (f_R \widetilde{\cup} g_R)(a)
\end{aligned}$$

and so $f_R \diamond g_R \diamond f_R \widetilde{\subseteq} f_R \widetilde{\cup} g_R$ Thus, we obtain that

$$f_R \diamond g_R \diamond f_R = f_R \widetilde{\cup} g_R.$$

Hence, (1) implies (2). It is clear that (2) implies (3).

Assume that (3) holds. Let E and Q any two-sided ideal and quasi-ideal of S, respectively and a be any element of $E \cap Q$ and $a \notin EQE$. Then, $a \in E$ and $a \in Q$ and there do not exist $x, z \in E$ and $y \in Q$ such that $a = xyz$. Since $\mathcal{S}_{E^c}$ and $\mathcal{S}_{Q^c}$ is an SU-ideal and SU-quasi-ideal of S, respectively, we have

$$\mathcal{S}_{E^c}(a) = \mathcal{S}_{Q^c}(a) = \emptyset.$$

and

$$(\mathcal{S}_{E^c} \diamond \mathcal{S}_{Q^c} \diamond \mathcal{S}_{E^c})(a) = U$$

But, this is a contradiction and so $a \in EQE$. Thus, $E \cap Q \subseteq EQE$. For the converse inclusion, let a be any element of EQE and $a \notin E \cap Q$. Then, there exist $x, z \in E$ and $y \in Q$ such that $a = xyz$. Thus,

$$(\mathcal{S}_{E^c} \widetilde{\cup} \mathcal{S}_{Q^c})(a) = U$$

and

$$(\mathcal{S}_{E^c} \diamond \mathcal{S}_{Q^c} \diamond \mathcal{S}_{E^c})(a) = \emptyset$$

But this is a contradiction and so $a \in E \cap Q$. Thus, $EQE \subseteq E \cap Q$ and so $EQE = E \cap Q$. It follows from Theorem 7.3 that R is regular duo. Hence, (3) implies (1). This completes the proof. □

8. Intra-regular rings

In this section, an intra-regular ring is characterized in terms of SU-interior ideals, SU-quasi-ideals and SU-generalized-bi-ideals. A ring R is called *intra-regular* [20] if for every element a of R there exist elements x_i and y_i in R such that

$$a = \sum_{i=1}^{n} x_i a^2 y_i = \sum_{i=1}^{n} (x_i a)(a y_i).$$

Proposition 8.1. *4.4 For a soft set f_R of an intra-regular ring R, the following conditions are equivalent:*

1) f_R is an SU-ideal of R.

2) f_R is an SU-interior ideal of R.

Proof. (1) implies (2) is clear. Assume that (2) holds. Let a and b be any elements of R. Then, since R is intra-regular, there exist elements x_i, y_i, u_i and v_i in R such that $a = \sum_{i=1}^{n} x_i a^2 y_i$ and $b = \sum_{i=1}^{n} u_i b^2 v_i$ for all $1 \leq i \leq n$. Since f_R is an SU-interior ideal of R, we have

$$f_R(ab) = f_R((\sum_{i=1}^{n} x_i a^2 y_i)b) = f_R(\sum_{i=1}^{n} x_i a^2 y_i b) \subseteq \bigcap_{1\leq i\leq n} f_R(x_i a^2 y_i b) = \bigcap_{1\leq i\leq n} f_R((x_i a)ay(_i b)) \subseteq \bigcap_{1\leq i\leq n} f_R(a) = f_R(a)$$

and

$$f_R(ab) = f_R(a(\sum_{i=1}^{n} u_i b^2 v_i)) = f_R(\sum_{i=1}^{n} a u_i b^2 v_i) \subseteq \bigcap_{1\leq i\leq n} f_R(a u_i b^2 v_i) = \bigcap_{1\leq i\leq n} f_R((a u_i b)b(v_i)) \subseteq \bigcap_{1\leq i\leq n} f_R(b) = f_R(b)$$

Hence, f_R is an SU-ideal of R and (2) implies (1). □

Theorem 8.2. *For a ring R, the following conditions are equivalent:*

1) R is intra-regular.

2) $f_R(a) = f_R(a^2)$ for all SU-interior ideal of R and for all $a \in R$.

Proof. First assume that (1) holds. Let f_R be any SU-interior ideal of R and a any element of R. Since R is intra-regular, there exist elements x_i and y_i in R such that $a = \sum_{i=1}^{n} x_i a^2 y_i$. Thus, for all $1 \leq i \leq n$,

$$f_R(a) = f_R(\sum_{i=1}^{n} x_i a^2 y_i) \subseteq \bigcap_{1\leq i\leq n} f_R(x_i a^2 y_i) \subseteq \bigcap_{1\leq i\leq n} f_R(x_i a^2) \subseteq \bigcap_{1\leq i\leq n} f_R(a^2) = \bigcap_{1\leq i\leq n} f_R(aa) \subseteq \bigcap_{1\leq i\leq n} f_R(a) = f_R(a)$$

so, we have $f_R(a) = f_R(a^2)$. Hence, (1) implies (3). Now assume that (3) holds. It is known that $I[a^2]$ is an interior-ideal of R. Thus, the soft characteristic function $\mathcal{S}_{(I[a^2])^c}$ is an SU-interior ideal of R. Since $a^2 \in I[a^2]$, we have;

$$\mathcal{S}_{(I[a^2])^c}(a) = \mathcal{S}_{(I[a^2])^c}(a^2) = \emptyset$$

Thus, $a \in I[a^2] = m\{a^2\} + n\{a^4\} + Sa^2S$. Here, R is intra-regular. Thus, (3) implies (1). This completes the proof. □

Theorem 8.3. *Let R be an intra-regular ring. Then, for every SU-interior ideal f_R of R,*

$$f_R(ab) = f_R(ba)$$

for all $a, b \in R$.

Proof. Let f_R be an SU-ideal of an intra-regular ring R. Then, by Theorem 8.2, we have;

$$f_R(ab) = f_R((ab)^2) = f_R(a(ba)b) \subseteq f_R(ba) = f_R((ba)^2) = f_R(b(ab)a) \subseteq f_R(ab)$$

so, we have $f_R(ab) = f_R(ba)$. This completes the proof. □

Theorem 8.4. *[20] A ring R is regular and intra-regular if and only if every quasi-ideal of R is idempotent.*

Theorem 8.5. *For a ring R, the following conditions are equivalent:*

1) *R is both regular and intra-regular.*

2) *$f_R \diamond f_R = f_R$ for every SU-quasi-ideal f_R of R. (That is, every SU-quasi-ideal of R is idempotent).*

3) *$f_R \diamond f_R = f_R$ for every SU-bi-ideal f_R of R. (That is, every SU-bi-ideal of R is idempotent).*

4) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-quasi-ideals f_R and g_R of R.*

5) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-quasi-ideal f_R and SU-bi-ideal g_R of R.*

6) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-quasi-ideal f_R and for every SU-generalized bi-ideal g_R of R.*

7) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-bi-ideal f_R and for every SU-quasi-ideal g_R of R.*

8) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-bi-ideals f_R and g_R of R.*

9) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-bi-ideal f_R and for every SU-generalized bi-ideal g_R of R.*

10) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-generalized-bi-ideal f_R and for every SU-quasi-ideal g_R of R.*

11) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-generalized-bi-ideal f_R and for every SU-bi-ideal g_R of R.*

12) *$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-generalized bi-ideals f_R and g_R of R.*

Proof. First assume that (1) holds. In order to show that (12) holds, let f_R and g_R be SU-generalized bi-ideals of R and $a \in R$. Since R is intra-regular, there exist elements y_i and z_i in R such that $a = \sum_{i=1}^{n} y_i a^2 z_i$ for every element a of R. Thus,

$$a = axa = axaxa = ax(\sum_{i=1}^{n} y_i aaz_i)xa = \sum_{i=1}^{n} (axy_ia)(az_ixa)$$

Then, for all $1 \leq i \leq n$, we have

$$\begin{aligned}
(f_R \diamond g_R)(a) &= \bigcap_{a=\sum_{i=1}^{m} b_ic_i} (f_R(b_i) \cup g_R(c_i)) \\
&\subseteq f_R(a(xy_i)a) \cup g_R(a(z_ix)a) \\
&\subseteq f_R(a) \cup g_R(a) \\
&= (f_R \widetilde{\cup} g_R)(a)
\end{aligned}$$

and so we have $f_R \widetilde{\cup} g_R \widetilde{\supseteq} f_R \diamond g_R$. This shows that (1) implies (12).

It is obvious that (12) implies (11), (11) implies (10), (10) implies (4), (4) implies (2) and (12) implies (9), (9) implies (8), (8) implies (7), (7) implies (4), (12) implies (6), (6) implies (5), (5) implies (4) and (8) implies (3) and (3) implies (2).

Assume that (2) holds. Let Q be quasi-ideal of S and a be any element of Q. Then, $QQ \subseteq Q$ always holds. We show that $Q \subseteq QQ$. Conversely, let $x \in Q$ and $x \notin QQ$. Then, there do not exist $y, z \in Q$ such that $x = yz$. Since Q is a quasi-ideal of S, the soft characteristic function $\mathcal{S}_{Q^c}$ is an SU-quasi-ideal of S. So we have, $\mathcal{S}_{Q^c})(x) = \emptyset$ and

$$(\mathcal{S}_{Q^c} \diamond \mathcal{S}_{Q^c})(x) = \bigcap_{x=yz} (\mathcal{S}_{Q^c}(y) \cup \mathcal{S}_{Q^c}(z)) = U$$

But, this contradicts with our hypothesis. So, $a \in QQ$. Thus, $Q \subseteq QQ$ and so $Q = QQ = Q^2$. It follows that Q is both regular and intra-regular, so (2) implies (1) by Theorem 8.4. □

Theorem 8.6. *[19] For a ring R the following conditions are equivalent:*

1) R is intra-regular.

2) $g_R \widetilde{\cup} f_R \widetilde{\supseteq} g_R \diamond f_R$ for every SU-right ideal f_R of R and SU-left ideal g_R of R over U.

Theorem 8.7. *For a ring R the following conditions are equivalent:*

1) R is both regular and intra-regular.

2) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-right ideal f_R and for every SU-left ideal g_R of R.

3) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-right ideal f_R and for every SU-quasi-ideal g_R of R.

4) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-right ideal f_R and for every SU-bi-ideal g_R of R.

5) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-right ideal f_R and for every SU-generalized bi-ideal g_R of R.

6) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-left ideal f_R and for every SU-quasi-ideal g_R of R.

7) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-left ideal f_R and for every SU-bi-ideal g_R of R.

8) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-left ideal f_R and for every SU-generalized bi-ideal g_R of R.

9) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-quasi-ideals f_R and g_R of R.

10) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-quasi-ideal f_R and for every SU-bi-ideal g_R of R.

11) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-quasi-ideal f_R and for every SU-generalized bi-ideal g_R of R.

12) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-bi-ideals f_R and g_R of R.

13) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-bi-ideal f_R and for every SU-generalized bi-ideal g_R of R.

14) $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ for every SU-generalized bi-ideals f_R and g_R of R.

Proof. Assume that (1) holds. Let f_R and g_R be any SU-generalized bi-ideals of R. Then, it follows by Theorem 8.6 that $f_R \widetilde{\cup} g_R \widetilde{\supseteq} f_R \diamond g_R$. Moreover, we have

$$f_R \widetilde{\cup} g_R = g_R \widetilde{\cup} f_R \widetilde{\supseteq} g_R \diamond f_R.$$

Thus, we have $f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R)$ and so (1) implies (14). It is obvious that (14) implies (13), (13) implies (12), (12) implies (9), (9) implies (6) and (6) implies (2) and (14) implies (11), (11) implies (10), (10) implies (9) and (14) implies (8), (8) implies (7), (7) implies (6) and (14) implies (5), (5) implies (4), (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_R and g_R be any SU-right ideal and SU-left ideal of R, respectively. Then,

$$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R) \widetilde{\supseteq} g_R \diamond f_R$$

It follows by Theorem 8.6 that R is intra-regular. On the other hand,

$$f_R \widetilde{\cup} g_R \widetilde{\supseteq} (f_R \diamond g_R) \widetilde{\cup} (g_R \diamond f_R) \widetilde{\supseteq} f_R \diamond g_R.$$

Since $f_R \diamond g_R \widetilde{\supseteq} f_R \widetilde{\cup} g_R$ always holds, we have $f_R \diamond g_R = f_R \widetilde{\cup} g_R$. Thus, it follows by Theorem 2.18 that R is regular. Thus, (2) implies (1). $\square$

9. Strongly regular rings

In this section, a strongly regular ring is characterized in terms of SU-ideals. An element a of R is called a *strongly regular* if there exists an element $x \in R$ such that

$$a = xa^2 = a^2x.$$

for all $a \in R$. Such a ring is regular and duo.

Theorem 9.1. *For a ring R the following conditions are equivalent:*

1) R is strongly regular.

2) Every quasi-ideal of R is semiprime.

3) Every bi-ideal of R is semiprime.

4) Every generalized bi-ideal of R is semiprime.

5) Every SU-quasi-ideal of R is soft union semiprime.

6) Every SU-bi-ideal of R is soft union semiprime.

7) Every SU-generalized bi-ideal of R is soft union semiprime.

8) $f_R(a) = f_R(a^2)$ *for every SU-quasi-ideal* f_R *of R and for all* $a \in R$.

9) $f_R(a) = f_R(a^2)$ *for every SU-bi-ideal* f_R *of R and for all* $a \in R$.

10) $f_R(a) = f_R(a^2)$ *for every SU-generalized bi-ideal* f_R *of R and for all* $a \in R$.

Proof. First assume that (1) holds. Let f_R be any SU-generalized bi-ideal of R. Since R is strongly regular, there exists an element $x \in R$ such that $a = a^2xa^2$. Thus, we have

$$f_R(a) = f_R(a^2xa^2) \subseteq f_R(a^2) \cup f_R(a^2) = f_R(a^2) = f_R(aa) = f_R(a(a^2xa^2) =$$

$$f_R(a(a^2xa)a) \subseteq f_R(a) \cup f_R(a) = f_R(a)$$

and so, $f_R(a) = f_R(a^2)$. Thus (1) implies (10).

It is clear that (10) implies (9), (9) implies (8), (8) implies (5) and (10) implies (7), (7) implies (6), (6) implies (5) and that (10) implies (4), (4) implies (3) and (3) implies (2).

Assume that (5) holds. Let Q be any quasi-ideal of S and $a^2 \in Q$ and $a \notin Q$. Since the soft characteristic function $\mathcal{S}_{Q^c}$ is an SU-quasi-ideal of S, it is soft union semiprime by hypothesis. Thus,

$$\mathcal{S}_{Q^c}(a) = U \subseteq \mathcal{S}_{Q^c}(a^2) = \emptyset$$

But, this is a contradiction. Hence, $a \in Q$ and so Q is semiprime. Thus (5) implies (2).

Finally assume that (2) holds. Let a be any element of R. Then, since the principal ideal $Q[a^2]$ generated by a^2 is quasi-ideal and so by assumption semiprime and since $a^2 \in Q[a^2]$,

$$\mathcal{S}_{(Q[a^2])^c}(a) = \mathcal{S}_{(Q[a^2])^c}(a^2) = \emptyset$$

implying that

$$a \in Q[a^2] = m\{a^2\} + n\{a^4\} + (a^2S \cap Sa^2).$$

Hence, R is strongly regular. Thus (2) implies (1). □

10. Conclusion

In this paper, the concepts of soft union interior ideals, soft union quasi-ideals and soft union generalized bi-ideals of rings have been introduced and studied. Moreover, regular, regular duo, intra-regular and strongly regular rings have been characterized by the properties of these soft union ideals. Based on these results, some further work can be done on the properties of soft union rings, which may be useful to characterize the classical rings, especially in the mean of regularity.

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