Soft Union Interior Ideals, Quasi-Ideals and Generalized Bi-Ideals of Rings

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Abstract. In this paper, soft union interior ideals, quasi-ideals and generalized bi-ideals of rings are defined and their properties are obtained and the interrelations of them are given. Moreover regular, regular duo, intra-regular and strongly regular rings are characterized in terms of these soft union ideals. This paper is a following study of [19].

1. Introduction

Probability theory, fuzzy set theory, rough set theory, vague set theory and the interval mathematics are useful approaches to describe uncertainty. However, each of these theories has its inherent difficulties. Molodtsov [13] proposed a completely new approach for modeling vagueness and uncertainty, which is called soft set theory. Since then, many related concepts about soft set operations, have undergone tremendous studies. Maji et al. [12] presented some definitions on soft sets and Ali et al. [3] introduced several operations of soft sets and Sezgin and Atagün [14] studied on soft set operations, as well. However, soft set theory have found its wide-ranging applications in the mean of algebraic structures such as groups [2, 15], semirings [8], rings [1], BCK/BCI-algebras [9–11], BL-algebras [22], near-rings [16] and soft substructures and union soft substructures [4, 17].

In [19], Sezgin Sezer made a new approach to the classical ring theory via soft set theory with the concept of soft union rings. Soft union rings, soft union left (right, two-sided) ideals, bi-ideals and soft union semiprime ideals of rings are defined, their basic properties are obtained and regular, regular duo, intra-regular and strongly regular rings are characterized by the properties of these soft union ideals in [19]. This paper is a following study of [19]. In this paper, soft union interior ideals, quasi-ideals, generalized bi-ideals of rings are defined, their basic properties with respect to soft set operations and soft int-uni product defined in [19] are obtained and the interrelations of them are investigated. Furthermore, regular, regular duo, intra-regular and strongly regular rings are characterized by the properties of these soft union ideals.

2. Preliminaries

In this section, we recall some basic notions relevant to rings and soft sets. Throughout this paper, $R$ denotes a ring. A nonempty subgroup $A$ of $R$ is called a right ideal of $R$ if $AR \subseteq A$ and is called a left ideal of $A$.

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R if RA ⊆ A. By two-sided ideal (or simply ideal), we mean a subset of R, which is both a left and right ideal of R. An additive subgroup (B, +) of R is called a bi-ideal of R if BRB ⊆ B. An additive subgroup (I, +) of R is called an interior ideal of R if RIR ⊆ X. An additive subgroup (Q, +) of R is called a quasi ideal of R if QR ∪ RQ ⊆ Q. A subset P of a ring R is called semiprime if ∀a ∈ R, a² ∈ P implies that a ∈ P. A semilattice is a structure S = (S, ⋆), where “⋆” is an infix binary operation, called the semilattice operation, such that “⋆” is associative, commutative and idempotent. From now on, U refers to an initial universe, E is a set of parameters, P(U) is the power set of U and A, B, C ⊆ E.

**Definition 2.1.** ([6, 13]) A soft set fA over U is a set defined by

\[ f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A. \]

Here fA is also called an approximate function. A soft set over U can be represented by the set of ordered pairs

\[ f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}. \]

It is clear to see that a soft set is a parametrized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by S(U).

**Definition 2.2.** [6] Let fA, fB ∈ R(U). Then, fA is called a subsoft set of fB and denoted by fA ⊆ fB, if fA(x) ⊆ fB(x) for all x ∈ E.

**Definition 2.3.** [6] Let fA, fB ∈ R(U). Then, union of fA and fB, denoted by fA ∪ fB, is defined as fA ∪ fB = fA∪B, where fA∪B(x) = fA(x) ∪ fB(x) for all x ∈ E.

**Definition 2.4.** [6] Let fA, fB ∈ R(U). Then, intersection of fA and fB, denoted by fA ∩ fB, is defined as fA ∩ fB = fA∩B, where fA∩B(x) = fA(x) ∩ fB(x) for all x ∈ E.

**Definition 2.5.** [6] Let fA, fB ∈ R(U). Then, ∧-product of fA and fB, denoted by fA ∧ fB, is defined as fA ∧ fB = fA∧B, where fA∧B(x, y) = fA(x) ∩ fB(y) for all (x, y) ∈ E × E.

**Definition 2.6.** [7] Let fA and fB be soft sets over the common universe U and Ψ be a function from A to B. Then, soft anti image of fA under Ψ, denoted by Ψ*(fA), is a soft set over U by

\[ (Ψ∗(f_A))(b) = \begin{cases} \bigcap \{f_A(a) \mid a \in A \text{ and } Ψ(a) = b\}, & \text{if } Ψ^{-1}(b) \neq \emptyset, \\ \emptyset, & \text{otherwise} \end{cases} \]

for all b ∈ B. And soft pre-image (or soft inverse image) of fB under Ψ, denoted by Ψ⁻¹(fB), is a soft set over U by (Ψ⁻¹(f_B))(a) = fB(Ψ(a)) for all a ∈ A.

**Definition 2.7.** [18] Let fA be a soft set over U and α ⊆ U. Then, lower α-inclusion of fA, denoted by L(fA; α), is defined as

\[ L(f_A : α) = \{x \in A \mid f_A(x) \supseteq α\}. \]

**Definition 2.8.** [19] Let fR and gR be soft sets over the common universe U. Then, soft intersection-union product fR ⊙ gR is defined by

\[
(f_R \ODOT g_R)(x) = \bigcap_{i=1}^{m} (f_R(a_i) \cup g_R(b_i))
\]

if \( x = \sum_{i=1}^{m} a_i b_i \) and \( a_i b_i \neq 0 \) for all \( 1 \leq i \leq m \). Otherwise, define

\[
(f_R \ODOT g_R)(x) = U.
\]
Here note that if \( R \) is a division ring and the multiplicative identity element of \( R \) is \( 1_R \), then \( x = x \cdot 1_R = 1_R \cdot x \), and so \( (f_R \circ g_R)(x) \neq \emptyset \) for all \( x \in R \).

For the sake of brevity, soft intersection-union product is abbreviated by soft int-uni product in what follows.

**Theorem 2.9.** \( [19] \) Let \( f_R, g_R, h_R \in \mathcal{R}(U) \). Then,

i) \( (f_R \circ g_R) \circ h_R = f_R \circ (g_R \circ h_R) \).

ii) \( f_R \circ g_R \neq g_R \circ f_R \), generally. However, if \( R \) is commutative, then \( f_R \circ g_R = g_R \circ f_R \).

iii) \( f_R \circ (g_R \cap h_R) = (f_R \circ g_R) \cap (f_R \circ h_R) \) and \( (f_R \cap g_R) \circ h_R = (f_R \circ h_R) \cap (g_R \circ h_R) \).

iv) \( f_R \circ (g_R \cup h_R) = (f_R \circ g_R) \cup (f_R \circ h_R) \) and \( (f_R \cup g_R) \circ h_R = (f_R \circ h_R) \cup (g_R \circ h_R) \).

v) If \( f_R \subseteq g_R \), then \( f_R \circ h_R \subseteq g_R \circ h_R \) and \( h_R \circ f_R \subseteq h_R \circ g_R \).

vi) If \( t_R, l_R \in \mathcal{S}(U) \) such that \( t_R \subseteq f_R \) and \( l_R \subseteq g_R \), then \( t_R \circ l_R \subseteq f_R \circ g_R \).

**Definition 2.10.** \( [19] \) Let \( X \) be a subset of \( S \). We denote by \( S_X \) the soft characteristic function of the complement \( X \) and define as

\[
S_X(x) = \begin{cases} 
0, & \text{if } x \in X, \\
\emptyset, & \text{if } x \in S \setminus X 
\end{cases}
\]

**Theorem 2.11.** \( [19] \) Let \( X \) and \( Y \) be nonempty subsets of a ring \( R \). Then, the following properties hold:

i) If \( Y \subseteq X \), then \( S_X \subseteq S_Y \).

ii) \( S_X \cap S_Y = S_{X \cap Y} \), \( S_X \cup S_Y = S_{X \cup Y} \).

**Definition 2.12.** \( [21] \) A soft set \( f_R \) over \( U \) is called a soft union ring of \( R \), if

i) \( f_R(x + y) \subseteq f_R(x) \cup f_R(y) \)

ii) \( f_R(x) \subseteq f_R(-x) \)

iii) \( f_R(xy) \subseteq f_R(x) \cup f_R(y) \)

for all \( x, y \in R \).

**Definition 2.13.** \( [21] \) A soft set \( f_R \) over \( U \) is called a soft union left (right) ideal of \( R \) over \( U \) if

i) \( f_R(x - y) \subseteq f_R(x) \cup f_R(y) \)

ii) \( f_R(xy) \subseteq f_R(y) \cup f_R(xy) \subseteq f_R(x) \)

for all \( x, y \in R \). A soft set over \( U \) is called a soft union two-sided (soft union ideal) of \( R \) if it is both soft union left and soft union right ideal of \( R \) over \( U \).

**Definition 2.14.** \( [19] \) An SU-ring \( f_R \) over \( U \) is called a soft union bi-ideal of \( R \) over \( U \) if

\[
f_R(xz) \subseteq f_R(x) \cup f_R(z)
\]

for all \( x, y, z \in R \).

For the sake of brevity, soft union ring, soft union right (left, two-sided) ideal and soft union bi-ideal are abbreviated by SU-ring, SU-right (left, two-sided) ideal and SU-bi-ideal, respectively.

It is easy to see that if \( f_R(x) = \emptyset \) for all \( x \in R \), then \( f_R \) is an SU-ring (right ideal, left ideal, ideal, bi-ideal) of \( R \) over \( U \). We denote such a kind of SU-ring (right ideal, left ideal, ideal, bi-ideal) by \( \emptyset \) \( [19] \).
Lemma 2.15. Let $f_R$ be any SU-ring over $U$. Then, we have the followings:

i) $\overline{\theta} \circ f_R \subset \overline{\theta}$. (If $R$ is regular, then $\overline{\theta} \circ \overline{\theta} = \overline{\theta}$).

ii) $f_R \circ \overline{\theta} \subset \overline{\theta}$ and $\overline{\theta} \circ f_R \supset \overline{\theta}$.

iii) $f_R \overline{\theta} = f_R$ and $f_R \overline{\theta} = \overline{\theta}$.

Theorem 2.16. [19] Let $X$ be a nonempty subset of a ring $R$. Then, $X$ is a subring (left, right, two-sided ideal, bi-ideal) of $R$ if and only if $S_X$ is an SU-ring (left, right, two-sided ideal, bi-ideal) of $R$.

Proposition 2.17. [19] Let $f_R$ be a soft set over $U$ and $f_R(x - y) \subseteq f_R(x) \cup f_R(y)$ for all $x, y \in R$. Then, we have the followings:

i) $f_R$ is an SU-ring over $U$ if and only if $f_R \circ f_R \supset f_R$.

ii) $f_R$ is an SU-left (right) ideal of $R$ over $U$ if and only if $\overline{\theta} \circ f_R \supset (f_R \circ \overline{\theta} f_R)$.

iii) $f_R$ is an SU-bi-ideal of $R$ over $U$ if and only if $f_R \circ f_R \supset f_R$ and $f_R \circ \overline{\theta} \supset f_R$.

Theorem 2.18. [19] For a ring $R$ the following conditions are equivalent:

1) $R$ is regular.

2) $f_R \circ g_R = f_R \overline{\theta} g_R$ for every SU-right ideal $f_R$ of $R$ over $U$ and SU-left ideal $g_R$ of $R$ over $U$.

3. Soft union interior ideals of rings

In this section, soft union interior ideals of rings is defined and their basic properties with respect to soft operations and soft int-uni product are studied.

Definition 3.1. Let $f_R$ be an SU-ring over $U$. Then, $f_R$ is called a soft union interior ideal of $R$, if

$$f_R(xay) \subseteq f_R(a)$$

for all $x, y, a \in R$.

Corollary 3.2. Let $a = \sum_{i=1}^{m} x_i y_i z_i$ and $f_R$ be an SU-interior ideal over $U$. Then, $f_R(a) = f_R(\sum_{i=1}^{m} x_i y_i z_i) \subseteq f_R(y_i)$ for all $1 \leq i \leq m$.

For the sake of brevity, soft union interior ideal is abbreviated by SU-interior ideal in what follows.

Example 3.3. Consider the ring $R = \mathbb{Z}_6$ and let $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx = \{e, x, y, xy\}\}$. Let $f_R$ be the universal set and $f_R$ be soft set over $U$ such that

$$f_R(0) = \{x\}, \ f_R(1) = \{e, x, y\}, \ f_R(2) = \{e, y\}, \ f_R(3) = \{e, x, yx\}, \ f_R(4) = \{e, y\}, \ f_R(5) = \{e, x, y\}.$$ 

Then, one can easily show that $f_R$ is an SU-interior ideal over $U$.

Now, let $U = S_3$ be the symmetric group. If we construct a soft set $g_R$ over $U$ such that

$$g_R(0) = \{(1), (12), (13)\}, \ g_R(1) = \{(1)\}, \ g_R(2) = \{(1), (12)\}, \ g_R(3) = \{(1)\}$$

then,

$$g_R(2 \cdot 2 \cdot 3) = g_R(0) \not\subseteq g_R(2)$$

then, $g_R$ is not an SU-interior ideal over $U$.
It is easy to see that if \( f_R(x) = \emptyset \) for all \( x \in R \), then \( f_R \) is an SU-interior ideal over \( U \). We denote such a kind of SU-interior ideal by \( \tilde{\theta} \). It is obvious that \( \tilde{\theta} = S_R \), i.e. \( \tilde{\theta}(x) = \emptyset \) for all \( x \in R \).

**Theorem 3.4.** Let \( f_R \) be a soft over \( U \). Then, \( f_R \) is an SU-bi-ideal of \( R \) over \( U \) if and only if \( f_R(x - y) \subseteq f_R(x) \cup f_R(y) \), \( f_R \circ f_R \supseteq f_R \) and \( \tilde{\theta} \circ f_R \supseteq \tilde{\theta} f_R \).

**Proof.** First assume that \( f_R \) is an SU-interior-ideal of \( R \) over \( U \). Since \( f_R \) is an SU-ring over \( U \), by Theorem 3.4 we have \( f_R(x - y) \subseteq f_R(x) \cup f_R(y) \) and \( f_R \circ f_R \supseteq f_R \). Let \( x \in R \). In the case, when \( (\tilde{\theta} \circ f_R \circ \tilde{\theta})(a) = a \), then it is obvious that

\[
(\tilde{\theta} \circ f_R \circ \tilde{\theta})(a) \supseteq f_R(a), \quad \text{thus} \quad \tilde{\theta} \circ f_R \circ \tilde{\theta} f_R.
\]

Otherwise, we have

\[
(\tilde{\theta} \circ f_R \circ \tilde{\theta})(x) = ((\tilde{\theta} \circ f_R) \circ \tilde{\theta})(x) = \bigcap x=\sum_{i=1}^{m} a_i b_i \subseteq \bigcup (\tilde{\theta} \circ f_R)(a_i) \cup \tilde{\theta}(b_i) \subseteq \bigcup f_R(b_i) = \sum_{i=1}^{m'} a_i b_i c_i \subseteq f_R(x)
\]

Thus, \( \tilde{\theta} \circ f_R \circ \tilde{\theta} f_R \). Here, note that if \( x \neq \sum_{i=1}^{m} a_i b_i \), then \( (\tilde{\theta} \circ f_R)(x) = U \), and so \( (\tilde{\theta} \circ f_R \circ \tilde{\theta})(x) = U \supseteq f_R(x) \).

Conversely, assume that \( \tilde{\theta} \circ f_R \circ \tilde{\theta} f_R \). Let \( x, a, y \) be any element of \( R \). Then, we have:

\[
f_R(xay) \subseteq (\tilde{\theta} \circ f_R \circ \tilde{\theta})(xay) = \bigcap xay=\sum_{i=1}^{m} x_i y_i \cup (\tilde{\theta} \circ f_R)(x_i) \cup \tilde{\theta}(y_i)
\]

(1)
The following theorem shows that the converse of Proposition 3.8 holds for a regular ring. Hence, $f_R$ is an SU-interior ideal over $U$. This completes the proof. □

**Corollary 3.5.** Let $f_R$ be a soft set. Then the following conditions are equivalent:

1) $f_R \circ θ \circ f_R \supseteq f_R$.

2) $f_R(\bigcup_{i=1}^{m} x_{i}y_{i}z_{i}) \subseteq f_R(y_{i})$ for all $1 \leq i \leq m$.

**Theorem 3.6.** A non-empty subset $I$ of a ring $R$ is an interior ideal of $R$ if and only if the soft subset $f_R$ defined by

$$f_R(x) = \begin{cases} \alpha, & \text{if } x \in R \setminus I, \\ \beta, & \text{if } x \in I \end{cases}$$

is an SU-interior ideal, where $α, β \subseteq U$ such that $α \supseteq β$.

**Proof.** Suppose $I$ is an interior ideal of $R$ and $x, y, a, b \in R$. If $a, b \in I$, then $a - b \in I$. Hence, $f_R(a - b) = f_R(a) = f_R(b) = β$ and so, $f_R(a - b) \subseteq f_R(a) \cup f_R(b)$. If $a, b \not\in I$, then $a - b \not\in I$ or $a - b \not\in I$. In any case, $f_R(a - b) \subseteq f_R(a) \cup f_R(b) = β$. This implies that $f_R(a) = β$. If $a \not\in I$, then $xay \in I$. Hence, $f_R(xay) = f_R(a) = β$. If $a \in I$, then $xay \not\in I$ or $xay \not\in I$. In any case, $f_R(xay) \subseteq f_R(a) = β$. Thus, $f_R$ is an SU-interior ideal of $S$.

Conversely assume that $f_R$ is an SU-interior ideal of $R$. Let $a, b \in I$ and $x, y \in R$. Then, $f_R(a - b) \subseteq f_R(a) \cup f_R(b) = β$. This implies that $f_R(a - b) = β$. Hence, $a - b \in I$. Now, $f_R(xay) \subseteq f_R(a) = β$. This implies that $f_R(xay) = β$. Hence, $xay \in I$ and so $I$ is an interior ideal of $R$. □

**Theorem 3.7.** Let $X$ be a nonempty subset of a ring $R$. Then, $X$ is an interior ideal of $R$ if and only if $S_X$ is an SU-interior ideal of $R$.

**Proof.** Since

$$S_X(x) = \begin{cases} U, & \text{if } x \in R \setminus X, \\ ∅, & \text{if } x \in X \end{cases}$$

and $U \supseteq ∅$, the rest of the proof follows from Theorem 3.6. □

It is obvious that every two-sided ideal of $R$ is an interior ideal of $R$. Moreover, we have the following:

**Proposition 3.8.** Let $f_R$ be a soft set over $U$. Then, if $f_R$ is an SU-ideal of $R$ over $U$, $f_R$ is an SU-interior ideal of $R$ over $U$.

**Proof.** Let $f_R$ be an SU-ideal of $R$ over $U$ and $x, y \in R$. Then,

$$f_R(xyz) = f_R((xyz)z) \subseteq f_R(xy) \subseteq f_R(y).$$

Hence, $f_R$ is an SU-interior ideal of $R$ over $U$. □

The following theorem shows that the converse of Proposition 3.8 holds for a regular ring.
Theorem 3.9. Let \( f_R \) be a soft set over \( U \), where \( R \) is a regular ring. Then, the following conditions are equivalent:

1) \( f_R \) is an SU-ideal of \( R \) over \( U \).

2) \( f_R \) is an SU-interior ideal of \( R \) over \( U \).

Proof. By Proposition 3.8, it suffices to prove that (2) implies (1). Assume that (2) holds. Let \( a, b \) be any elements of \( R \). Then, since \( R \) is regular, there exist elements \( x \) and \( y \) in \( R \) such that

\[
a = axa \quad \text{and} \quad b = byb.
\]

Then, since \( f_R \) is an interior ideal of \( R \), we have

\[
f_R(ab) = f_R((axa)b) = f_R((ax)a(b)) \subseteq f_R(a),
\]

and

\[
f_R(ab) = f_R(a(byb)) = f_R((a)b(yb)) \subseteq f_R(b).
\]

This means that \( f_R \) is an SU-ideal of \( R \). Thus, (2) implies (1). \( \square \)

Proposition 3.10. Let \( R \) be a division ring and \( f_R \) be a soft set over \( U \). Then, \( f_R \) is an SU-ideal of \( R \) if and only if \( f_R \) is an SU-interior ideal of \( R \).

Proof. The necessity is clear by Proposition 3.8. Now let us show the sufficiency. For \( x, y \in R \), \( f_R(xy) = f_R(xye) \subseteq f_R(y) \) and \( f_R(xy) = f_R(exy) \subseteq f_R(x) \). Thus, \( f_R \) is an SU-ideal of \( R \). \( \square \)

It is known that a ring \( R \) is called left (right) simple if it contains no proper left (right) ideal of \( R \) and is called simple if it contains no proper ideal.

Definition 3.11. [19] A ring \( R \) is called soft left (right) union simple if every SU-left (right) ideal of \( R \) is a constant function and is called soft union simple if every SU-ideal of \( R \) is a constant function.

Theorem 3.12. [19] For a ring \( R \), the following conditions are equivalent:

1) \( R \) is simple.

2) \( R \) is soft union simple.

Theorem 3.13. For a regular ring \( R \), the following conditions are equivalent:

1) \( R \) is simple.

2) \( R \) is soft union simple.

3) Every SU-interior ideal of \( R \) is constant function.

Proof. The equivalence of (1) and (2) follows from Theorem 3.12. Assume that (2) holds. Let \( f_R \) be any SU-interior ideal of \( R \) and \( a \) and \( b \) be any element of \( R \). Then, since \( R \) is simple, it follows that there exist elements \( x \) and \( y \) in \( R \) such that

\[
a = xby.
\]

Then, since \( f_R \) is an SU-interior ideal of \( R \), we have

\[
f_R(a) = f_R(xby) \subseteq f_R(b).
\]

One can similarly show that \( f_R(b) \subseteq f_R(a) \). Thus, \( f_R(a) = f_R(b) \). Since \( a \) and \( b \) be any elements of \( R \), \( f_R \) is a constant function and so (2) implies (3). Since every SU-interior ideal of \( R \) is an SU-ideal of \( R \) by the regularity of \( R \), (3) implies (2). \( \square \)
Definition 3.14. [19] A soft set $f_R$ over $U$ is called soft union semiprime if for all $a \in R$,

$$f_R(a) \subseteq f_R(a^2).$$

Proposition 3.15. Let $f_R$ be a soft union semiprime SU-interior ideal of a ring $R$. Then, $f_R(a^n) \subseteq f_R(a^{n+1})$ for all positive integers $n$.

Proof. Let $n$ be any positive integer. Then,

$$f_R(a^n) \subseteq f_R(a^{2n}) \subseteq f_R(a^{3n-2}a^{n+1}) \subseteq f_R(a^{n+1}).$$

Proposition 3.16. Let $f_R$ and $f_T$ be SU-interior ideals over $U$. Then, $f_R \lor f_T$ is an SU-interior ideal of $R \times T$ over $U$.

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_2) \in R \times T$. Then,

$$f_{R \lor T}((x_1, y_1) - (x_2, y_2)) = f_{R \lor T}(x_1 - x_2, y_1 - y_2) = f_R(x_1 - x_2) \cup f_T(y_1 - y_2) \subseteq (f_R(x_1) \cup f_R(x_2)) \cup (f_T(y_1) \cup f_T(y_2)) = (f_R(x_1) \cup f_T(y_1)) \cup (f_R(x_2) \cup f_T(y_2)) = f_{R \lor T}(x_1, y_1) \cup f_{R \lor T}(x_2, y_2),$$

and

$$f_{R \lor T}((x_1, y_1)(x_2, y_2)(x_3, y_2)) = f_{R \lor T}(x_1x_2x_3, y_1y_2y_3) = f_R(x_1x_2x_3) \cup f_T(y_1y_2y_3) \subseteq f_R(x_2) \cup f_T(y_2) = f_{R \lor T}(x_2, y_2)$$

Therefore, $f_R \lor f_T$ is an SU-interior ideal of $R \times T$ over $U$.

Proposition 3.17. If $f_R$ and $h_R$ are SU-interior ideals of $R$ over $U$, then so is $f_R \lor h_R$.

Proof. Let $x, y, z \in R$. Then, we have

$$(f_R \lor h_R)(x - y) = f_R(x - y) \cup h_R(x - y) \subseteq (f_R(x) \cup f_R(y)) \cup (h_R(x) \cup h_R(y)) = (f_R(x) \cup h_R(x)) \cup (f_R(y) \cup h_R(y)) = (f_R \lor h_R)(x) \cup (f_R \lor h_R)(y)$$

and

$$(f_R \lor h_R)(xy) = f_R(xy) \cup h_R(xy) \subseteq (f_R(x) \cup f_R(y)) \cup (h_R(x) \cup h_R(y)) = (f_R(x) \cup h_R(x)) \cup (f_R(y) \cup h_R(y)) = (f_R \lor h_R)(x) \cup (f_R \lor h_R)(y)$$

and

$$(f_R \lor h_R)(xyz) = f_R(xyz) \cup h_R(xyz) \subseteq f_R(y) \cup h_R(y) = (f_R \lor h_R)(y)$$

Therefore, $f_R \lor h_R$ is an SU-interior ideal of $R$ over $U$. 

Proposition 3.18. Let $f_R$ be a soft set over $U$ and $\alpha$ be a subset of $U$ such that $\alpha \in \text{Im}(f_R)$, where $\text{Im}(f_R) = \{a \subseteq U : f_R(x) = a, \text{ for } x \in R\}$. If $f_R$ is an SU-interior ideal over $U$, then $L(f_R; \alpha)$ is an interior ideal of $R$.

Proof. Since $f_R(a) = a$ for some $x \in R$, then $\emptyset \neq L(f_R; \alpha) \subseteq R$. Let $a, b \in L(f_R; \alpha)$ and $x, y \in R$, then $f_R(a) \subseteq a$ and $f_R(b) \subseteq a$. We need to show that $a - b \in L(f_R; \alpha)$ and $xay \in L(f_R; \alpha)$ for all $a, b \in L(f_R; \alpha)$ and $x, y \in R$. Since $f_R$ is an SU-interior ideal of $R$ over $U$, it follows that $f_R(a - b) \subseteq f_R(a) \cup f_R(b)$ and $f_R(xay) \subseteq f_R(a)$ implying that $a - b \in L(f_R; \alpha)$ and $xay \in L(f_R; \alpha)$. Thus, the proof is completed. $\square$

Definition 3.19. Let $f_R$ be an SU-interior ideal over $U$. Then, the interior ideals $L(f_R; \alpha)$ are called lower $\alpha$-interior ideals of $f_R$.

Proposition 3.20. Let $f_R$ be a soft set over $U$, $L(f_R; \alpha)$ be lower $\alpha$-interior ideals of $f_R$ for each $\alpha \subseteq U$ and $\text{Im}(f_R)$ be an ordered set by inclusion. Then, $f_R$ is an SU-interior ideal of $R$ over $U$.

Proof. Let $a, b \in R$ and $f_R(a) = \alpha_1$ and $f_R(b) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $a \in L(f_R; \alpha_1)$ and $b \in L(f_R; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $a, b \in L(f_R; \alpha_1)$ and since $L(f_R; \alpha)$ is an interior ideal of $R$ for all $\alpha \subseteq U$, it follows that $a - b \in L(f_R; \alpha_1)$ and $xay \in L(f_R; \alpha_1)$. Hence, $f_R(a - b) \subseteq \alpha_1 \cap \alpha_2 = f_R(a) \cup f_R(b)$, and $f_R(xay) \subseteq f_R(a)$ implying that $a - b \in L(f_R; \alpha)$ and $xay \in L(f_R; \alpha)$. Thus, $f_R$ is an SU-interior ideal of $R$ over $U$. $\square$

Proposition 3.21. Let $f_R$ and $f_T$ be soft sets over $U$ and $\Psi$ be a ring isomorphism from $R$ to $T$. If $f_R$ is an SU-interior ideal of $R$ over $U$, then $\Psi^*(f_R)$ is an SU-interior ideal of $T$ over $U$.

Proof. Let $t_1, t_2, t_3 \in T$. Since $\Psi$ is surjective, then there exist $r_1, r_2, r_3 \in R$ such that $\Psi(r_1) = t_1, \Psi(r_2) = t_2, \Psi(r_3) = t_3$. Then,

$$\begin{align*}
(\Psi^*(f_R))(t_1 &- t_2) \\
= &\bigcap \{f_R(r) : r \in R, \Psi(r) = t_1 - t_2\} \\
= &\bigcap \{f_R(r) : r \in R, r = \Psi^{-1}(t_1 - t_2)\} \\
= &\bigcap \{f_R(r) : r \in R, r = \Psi^{-1}(\Psi(r_1 - r_2)) = r_1 - r_2\} \\
= &\bigcap \{f_R(r_1 - r_2) : r_1 \in R, \Psi(r_1) = t_i, i = 1, 2\} \\
\subseteq &\bigcap \{f_R(r_1) \cup f_R(r_2) : r_1 \in R, \Psi(r_1) = t_i, i = 1, 2\} \\
= &\bigcap \{f_R(r_1) \cup f_R(r_2) : r_2 \in R, \Psi(r_2) = t_2\} \\
= &\bigcap \{\Psi^*(f_R) : r_1 \cup r_2 = \Psi^*(f_R)\}(t_3)
\end{align*}$$

One can similarly show that $(\Psi^*(f_R))(t_1 t_2) \subseteq (\Psi^*(f_R))(t_1) \cup (\Psi^*(f_R))(t_2)$ Also

$$\begin{align*}
(\Psi^*(f_R))(t_1 t_2 t_3) \\
= &\bigcap \{f_R(s) : s \in R, \Psi(s) = t_1 t_2 t_3\} \\
= &\bigcap \{f_R(s) : s \in R, s = \Psi^{-1}(t_1 t_2 t_3)\} \\
= &\bigcap \{f_R(s) : s \in R, s = \Psi^{-1}(\Psi(s_1 s_2 s_3)) = s_1 s_2 s_3\} \\
= &\bigcap \{f_R(s_1 s_2 s_3) : s_1 \in R, \Psi(s_1) = t_i, i = 1, 2, 3\} \\
\subseteq &\bigcap \{f_R(s_2) : s_2 \in R, \Psi(s_2) = t_2\} \\
= &\Psi^*(f_R)(t_2)
\end{align*}$$

Hence, $\Psi^*(f_R)$ is an SU-interior ideal of $R$ over $U$. $\square$

Proposition 3.22. Let $f_R$ and $f_T$ be soft sets over $U$ and $\Psi$ be a ring homomorphism from $R$ to $T$. If $f_T$ is an SU-interior ideal of $T$ over $U$, then $\Psi^{-1}(f_T)$ is an SU-interior ideal of $R$ over $U$.

Proof. Let $r_1, r_2, r_3 \in R$. Then,

$$\begin{align*}
(\Psi^{-1}(f_T))(r_1 &- r_2) \\
= &f_T(\Psi(r_1 - r_2)) \\
= &f_T(\Psi(r_1) \cup \Psi(r_2)) \\
\subseteq &f_T(\Psi(r_1)) \cup f_T(\Psi(r_2)) \\
= &\Psi^{-1}(f_T)(r_1) \cup (\Psi^{-1}(f_T))(r_2)
\end{align*}$$
One can similarly show that \((\Psi^{-1}(f_1))(r_1r_2) \subseteq (\Psi^{-1}(f_1))(r_1) \cup (\Psi^{-1}(f_1))(r_2)\). Also
\[
(\Psi^{-1}(f_1))(r_1r_2r_3) = f_1(\Psi(r_1r_2r_3)) = f_1(\Psi(r_1)\Psi(r_2)\Psi(r_3)) \subseteq f_1(\Psi(r_3)) = (\Psi^{-1}(f_1))(r_2)
\]
Hence, \(\Psi^{-1}(f_1)\) is an SU-interior ideal over \(U\). \(\square\)

4. Soft union quasi-ideals of rings

In this section, soft union quasi-ideals are defined and their properties as regards soft set operations, soft int-uniform product and certain kinds of soft union ideals are studied.

**Definition 4.1.** A soft set over \(U\) is called a soft union quasi-ideal of \(R\) over \(U\) if \(f_R(x - y) \subseteq f_R(x) \cup f_R(y)\) and \((f_R \circ \bar{\theta})\cup(f_R \circ \bar{\theta}f_R)\).

For the sake of brevity, soft union quasi-ideal is abbreviated by SU-quasi-ideal in what follows.

**Proposition 4.2.** Every SU-quasi ideal of \(R\) is an SU-ring of \(R\).

**Proof.** Let \(f_R\) be any SU-quasi-ideal of \(R\). Then, \(f_R(x - y) \subseteq f_R(x) \cup f_R(y)\) and since \(f_R \circ \bar{\theta}\),
\[
f_R \circ f_R \circ \bar{\theta} \circ f_R \text{ and } f_R \circ \bar{\theta} \circ f_R \circ \bar{\theta}.
\]
Hence,
\[
f_R \circ f_R \circ \bar{\theta} \circ f_R \circ \bar{\theta} \circ f_R
\]
That is, \(f_R\) is an SU-ring over \(U\) by Proposition 2.17. \(\square\)

**Proposition 4.3.** Each one-sided SU-ideal of \(R\) is an SU-quasi-ideal of \(R\).

**Proof.** Let \(f_R\) be an SU-left ideal of \(R\). Then, \(f_R(x - y) \subseteq f_R(x) \cup f_R(y)\) and since \(\bar{\theta} \circ \bar{\theta}f_R\), we have
\[
(\bar{\theta} \circ f_R)\cup(\bar{\theta} \circ f_R) \subseteq \bar{\theta} \circ f_R \circ \bar{\theta} \circ f_R.
\]
Thus, \(f_R\) is an SU-quasi-ideal of \(R\). \(\square\)

**Proposition 4.4.** Every SU-quasi-ideal of \(R\) is an SU-bi-ideal of \(R\).

**Proof.** Let \(f_R\) be an SU-quasi-ideal of \(R\). Then, \(f_R(x - y) \subseteq f_R(x) \cup f_R(y)\),
\[
f_R \circ f_R = (f_R \circ f_R)\cup(f_R \circ f_R) \subseteq \bar{\theta} \circ f_R \circ \bar{\theta} \circ f_R
\]
and
\[
f_R \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta} \circ \bar{\theta}
\]
and so \(f_R \circ \bar{\theta} \circ f_R \circ \bar{\theta} \circ f_R\), as \(f_R\) is an SU-quasi-ideal of \(R\). Hence,
\[
f_R \circ \bar{\theta} \circ f_R \circ \bar{\theta} \circ f_R.
\]
Thus, \(f_R\) is an SU-bi-ideal of \(R\) by Proposition 2.17. \(\square\)

The following theorem shows that the converse of Proposition 4.4 holds for a regular ring. First, we have the following lemma:
Lemma 4.5. Let $f_R$ and $h_R$ be soft SU-rings (left, right) ideals, bi-ideals, interior ideals, quasi-ideals) of $R$ over $U$, where $R$ is a division ring. Then,

$$(f_R \circ h_R)(x - y) \subseteq (f_R \circ h_R)(x) \cup (f_R \circ h_R)(y)$$

for all $x, y \in R$.

Proof. Let $f_R$ and $h_R$ be soft sets over $U$ and $x, y \in R$. Then,

$$(f_R \circ h_R)(x) \cup (f_R \circ h_R)(y) = \bigcup_{x} \sum_{i=1}^{m} a_i b_i y = \bigcup_{y} \sum_{i=1}^{n} c_i d_i$$

$$= \bigcup_{x} \sum_{i=1}^{m} a_i b_i \cap \bigcup_{y} \sum_{i=1}^{n} c_i d_i$$

$$\supseteq \sum_{x+y} \bigcup_{i=1}^{k} (f_R(x_i) \cup h_R(y_i))$$

$$= (f_R \circ h_R)(x + y)$$

and

$$(f_R \circ h_R)(-x) = \bigcup_{-x} \sum_{i=1}^{m} a_i b_i$$

$$= \bigcup_{x} \sum_{i=1}^{m} (-a_i) b_i$$

$$= \bigcup_{x} \sum_{i=1}^{m} (-a_i) b_i$$

$$= (f_R \circ h_R)(x)$$

Thus, the proof is completed. \qed

Theorem 4.6. Let $f_R$ be a soft set over $U$, where $R$ is a regular ring. Then, the following conditions are equivalent:

1) $f_R$ is an SU quasi-ideal of $R$ over $U$.

2) $f_R$ is an SU bi-ideal of $R$ over $U$.

Proof. By Proposition 4.4, it suffices to prove that (2) implies (1). Assume that (2) holds. Let $f_R$ be an SU bi-ideal of $R$. Then, $\overline{\theta} \circ f_R$ (resp. $f_R \circ \overline{\theta}$) is an SU left (resp. right) ideal of $R$. In fact, $(\overline{\theta} \circ f_R)(x - y) \subseteq (\overline{\theta} \circ f_R)(x) \cup (\overline{\theta} \circ f_R)(y)$ by Lemma 4.5 and $\overline{\theta} \circ (\overline{\theta} \circ f_R) = f_R \circ (\overline{\theta} \circ \overline{\theta}) = f_R \circ (f_R \circ \overline{\theta}) = f_R \circ \overline{\theta} \circ f_R$. It follows by Theorem 2.18 that

$$(f_R \circ \overline{\theta} \cup \overline{\theta} \circ f_R) = (f_R \circ \overline{\theta}) \circ (f_R \circ \overline{\theta}) = f_R \circ (\overline{\theta} \circ \overline{\theta}) = f_R \circ \overline{\theta} \circ f_R \overline{f_R}$$

since $f_R$ is an SU bi-ideal of $R$. Thus, $f_R$ is an SU quasi-ideal of $R$ and (2) implies (1). \qed
Theorem 4.7. A non-empty subset \( Q \) of a ring \( R \) is a quasi-ideal of \( R \) if and only if the soft subset \( f_R \) defined by

\[
f_R(x) = \begin{cases} 
\alpha, & \text{if } x \in R \setminus Q, \\
\beta, & \text{if } x \in Q
\end{cases}
\]

is an SU-quasi-ideal, where \( \alpha, \beta \subseteq U \) such that \( \alpha \supseteq \beta \).

Proof. It is similar to Theorem 3.6. \( \square \)

Theorem 4.8. Let \( X \) be a nonempty subset of a ring \( R \). Then, \( X \) is a quasi-ideal of \( R \) if and only if \( S_X \) is an SU-quasi-ideal of \( R \) over \( U \).

Proof. It follows from Theorem 4.7. \( \square \)

Theorem 4.9. Let \( f_R \) and \( g_R \) be any SU-quasi-ideal of \( R \) over \( U \). Then, the soft int-uni product \( f_R \odot g_R \) is an SU-left ideal of \( R \).

Proof. Let \( f_R \) be an SU-quasi-ideal of \( R \). Then, \( f_R \) is an SU-bi-ideal by Proposition 4.4. Hence, \( f_R \odot \overline{\theta} \odot f_R \odot g_R \).

Moreover,

\[
(f_R \odot g_R)(x - y) \subseteq (f_R \odot g_R)(x) \cup (f_R \odot g_R)(y)
\]

and

\[
(f_R \odot g_R) \odot (f_R \odot g_R) = (f_R \odot g_R \odot f_R) \odot g_R \odot \overline{\theta} \odot f_R \odot g_R \odot \overline{\theta} \odot f_R \odot g_R
\]

and

\[
(f_R \odot g_R) \odot \overline{\theta} \odot (f_R \odot g_R) = (f_R \odot (g_R \odot \overline{\theta}) \odot f_R) \odot g_R \odot \overline{\theta} \odot f_R \odot g_R
\]

Thus, it follows that \( f_R \odot g_R \) is an SU-bi-ideal of \( R \) over \( U \). \( \square \)

Corollary 4.10. Let \( R \) be a regular ring and \( f_R, g_R \) be any SU-quasi-ideals of \( R \) over \( U \). Then, \( f_R \odot g_R \) is an SU-quasi-ideal of \( R \) over \( U \).

Proof. Follows from Theorem 4.6 and Theorem 4.9. \( \square \)

Proposition 4.11. Let \( f_R \) be any SU-right ideal of \( R \) and \( g_R \) be any SU-left ideal of \( R \). Then, \( f_R \odot g_R \) is an SU-quasi-ideal of \( R \).

Proof. Let \( f_R \) be any SU-right ideal of \( R \) and \( g_R \) be any SU-left ideal of \( R \). Then, one can easily show that

\[
((f_R \odot g_R)(x - y) \subseteq f_R(x) \cup g_R(y)
\]

as in the proof of Proposition 3.17. Moreover,

\[
((f_R \odot g_R) \odot \overline{\theta}) \cup (f_R \odot g_R) \odot g_R \odot \overline{\theta} \odot f_R \odot g_R
\]

by Theorem 2.18. \( \square \)

Proposition 4.12. Let \( R \) be a regular ring, \( f_R \) be any SU-right ideal of \( R \) and \( g_R \) be any SU-left ideal of \( R \). Then, \( f_R \odot g_R \) is an SU-quasi-ideal of \( R \).

Proof. Let \( R \) be a regular ring and \( f_R \) be an SU-right ideal of \( R \) and \( g_R \) be an SU-left ideal of \( R \). It follows by Proposition 4.11 that \( f_R \odot g_R \) is an SU-quasi-ideal of \( R \). Since \( R \) is regular,

\[
f_R \odot g_R = f_R \odot g_R
\]

by Theorem 2.18. Thus, \( f_R \odot g_R \) is an SU-quasi-ideal of \( R \). \( \square \)

Proposition 4.13. Let \( f_R \) and \( g_R \) be any SU-quasi-ideals of \( R \). Then, \( f_R \odot g_R \) is an SU-quasi-ideal of \( R \).
Proof. Let \( f_R \) and \( g_R \) be any \( SU \)-quasi-ideals of \( R \). Then, one can easily show that \((f_R \cup g_R)(x - y) \supseteq f_R(x) \cup g_R(y)\) as in the proof of Proposition 3.17. Also,

\[
((f_R \cup g_R) \circ \tilde{\theta}) \cup (f_R \cup g_R) \supseteq f_R \circ \tilde{\theta} \cup (f_R \circ \tilde{\theta}) \supseteq f_R
\]

and

\[
((f_R \cup g_R) \circ \tilde{\theta}) \cup (f_R \cup g_R) \supseteq (g_R \circ \tilde{\theta}) \cup (f_R \circ \tilde{\theta}) \supseteq g_R.
\]

Thus,

\[
((f_R \cup g_R) \circ \tilde{\theta}) \cup (f_R \cup g_R) \supseteq f_R \circ \tilde{\theta} \cup (f_R \circ \tilde{\theta}) \supseteq f_R \cup g_R.
\]

\( \Box \)

**Proposition 4.14.** Let \( f_R \) be a soft set over \( U \) and \( \alpha \) be a subset of \( U \) such that \( \alpha \in \text{Im}(f_R) \). If \( f_R \) is an \( SU \)-quasi-ideal of \( R \) over \( U \), then \( \mathcal{L}(f_R; \alpha) \) is a quasi-ideal of \( R \).

Proof. Since \( f_R(x) = \alpha \) for some \( x \in R \), then \( \emptyset \notin \mathcal{L}(f_R; \alpha) \subseteq R \). Let \( a \in (R \cdot \mathcal{L}(f_R; \alpha) \cup \mathcal{L}(f_R; \alpha) \cdot R) \). Then, there exist \( x, y \in \mathcal{L}(f_R; \alpha) \) and \( s, r \in R \) such that

\[ a = sx = yr. \]

Thus, \( f_R(x) \subseteq \alpha \) and \( f_R(y) \subseteq \alpha \). Hence, \( f_R(x - y) \subseteq f_R(x) \cup f_R(y) \subseteq \alpha \), implying that \( x - y \in \mathcal{L}(f_R; \alpha) \). Moreover,

\[
(\tilde{\theta} \circ f_R)(a) = \sum_{i=1}^{m} \{ \tilde{\theta}(c_i) \cup f_R(d_i) \} \subseteq \tilde{\theta}(s) \cup f_R(x) = f_R(x) \subseteq \alpha
\]

and

\[
(f_R \circ \tilde{\theta})(a) = \sum_{i=1}^{m} \{ f_R(k_i) \circ \tilde{\theta}(t_i) \} \subseteq f_R(y) \cup \tilde{\theta}(r) = f_R(y) \subseteq \alpha
\]

Since \( f_R \) is an \( SU \)-quasi-ideal of \( R \), we have

\[ f_R(a) \subseteq (\tilde{\theta} \circ f_R)(a) \cup (f_R \circ \tilde{\theta})(a) \subseteq \alpha, \]

thus \( a \in \mathcal{L}(f_R; \alpha) \). This shows that \( \mathcal{L}(f_R; \alpha) \) is a quasi-ideal of \( R \). \( \Box \)

**Definition 4.15.** Let \( f_R \) be an \( SU \)-quasi-ideal of \( R \) over \( U \). Then, the quasi-ideals \( \mathcal{L}(f_R; \alpha) \) are called lower \( \alpha \)-quasi-ideals of \( f_R \).

**Proposition 4.16.** Let \( f_R \) be any \( SU \)-quasi-ideal of a commutative ring \( R \) and \( a \) be any element of \( A \). Then,

\[ f_R(a^n) \subseteq f_R(a^{n+1}) \]

for every positive integer \( n \).
Proof. For any positive integer \( n \), we have

\[
(f_R \circ \tilde{\theta})(a^{n+1}) = \bigcap_{i=1}^{a^{n+1}} (f_R(x_i) \cup \tilde{\theta}(y_i)) = f_R(a^n) \cup \tilde{\theta}(a) = f_R(a^n).
\]

Similarly,

\[
(\tilde{\theta} \circ f_R)(a^{n+1}) \subseteq f_R(a^n).
\]

Thus, since \( f_R \) is an SU-quasi-ideal of \( R \)

\[
f_R(a^{n+1}) \subseteq \bigcup (f_R(x) \cup f_R(y)) \subseteq f_R(a^n) \cup f_R(a^n) = f_R(a^n)
\]

This completes the proof. \( \square \)

5. Soft union generalized bi-ideals of rings

In this section, soft union generalized bi-ideals are defined and their properties as regards soft set operations and soft int-uni product are studied.

Definition 5.1. A soft set over \( U \) is called a soft union generalized bi-ideal of \( R \) over \( U \) if \( f_R(x - y) \subseteq f_R(x) \cup f_R(y) \) and \( f_R(xyz) \subseteq f_R(x) \cup f_R(z) \) for all \( x, y, z \in R \).

For the sake of brevity, soft union generalized bi-ideal is abbreviated by SU-generalized bi-ideal in what follows.

It is clear that every SU-bi-ideal of \( R \) is an SU-generalized bi-ideal of \( R \), but the converse of this statement does not hold in general. The following theorem shows that the converse of this holds for a regular ring.

Proposition 5.2. Every SU-generalized bi-ideal of a regular ring is an SU-bi-ideal of \( R \).

Proof. Let \( f_R \) be an SU-generalized bi-ideal of \( R \) and let \( a \) and \( b \) be any element of \( R \). Then, since \( R \) is regular, there exists an element \( x \in R \) such that \( b = bx \). Thus, we have

\[
f_R(ab) = f_R(a(bx)) = f_R(a(bx)b) \subseteq f_R(a) \cup f_R(b).
\]

This implies that \( f_R \) is an SU-ring of \( R \) and so \( f_R \) is an SU-bi-ideal of \( R \). \( \square \)

Theorem 5.3. Let \( f_R \) be a soft set over \( U \). Then, \( f_R \) is an SU-generalized bi-ideal of \( R \) over \( U \) if and only if \( f_R(x - y) \subseteq f_R(x) \cup f_R(y) \) and \( f_R \circ \tilde{\theta} \circ f_R \subseteq f_R \).
Corollary 5.4. Let $f_R$ be a soft set. Then the following conditions are equivalent:

1) $f_R \circ \overline{\theta} \circ f_R$.
Proposition 5.15. Let $f$ be a SU-generalized bi-ideal of $R$ over $U$, then so is $\Psi$.

Theorem 5.5. A non-empty subset $G$ of a ring $R$ is a generalized bi-ideal of $R$ if and only if the soft subset $f_R \in \mathcal{F}(R)$ defined by

$$f_R(x) = \begin{cases} \alpha, & \text{if } x \in R \setminus G, \\ \beta, & \text{if } x \in G \end{cases}$$

is an SU-generalized bi-ideal, where $\alpha, \beta \in U$ such that $\alpha \geq \beta$.

Theorem 5.6. Let $X$ be a nonempty subset of a ring $R$. Then, $X$ is a generalized bi-ideal of $R$ if and only if $S_X$ is an SU-generalized bi-ideal of $R$ over $U$.

It is known that every left (right, two sided) ideal of a ring $R$ is a bi-ideal of $R$. Moreover, we have the following:

Theorem 5.7. Every SU-left (right, two sided) ideal of a ring $R$ over $U$ is an SU-generalized bi-ideal of $R$ over $U$.

Proof. Let $f_R$ be an SU-left (right, two sided) ideal of $R$ over $U$ and $x, y, z \in R$. Then,

$$f_R(xyz) \subseteq f_R((xy)z) \subseteq f_R(z) \subseteq f_R(x) \cup f_R(z)$$

Thus, $f_R$ is an SU-generalized bi-ideal of $R$. \qed

Theorem 5.8. Let $f_R$ be any soft subset of a ring $R$ and $g_R$ be any SU-bi-ideal of $R$ over $U$. Then, the soft int-uni products $f_R \circ g_R$ and $g_R \circ f_R$ are SU-generalized bi-ideals of $R$ over $U$.

Proof. The proof is given for $f_R \circ g_R$. One can easily show that $(f_R \circ g_R)(x-y) \subseteq (f_R \circ g_R)(x) \cup (f_R \circ g_R)(y)$ for all $x, y \in R$ as shown in the proof of Lemma 4.5. Moreover,

$$(f_R \circ g_R) \circ \overline{\mathcal{O}} \circ (f_R \circ g_R) = f_R \circ (g_R \circ \overline{\mathcal{O}} \circ f_R \circ g_R) \supseteq f_R \circ (g_R \circ \overline{\mathcal{O}} \circ g_R) \supseteq f_R \circ g_R$$

It follows that $f_R \circ g_R$ is an SU-generalized bi-ideal of $R$ over $U$. It can be seen in a similar way that $g_R \circ f_R$ is an SU-generalized bi-ideal of $R$ over $U$. This completes the proof. \qed

Proposition 5.9. Let $f_R$ and $f_T$ be SU-generalized bi-ideals over $U$. Then, $f_R \vee f_T$ is an SU-generalized bi-ideal of $R \times T$ over $U$.

Proposition 5.10. If $f_R$ and $h_R$ are two SU-generalized bi-ideals of $R$ over $U$, then so is $f_R \cup h_R$ of $R$ over $U$.

Proposition 5.11. Let $f_R$ be a soft set over $U$ and $\alpha$ be a subset of $U$ such that $\alpha \in \mathcal{I}(f_R)$. If $f_R$ is an SU-generalized bi-ideal of $R$ over $U$, then $\mathcal{L}(f_R; \alpha)$ is a generalized bi-ideal of $R$.

Definition 5.12. If $f_R$ is an SU-generalized bi-ideal of $R$ over $U$, then generalized bi-ideals $\mathcal{L}(f_R; \alpha)$ are called lower $\alpha$ generalized bi-ideals of $f_R$.

Proposition 5.13. Let $f_R$ be a soft set over $U$, $\mathcal{L}(f_R; \alpha)$ be lower $\alpha$ generalized bi-ideals of $f_R$ for each $\alpha \subseteq U$ and $\mathcal{I}(f_R)$ be an ordered set by inclusion. Then, $f_R$ is an SU-generalized bi-ideal of $R$ over $U$.

Proposition 5.14. Let $f_R$ and $f_T$ be soft sets over $U$ and $\Psi$ be a ring isomorphism from $R$ to $T$. If $f_R$ is an SU-generalized bi-ideal of $R$ over $U$, then so is $\Psi(f_R)$ of $T$ over $U$.

Proposition 5.15. Let $f_R$ and $f_T$ be soft sets over $U$ and $\Psi$ be a ring homomorphism from $R$ to $T$. If $f_T$ is an SU-generalized bi-ideal of $T$ over $U$, then so is $\Psi^{-1}(f_T)$ of $R$ over $U$. 
6. Regular rings

In this section, regular ring is characterized in terms of $SU$-interior ideals, $SU$-quasi-ideals and $SU$-generalized-bi-ideals.

**Proposition 6.1.** [5] For a ring $R$, the following conditions are equivalent:
1) $R$ is regular.
2) $RL = R \cap L$ for every right ideal $R$ and left ideal $L$ of $R$.
3) $ARA = A$ for every quasi-ideal $A$ of $R$.

**Theorem 6.2.** For a ring $R$, the following conditions are equivalent:
1) $R$ is regular.
2) $fR \circ \tilde{\theta} \circ fR$ for every SU-generalized bi-ideal $fR$ of $R$ over $U$.
3) $fR = fR \circ \tilde{\theta} \circ fR$ for every SU-bi-ideal $fR$ of $R$ over $U$.
4) $fR = fR \circ \tilde{\theta} \circ fR$ for every SU-quasi-ideal $fR$ of $R$ over $U$.

**Proof.** First assume that (1) holds. Let $fR$ be any SU-generalized bi-ideal of $R$ over $U$, and $R$ be any element of $R$. Then, since $R$ is regular, there exists an element $x \in R$ such that $s = sx$. Thus, we have:

\[
(fR \circ \tilde{\theta} \circ fR)(s) = (fR \circ \tilde{\theta})(s) = \bigcup_{i=1}^{n} (fR \circ \tilde{\theta})(a_i) \cup fR(b_i)
\]

\[
\subseteq (fR \circ \tilde{\theta})(sx) \cup fR(s)
\]

\[
= \bigcup_{i=1}^{n} ((fR(n_i) \cup \tilde{\theta}(k_i)) \cup fR(s)
\]

\[
= fR(s)
\]

and so, we have $fR \circ \tilde{\theta} \circ fR \subseteq fR$. Since $fR$ is an SU-generalized bi-ideal of $R$, $fR \circ \tilde{\theta} \circ fR \subseteq fR$. Thus, $fR \circ \tilde{\theta} \circ fR = fR$ which means that (1) implies (2).

(2) implies (3) and (3) implies (4) is obvious. Assume that (4) holds. In order to show that $R$ is regular, we need to illustrate that $ARA = A$ for every quasi-ideal $A$ of $R$. Let $A$ be any quasi-ideal of $R$. Then, since $ARA \subseteq A(RR) \cup (RR)A \subseteq AR \cup RA \subseteq A$,

$ARA \subseteq A$. Therefore, it is enough to show that $A \subseteq ARA$. Conversely, let $a \in A$ and $a \notin ARA$. Then, by Theorem 5.6, the soft characteristic function $S_{K'}$ of $A$ is an SU-quasi-ideal of $S$. Thus, $(S_{K'})(a) = \emptyset$. Since $a \notin ASA$, this means that there do not exist $x, z \in A$ and $y \in R$ such that $a = xyz$. Since $\tilde{\theta}$ is an SU-quasi ideal of $S$, we have,

\[
(S_{K'} \circ \tilde{\theta} \circ S_{K'})(a) = U
\]

But this is a contradiction. Hence $A = ARA$. It follows by Proposition 3.13 that $R$ is regular, so (4) implies (1). \[\Box\]
Theorem 6.3. For a ring $R$ the following conditions are equivalent:

1) $R$ is regular.

2) $f_R \cup g_R = f_R \circ g_R \circ f_R$ for every SU-quasi-ideal $f_R$ of $R$ and SU-ideal $g_R$ of $R$ over $U$.

3) $f_R \cup g_R = f_R \circ g_R \circ f_R$ for every SU-quasi-ideal $f_R$ of $R$ and SU-interior ideal $g_R$ of $R$ over $U$.

4) $f_R \cup g_R = f_R \circ g_R \circ f_R$ for every SU-bi-ideal $f_R$ of $R$ and SU-ideal $g_R$ of $R$ over $U$.

5) $f_R \cup g_R = f_R \circ g_R \circ f_R$ for every SU-bi-ideal $f_R$ of $R$ and SU-interior ideal $g_R$ of $R$ over $U$.

6) $f_R \cup g_R = f_R \circ g_R \circ f_R$ for every SU-generalized bi-ideal $f_R$ of $R$ and SU-ideal $g_R$ of $R$ over $U$.

7) $f_R \cup g_R = f_R \circ g_R \circ f_R$ for every SU-generalized bi-ideal $f_R$ of $R$ and SU-interior ideal $g_R$ of $R$ over $U$.

Proof. First assume that (1) holds. Let $f_R$ be any SU-generalized bi-ideal and $g_R$ be any SU-interior ideal of $R$ over $U$. Then,

$$f_R \circ g_R \circ f_R \circ \tilde{\theta} \circ f_R \circ \tilde{\theta}$$

and

$$f_R \circ g_R \circ f_R \circ \tilde{\theta} \circ g_R \circ \tilde{\theta}$$

so $f_R \circ g_R \circ f_R \circ \tilde{\theta} \circ g_R \circ \tilde{\theta}$. To show that $f_R \cup g_R \circ f_R \circ g_R \circ f_R$ holds, let $s$ be any element of $R$. Since $R$ is regular, there exists an element $x$ in $R$ such that

$$s = sx(s = sx(sx))$$

Since $g_R$ is an SU-interior ideal of $R$, we have

$$(f_R \circ g_R \circ f_R)(s) = [f_R \circ (g_R \circ f_R)](s) = \sum_{i=1}^{m} [f_R(t_i) \cup (g_R \circ f_R)(t_i)]$$

$$(f_R \circ g_R \circ f_R)(s) \subseteq f_R(s) \cup (g_R \circ f_R)(ssxs)$$

$$= f_R(s) \cup \left\{ \sum_{i=1}^{m} [g_R(y_i) \cup f_R(z_i)] \right\}$$

$$= f_R(s) \cup g_R(ssxs) \cup f_R(s)$$

$$\subseteq f_R(s) \cup g_R(s) \cup f_R(s)$$

$$(f_R \cup g_R)(s)$$

so we have $f_R \cup g_R \subseteq f_R \circ g_R \circ f_R$. Thus we obtain that $f_R \cup g_R = f_R \circ g_R \circ f_R$, hence (1) implies (7).

It is clear that (7) implies (5), (5) implies (3), and that (3) implies (2). Also, (7) implies (6), (6) implies (4) and (4) implies (2) is obvious.

Assume that (2) holds. In order to show that $R$ is regular, it is enough to show that $f_R = f_R \circ \tilde{\theta} \circ f_R$ for all SU-quasi-ideal $f_R$ of $R$ over $U$ by Theorem 6.2. Since $\tilde{\theta}$ is an SU-ideal of $R$, we have

$$f_R = f_R \circ \tilde{\theta} \circ f_R.$$ 

Thus, $R$ is regular and (2) implies (1). This completes the proof. □

Theorem 6.4. For a ring $R$ the following conditions are equivalent:
1) $R$ is regular.

2) $f_R \cup g_R \supseteq f_R \circ g_R$ for every $SU$-quasi-ideal $f_R$ of $R$ and $SU$-left ideal $g_R$ of $R$ over $U$.

3) $f_R \cup g_R \supseteq f_R \circ g_R$ for every $SU$-bi-ideal $f_R$ of $R$ and $SU$-left ideal $g_R$ of $R$ over $U$.

4) $f_R \cup g_R \supseteq f_R \circ g_R$ for every $SU$-generalized bi-ideal $f_R$ of $R$ and $SU$-left ideal $g_R$ of $R$ over $U$.

Proof. First assume that (1) holds. Let $f_R$ be any $SU$-generalized bi-ideal and $g_R$ be any $SU$-left ideal of $R$ over $U$. Let $s$ be any element of $R$. Then, since $R$ is regular, there exists an element $x$ in $R$ such that $s = xs$. Thus, we have

$$(f_R \circ g_R)(s) = \bigcup_{i=1}^{m} (f_R(a_i) \cup g_R(b_i))$$

$$\subseteq f_R(s) \cup g_R(xs) \subseteq (f_R(s) \cup g_R(s)) = (f \cup g_R)(s)$$

Thus, $f_R \circ g_R \subseteq f_R \cup g_R$. Hence, we obtain that (1) implies (4).

It is clear that (4) implies (3), (3) implies (2). Assume that (2) holds. Since $f_R \cup g_R \supseteq f_R \circ g_R$ always holds for every $SU$-right ideal of $R$ is an $SU$-quasi-ideal of $R$, we have $f_R \cup g_R = f_R \circ g_R$ for every $SU$-right ideal $f_R$ and $SU$-left ideal $g_R$ of $R$. Thus, it follows by Theorem 2.18 that $R$ is regular and (2) implies (1).

**Theorem 6.5.** For a ring $R$ the following conditions are equivalent:

1) $R$ is regular.

2) $h_R \cup f_R \cup g_R = h_R \circ f_R \circ g_R$ for every $SU$-right ideal $h_R$, every $SU$-quasi-ideal $f_R$ and every $SU$-left ideal $g_R$ of $R$.

3) $h_R \cup f_R \cup g_R = h_R \circ f_R \circ g_R$ for every $SU$-right ideal $h_R$, every $SU$-bi-ideal $f_R$ and every $SU$-left ideal $g_R$ of $R$.

4) $h_R \cup f_R \cup g_R = h_R \circ f_R \circ g_R$ for every $SU$-right ideal $h_R$, every $SU$-generalized bi-ideal $f_R$ and every $SU$ left-ideal $g_R$ of $R$.

Proof. Assume that (1) holds. Let $h_R$, $f_R$ and $g_R$ be any $SU$-right ideal, $SU$-generalized bi-ideal and $SU$-left ideal of $R$, respectively. Let $a$ be any element of $R$. Since $R$ is regular, there exists an element $x$ in $R$ such that $a = axa$. Hence, we have:

$$(h_R \circ f_R \circ g_R)(a) = [h_R \circ (f_R \circ g_R)](a)$$

$$= \bigcup_{i=1}^{m} [h_R(y_i) \cup (f_R \circ g_R)(z_i)]$$

$$\subseteq h_R(ax) \cup (f_R \circ g_R)(a) \subseteq h_R(ax) \cup \bigcup_{i=1}^{m} [f_R(p_i) \cup g_R(q_i)]$$

$$= h_R(ax) \cup \bigcup_{i=1}^{m} [f_R(p_i) \cup g_R(q_i)]$$

$$\subseteq h_R(a) \cup (f_R(a) \cup g_R(ax)) \subseteq h_R(a) \cup (f_R(a) \cup g_R(a))$$

$$= (h_R \cup f_R \cup g_R)(a)$$
Let $R$ be a regular ring and $f$ be an SU-right ideal and $SU$-left ideal of $R$, respectively. It is obvious that

$$h_R \circ g_R \supseteq h_R \cup g_R.$$  

Since $\bar{\theta}$ itself is an SU-quasi-ideal of $R$, by assumption we have:

$$h_R \cup g_R = h_R \cup \bar{\theta} \cup g_R \circ \bar{\theta} \circ g_R \circ h_R.$$  

It follows that $h_R \cup g_R \supseteq h_R \circ g_R$ for every SU-right ideal $h_R$ and SU-left ideal $g_R$ of $R$. It follows by Theorem 2.18 that $R$ is regular. Hence, (2) implies (1). This completes the proof. \(\square\)

**Proposition 6.6.** [19] A ring $R$ is regular if and only if every SU-left (right, two-sided) ideal of $R$ is idempotent.

**Proposition 6.7.** Let $R$ be a regular ring and $f_R$ be an SU-quasi-ideal of $R$. Then,

$$(\bar{\theta} \circ f_R) \cup (f_R \circ \bar{\theta}) = f_R.$$  

**Proof.** Let $f_R$ be any SU-quasi-ideal of $R$. Then, $(\bar{\theta} \circ f_R) \cup (f_R \circ \bar{\theta}) \supseteq f_R$. Thus, it suffices to show that $f_R \supseteq (\bar{\theta} \circ f_R) \cup f_R$ is an SU-right ideal of $R$. In fact,

$$(\bar{\theta} \circ f_R) \cup (f_R \circ \bar{\theta}) = (\bar{\theta} \circ f_R) \cup ((\bar{\theta} \circ f_R) \circ \bar{\theta} \circ f_R) = (\bar{\theta} \circ f_R) \cup (\bar{\theta} \circ f_R) \cup (\bar{\theta} \circ f_R) = (\bar{\theta} \circ f_R) \cup (\bar{\theta} \circ f_R) \cup (\bar{\theta} \circ f_R) \cup (\bar{\theta} \circ f_R).$$  

And $(f_R \cup (\bar{\theta} \circ f_R))(x - y) \subseteq (f_R \cup (\bar{\theta} \circ f_R))(x) \cup (f_R \cup (\bar{\theta} \circ f_R))(y)$. Since $R$ is regular, every SU-left (right) ideal of $R$ is idempotent by Proposition 6.6. Thus, we have

$$f_R \supseteq (\bar{\theta} \circ f_R) \cup (f_R \circ \bar{\theta}) = f_R.$$  

that is to say $f_R \supseteq \bar{\theta} \circ f_R$. Similarly, one can show that $f_R \supseteq \bar{\theta} \circ f_R$. Thus, $f_R \supseteq (\bar{\theta} \circ f_R) \cup (f_R \circ \bar{\theta})$ and so,

$$(\bar{\theta} \circ f_R) \cup (f_R \circ \bar{\theta}) = f_R.$$  

\(\square\)

**Theorem 6.8.** Let $f_R$ be a soft set and $R$ be a regular ring. Then, the following conditions are equivalent:

1) $f_R$ is an SU-quasi-ideal of $R$.

2) $f_R$ may be presented in the form $f_R = g_R \circ h_R$, where $g_R$ is an SU-right ideal and $h_R$ is an SU-left ideal of $R$.

**Proof.** Assume that (1) holds. Since $R$ is regular, it follows by Theorem 6.2 that $f_R = f_R \circ \bar{\theta} \circ f_R$, where $f_R$ is an SU-quasi-ideal of $R$. Thus,

$$f_R = f_R \circ \bar{\theta} \circ f_R = f_R \circ (\bar{\theta} \circ \bar{\theta}) \circ f_R = (f_R \circ \bar{\theta}) \circ (\bar{\theta} \circ f_R).$$  

Since $f_R \circ \bar{\theta}$ is an SU-right ideal of $R$ and $\bar{\theta} \circ f_R$ is an SU-left ideal of $R$, (1) implies (2).

Conversely, assume that $f_R = g_R \circ h_R$, where $g_R$ is an SU-right ideal and $h_R$ is an SU-left ideal of $R$. Then, by Proposition 4.11, $g_R \circ h_R$ is an SU-quasi-ideal of $R$. \(\square\)
Proposition 6.9. Let $R$ be a regular ring and $f_R$ be an $SU$-quasi-ideal of $R$. Then, $(f_R)^2 = (f_R)^3$.

Proof. Let $R$ be a regular ring and $f_R$ be an $SU$-quasi-ideal of $R$. Then, by Corollary 4.10, $(f_R)^2$ is an $SU$-quasi-ideal of $R$ and by Theorem 6.2,

$$(f_R)^2 = (f_R)^2 \circ \overline{θ} \circ (f_R)^2 = f_R \circ f_R \circ \overline{θ} \circ f_R = f_R \circ (f_R \circ \overline{θ} \circ f_R) \circ f_R = f_R \circ f_R \circ f_R = (f_R)^3$$

7. Regular duo rings

In this section, a left (right) duo ring is characterized in terms of $SU$-ideals. A ring $R$ is called left (right) duo if every left (right) ideal of $R$ is a two-sided ideal of $R$. A ring $R$ is duo if it is both left and right duo.

Definition 7.1. A ring $R$ is called soft left (right) duo if every $SU$-left (right) ideal of $R$ is an $SU$-ideal of $R$ and is called soft duo, if it is both soft left and soft right duo.

Theorem 7.2. [19] For a regular ring $R$, the following conditions are equivalent:

1) $R$ is duo.
2) $R$ is soft duo.

Theorem 7.3. [20] For a ring $R$, the following conditions are equivalent:

1) $R$ is a regular duo ring.
2) $A \cap B = AB$ for every left ideal $A$ and every right ideal $B$ of $R$.
3) $Q^2 = Q$ for every quasi-ideal of $R$. (That is, every quasi-ideal is idempotent.)
4) $EQE = E \cap Q$ for every ideal $E$ and every quasi-ideal $Q$ of $R$.

Theorem 7.4. For a ring $R$, the following conditions are equivalent:

1) $R$ is a regular duo ring.
2) $R$ is a regular soft duo ring.
3) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-bi-ideals $f_R$ and $g_R$ of $R$.
4) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-bi-ideal $f_R$ and for all $SU$-quasi-ideal $g_R$ of $R$.
5) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-bi-ideal $f_R$ and for all $SU$-right ideal $g_R$ of $R$.
6) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-quasi-ideal $f_R$ and for all $SU$-bi-ideal $g_R$ of $R$.
7) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-quasi-ideals $f_R$ and $g_R$ of $R$.
8) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-quasi-ideal $f_R$ and for all $SU$-right ideal $g_R$ of $R$.
9) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-left ideal $f_R$ and for all $SU$-bi-ideal $g_R$ of $R$.
10) $f_R \circ g_R = f_R \overline{θ} g_R$ for all $SU$-left ideal $f_R$ and for all $SU$-right ideal $g_R$ of $R$.
11) $f_R \circ g_R = f_R \overline{θ} g_R$ and $h_R \circ k_R = h_R \overline{θ} k_R$ for all $SU$-right ideals $f_R$ and $g_R$ of $R$ and for all $SU$-left ideal $h_R$ and $k_R$ of $R$.
12) Every $SU$-quasi-ideal of $R$ is idempotent.
Proof. The equivalence of (1) and (2) follows from Theorem 7.3. Assume that (2) holds. Let $f_R$ and $g_R$ be any $SU$-bi-ideals of $R$. Then, $f_R$ is an $SU$-right ideal of $R$ and $g_R$ is an $SU$-left ideal of $R$. Since $R$ is regular,

$$f_R \circ g_R = f_R \overline{g_R}$$

Thus, (2) implies (3). It is clear that (3) implies (4), (4) implies (5), (5) implies (8), (8) implies (11), (11) implies (3), (3) implies (6), (6) implies (7), (7) implies (8) and (6) implies (9), (9) implies (10), (10) implies (11).

Assume that (11) holds. Let $A$ and $B$ be any left ideal and right ideal of $R$, respectively. Let $a$ be any element of $A \cap B$ and $a \notin AB$. Then, $a \in A$ and $a \in B$ and there do not exist $x \in A$ and $y \in B$ such that $a = xy$. Since $S_A$ and $S_B$ is an $SU$-left ideal and $SU$-right ideal of $S$, respectively, we have

$$S_A(a) = S_B(a) = \emptyset.$$ and

$$(S_A \circ S_B)(a) = U$$

But this is a contradiction, so $a \in AB$. Thus, $A \cap B \subseteq AB$. For the converse inclusion, let $a$ be any element of $AB$ and $a \notin A \cap B$. Then, there exist $y \in A$ and $z \in B$ such that $a = yz$. Thus,

$$(S_A \cup S_B)(a) = U$$

and

$$(S_A \cup S_B)(a) = \bigcap_{i=1}^{k} (S_A(m_i) \cup S_B(n_i)) \subseteq (S_A(y) \cup S_B(z)) = \emptyset.$$ Hence, $(S_A \circ S_B)(a) = \emptyset$. But this is a contradiction. This implies that $a \in A \cap B$ and that $AB \subseteq A \cap B$. Thus, we have $AB = A \cap B$. It follows by Theorem 7.3 that $R$ is a regular duo ring. Thus (11) implies (1). It is clear that (7) implies (12) by taking $g_R = f_R$.

Conversely, assume that (12) holds. Let $Q$ be any quasi-ideal of $S$ and $a$ be any element of $Q$ and $a \notin QQ$. Then, $S_Q$ is an $SU$-quasi-ideal of $S$. Thus, we have $S_Q(a) = \emptyset$ and since there do not exist $y, z \in Q$ such that $a = yz$,

$$(S_Q \circ S_Q)(a) = U$$

But this is a contradiction. Hence, we have $a \in Q^2$ and $Q \subseteq Q^2$. Since the converse inclusion always holds, $Q = Q^2$. It follows by Theorem 7.3 that $R$ is a regular duo ring and that (12) implies (1). This completes the proof. □

Theorem 7.5. For a ring $R$, the following conditions are equivalent:

1) $R$ is a regular duo ring.
2) $f_R \circ g_R \circ f_R = f_R \overline{g_R}$ for every $SU$-ideal $f_R$ and every $SU$-bi-ideal $g_R$ of $R$.
3) $f_R \circ g_R \circ f_R = f_R \overline{g_R}$ for every $SU$-ideal $f_R$ and every $SU$-quasi-ideal $g_R$ of $R$.

Proof. First assume that (1) holds. Let $f_R$ and $g_R$ be any $SU$-bi-ideal and any $SU$-ideal of $R$, respectively. Then, we have

$$f_R \circ g_R \circ (f_R \circ \overline{g_R}) \circ \overline{f_R} = f_R \circ (\overline{g_R} \circ f_R) \circ \overline{g_R} \circ \overline{f_R}.$$ On the other hand, since $R$ is regular and duo, $f_R$ is an $SU$-ideal of $R$. Hence, we have

$$f_R \circ g_R \circ f_R \overline{g_R} \circ \overline{f_R} \overline{g_R}$$

and so

$$f_R \circ g_R \circ f_R \overline{g_R} \circ \overline{f_R} \overline{g_R}$$
In order to show the converse inclusion, let \( a \) be any element of \( R \). Then, since \( R \) is regular, there exists an element \( x \) in \( R \) such that
\[
a = axa = (xa)x
\]
Thus, we have
\[
(f_R \circ g_R \circ f_R)(a) = \bigcap_{a=\sum_{i=1}^{n} x_i y_i} \bigl( f_R(x_i) \cup (g_R \circ f_R)(y_i) \bigr)
\]
\[
\subseteq f_R(ax) \cup (g_R \circ f_R)(axa)
\]
\[
= f_R(ax) \cup \bigl( \bigcap_{ax=\sum_{i=1}^{n} p_i q_i} \bigl[ g_R(p_i) \circ f_R(q_i) \bigr] \bigr)
\]
and so \( f_R \circ g_R \circ f_R \subseteq f_R \cup g_R \). Thus, we obtain that
\[
f_R \circ g_R \circ f_R = f_R \cup g_R.
\]
Hence, (1) implies (2). It is clear that (2) implies (3).

Assume that (3) holds. Let \( E \) and \( Q \) any two-sided ideal and quasi-ideal of \( S \), respectively and \( a \) be any element of \( E \cap Q \) and \( a \notin EQE \). Then, \( a \in E \) and \( a \in Q \) and there do not exist \( x, z \in E \) and \( y \in Q \) such that \( a = xyz \). Since \( S_E \) and \( S_Q \) is an \( SU \)-ideal and \( SU \)-quasi-ideal of \( S \), respectively, we have
\[
S_E(a) = S_Q(a) = \emptyset.
\]
and
\[
(S_E \circ S_Q \circ S_E)(a) = U.
\]
But, this is a contradiction and so \( a \in EQE \). Thus, \( E \cap Q \subseteq EQE \). For the converse inclusion, let \( a \) be any element of \( EQE \) and \( a \notin E \cap Q \). Then, there exist \( x, z \in E \) and \( y \in Q \) such that \( a = xyz \). Thus,
\[
(S_E \cup S_Q)(a) = U
\]
and
\[
(S_E \circ S_Q \circ S_E)(a) = \emptyset.
\]
But this is a contradiction and so \( a \in E \cap Q \). Thus, \( EQE \subseteq E \cap Q \) and so \( EQE = E \cap Q \). It follows from Theorem 7.3 that \( R \) is regular duo. Hence, (3) implies (1). This completes the proof.

8. Intra-regular rings

In this section, an intra-regular ring is characterized in terms of \( SU \)-interior ideals, \( SU \)-quasi-ideals and \( SU \)-generalized-bi-ideals. A ring \( R \) is called intra-regular [20] if for every element \( a \) of \( R \) there exist elements \( x_i \) and \( y_i \) in \( R \) such that
\[
a = \sum_{i=1}^{n} x_i a^2 y_i = \sum_{i=1}^{n} (x_i a)(ay_i).
\]
Proposition 8.1. 4.4 For a soft set $f_R$ of an intra-regular ring $R$, the following conditions are equivalent:

1) $f_R$ is an SU-ideal of $R$.

2) $f_R$ is an SU-interior ideal of $R$.

Proof. (1) implies (2) is clear. Assume that (2) holds. Let $a$ and $b$ be any elements of $R$. Then, since $R$ is intra-regular, there exist elements $x_i, y_i, u_i$ and $v_i$ in $R$ such that $a = \sum_{i=1}^{n} x_i a^2 y_i$ and $b = \sum_{i=1}^{n} u_i b^2 v_i$ for all $1 \leq i \leq n$. Since $f_R$ is an SU-interior ideal of $R$, we have

$$f_R(ab) = f_R(\sum_{i=1}^{n} x_i a^2 y_i) = f_R(\sum_{i=1}^{n} x_i a^2 y_i b) \subseteq \bigcap_{1 \leq i < n} f_R(x_i a^2 y_i b) \subseteq \bigcap_{1 \leq i < n} f_R(a) = f_R(a)$$

and

$$f_R(ab) = f_R(a(\sum_{i=1}^{n} u_i b^2 v_i)) = f_R(\sum_{i=1}^{n} au_i b^2 v_i) \subseteq \bigcap_{1 \leq i < n} f_R(au_i b^2 v_i) \subseteq \bigcap_{1 \leq i < n} f_R(b) = f_R(b)$$

Hence, $f_R$ is an SU-ideal of $R$ and (2) implies (1). □

Theorem 8.2. For a ring $R$, the following conditions are equivalent:

1) $R$ is intra-regular.

2) $f_R(a) = f_R(a^2)$ for all SU-interior ideal of $R$ and for all $a \in R$.

Proof. First assume that (1) holds. Let $f_R$ be any SU-interior ideal of $R$ and $a$ any element of $R$. Since $R$ is intra-regular, there exist elements $x_i$ and $y_i$ in $R$ such that $a = \sum_{i=1}^{n} x_i a^2 y_i$. Thus, for all $1 \leq i \leq n,$

$$f_R(a) = f_R(\sum_{i=1}^{n} x_i a^2 y_i) \subseteq \bigcap_{1 \leq i < n} f_R(x_i a^2 y_i) \subseteq \bigcap_{1 \leq i < n} f_R(a^2) \subseteq \bigcap_{1 \leq i < n} f_R(aa) \subseteq \bigcap_{1 \leq i < n} f_R(a) = f_R(a)$$

so, we have $f_R(a) = f_R(a^2)$. Hence, (1) implies (3). Now assume that (3) holds. It is known that $[a^2]$ is an interior-ideal of $R$. Thus, the soft characteristic function $S_{[a^2]}$ is an SU-ideal of $R$. Since $a^2 \in [a^2]$, we have;

$$S_{[a^2]}(a) = S_{[a^2]}(a^2) = \emptyset$$

Thus, $a \in [a^2] = m[a^2] + n[a^4] + Sa^2S$. Here, $R$ is intra-regular. Thus, (3) implies (1). This completes the proof. □

Theorem 8.3. Let $R$ be an intra-regular ring. Then, for every SU-interior ideal $f_R$ of $R$,

$$f_R(ab) = f_R(ba)$$

for all $a, b \in R$.

Proof. Let $f_R$ be an SU-ideal of an intra-regular ring $R$. Then, by Theorem 8.2, we have;

$$f_R(ab) = f_R((ab)^2) = f_R(a(ba)b) \subseteq f_R(ba) = f_R((ba)^2) = f_R(b(ab)a) \subseteq f_R(ab)$$

so, we have $f_R(ab) = f_R(ba)$. This completes the proof. □

Theorem 8.4. [20] A ring $R$ is regular and intra-regular if and only if every quasi-ideal of $R$ is idempotent.
Theorem 8.5. For a ring $R$, the following conditions are equivalent:

1) $R$ is both regular and intra-regular.

2) $f_R \circ f_R = f_R$ for every SU-quasi-ideal $f_R$ of $R$. (That is, every SU-quasi-ideal of $R$ is idempotent).

3) $f_R \circ f_R = f_R$ for every SU-bi-ideal $f_R$ of $R$. (That is, every SU-bi-ideal of $R$ is idempotent).

4) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-quasi-ideals $f_R$ and $g_R$ of $R$.

5) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-bi-ideal $f_R$ and $SU$-bi-ideal $g_R$ of $R$.

6) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-quasi-ideal $f_R$ and for every SU-generalized bi-ideal $g_R$ of $R$.

7) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-bi-ideal $f_R$ and for every SU-quasi-ideal $g_R$ of $R$.

8) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-bi-ideals $f_R$ and $g_R$ of $R$.

9) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-bi-ideal $f_R$ and for every SU-generalized bi-ideal $g_R$ of $R$.

10) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-generalized bi-ideal $f_R$ and for every SU-quasi-ideal $g_R$ of $R$.

11) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-generalized bi-ideal $f_R$ and for every SU-bi-ideal $g_R$ of $R$.

12) $f_R \bigcup g_R \bigcirc (f_R \circ g_R) \bigcup (g_R \circ f_R)$ for every SU-generalized bi-ideals $f_R$ and $g_R$ of $R$.

Proof. First assume that (1) holds. In order to show that (12) holds, let $f_R$ and $g_R$ be SU-generalized bi-ideals of $R$ and $a \in R$. Since $R$ is intra-regular, there exist elements $y_i$ and $z_i$ in $R$ such that $a = \sum_{i=1}^{n} y_i a z_i$ for every element $a$ of $R$. Thus,

$$a = axa = axa = ax(\sum_{i=1}^{n} y_i a z_i)xa = \sum_{i=1}^{n} (axy)(az,xa)$$

Then, for all $1 \leq i \leq n$, we have

$$(f_R \circ g_R)(a) = \bigcup_{x=a} \bigcup_{b_i c_i} (f_R(b_i) \cup g_R(c_i))$$

$$\subseteq f_R(a(xy)a) \cup g_R(a(z)a)$$

$$\subseteq f_R(a) \cup g_R(a)$$

and so we have $f_R \bigcup g_R \bigcirc f_R \circ g_R$. This shows that (1) implies (12).

It is obvious that (12) implies (11), (11) implies (10), (10) implies (4), (4) implies (2) and (12) implies (9), (9) implies (8), (8) implies (7), (7) implies (4), (12) implies (6), (6) implies (5), (5) implies (4) and (8) implies (3) and (3) implies (2).

Assume that (2) holds. Let $Q$ be quasi-ideal of $S$ and $a$ be any element of $Q$. Then, $QQ \subseteq Q$ always holds. We show that $Q \subseteq QQ$. Conversely, let $x \in Q$ and $x \notin QQ$. Then, there do not exist $y, z \in Q$ such that $x = yz$. Since $Q$ is a quasi-ideal of $S$, the soft characteristic function $S_Q$ is an SU-quasi-ideal of $S$. So we have, $S_Q(x) = \emptyset$ and

$$(S_Q \circ S_Q)(x) = \bigcup_{x \in QQ} (S_Q(y) \cup S_Q(z)) = U$$

But, this contradicts with our hypothesis. So, $a \in QQ$ Thus, $Q \subseteq QQ$ and so $Q = QQ = Q^2$. It follows that $Q$ is both regular and intra-regular, so (2) implies (1) by Theorem 8.4. $\Box$
Theorem 8.6. [19] For a ring $R$ the following conditions are equivalent:

1) $R$ is intra-regular.

2) $g_R \cup f_R \supset g_R \circ f_R$ for every SU-right ideal $f_R$ of $R$ and SU-left ideal $g_R$ of $R$ over $U$.

Theorem 8.7. For a ring $R$ the following conditions are equivalent:

1) $R$ is both regular and intra-regular.

2) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-right ideal $f_R$ and for every SU-left ideal $g_R$ of $R$.

3) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-right ideal $f_R$ and for every SU-quasi-ideal $g_R$ of $R$.

4) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-right ideal $f_R$ and for every SU-bi-ideal $g_R$ of $R$.

5) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-right ideal $f_R$ and for every SU-generalized bi-ideal $g_R$ of $R$.

6) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-left ideal $f_R$ and for every SU-quasi-ideal $g_R$ of $R$.

7) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-left ideal $f_R$ and for every SU-bi-ideal $g_R$ of $R$.

8) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-left ideal $f_R$ and for every SU-generalized bi-ideal $g_R$ of $R$.

9) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-quasi-ideals $f_R$ and $g_R$ of $R$.

10) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-quasi-ideal $f_R$ and for every SU-bi-ideal $g_R$ of $R$.

11) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-quasi-ideal $f_R$ and for every SU-generalized bi-ideal $g_R$ of $R$.

12) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-bi-ideals $f_R$ and $g_R$ of $R$.

13) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-bi-ideal $f_R$ and for every SU-generalized bi-ideal $g_R$ of $R$.

14) $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ for every SU-generalized bi-ideals $f_R$ and $g_R$ of $R$.

Proof. Assume that (1) holds. Let $f_R$ and $g_R$ be any SU-generalized bi-ideals of $R$. Then, it follows by Theorem 8.6 that $f_R \supset g_R \supset f_R \circ g_R$. Moreover, we have

$$f_R \supset g_R = g_R \supset f_R \circ g_R \circ f_R.$$ 

Thus, we have $f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R)$ and so (1) implies (14). It is obvious that (14) implies (13), (13) implies (12), (12) implies (11), (11) implies (10), (10) implies (9) and (14) implies (8), (8) implies (7), (7) implies (6) and (6) implies (5). Thus, (14) implies (1), (5) implies (4), (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let $f_R$ and $g_R$ be any SU-right ideal and SU-left ideal of $R$, respectively. Then,

$$f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R) \supset g_R \circ f_R.$$ 

It follows by Theorem 8.6 that $R$ is intra-regular. On the other hand,

$$f_R \supset g_R \supset (f_R \circ g_R) \cup (g_R \circ f_R) \supset f_R \circ g_R.$$ 

Since $f_R \circ g_R = f_R \cup g_R$ always holds, we have $f_R \circ g_R = f_R \cup g_R$. Thus, it follows by Theorem 2.18 that $R$ is regular. Thus, (2) implies (1). □
9. Strongly regular rings

In this section, a strongly regular ring is characterized in terms of SU-ideals. An element $a$ of $R$ is called a strongly regular if there exists an element $x \in R$ such that

$$a = xa^2 = a^3x.$$ 

for all $a \in R$. Such a ring is regular and duo.

**Theorem 9.1.** For a ring $R$ the following conditions are equivalent:

1) $R$ is strongly regular.

2) Every quasi-ideal of $R$ is semiprime.

3) Every bi-ideal of $R$ is semiprime.

4) Every generalized bi-ideal of $R$ is semiprime.

5) Every SU-quasi-ideal of $R$ is soft union semiprime.

6) Every SU-bi-ideal of $R$ is soft union semiprime.

7) Every SU-generalized bi-ideal of $R$ is soft union semiprime.

8) $f_R(a) = f_R(a^2)$ for every SU-quasi-ideal $f_R$ of $R$ and for all $a \in R$.

9) $f_R(a) = f_R(a^2)$ for every SU-bi-ideal $f_R$ of $R$ and for all $a \in R$.

10) $f_R(a) = f_R(a^2)$ for every SU-generalized bi-ideal $f_R$ of $R$ and for all $a \in R$.

**Proof.** First assume that (1) holds. Let $f_R$ be any SU-generalized bi-ideal of $R$. Since $R$ is strongly regular, there exists an element $x \in R$ such that $a = a^3xa^2$. Thus, we have

$$f_R(a) = f_R(a^2x^2) \subseteq f_R(a^2) \cup f_R(a^3) = f_R(a^2) = f_R(aa) = f_R(a(a^2x^2)) = f_R(a(a^2xa^2)) \subseteq f_R(a) \cup f_R(a) = f_R(a)$$

and so, $f_R(a) = f_R(a^2)$. Thus (1) implies (10).

It is clear that (10) implies (9), (9) implies (8), (8) implies (5) and (10) implies (7), (7) implies (6), (6) implies (5) and that (10) implies (4), (4) implies (3) and (3) implies (2).

Assume that (5) holds. Let $Q$ be any quasi-ideal of $S$ and $a^2 \in Q$ and $a \not\in Q$. Since the soft characteristic function $S_Q$ is an SU-quasi-ideal of $S$, it is soft union semiprime by hypothesis. Thus,

$$S_Q(a) = U \subseteq S_Q(a^2) = \emptyset$$

But, this is a contradiction. Hence, $a \in Q$ and so $Q$ is semiprime. Thus (5) implies (2).

Finally assume that (2) holds. Let $a$ be any element of $R$. Then, since the principal ideal $Q[a^2]$ generated by $a^2$ is quasi-ideal and so by assumption semiprime and since $a^2 \in Q[a^2]$,

$$S_{Q[a^2]}(a) = S_{Q[a^2]}(a^2) = \emptyset$$

implying that

$$a \in Q[a^2] = m[a^2] + n[a^4] + (a^2S \cap Sa^2).$$

Hence, $R$ is strongly regular. Thus (2) implies (1). □
10. Conclusion

In this paper, the concepts of soft union interior ideals, soft union quasi-ideals and soft union generalized bi-ideals of rings have been introduced and studied. Moreover, regular, regular duo, intra-regular and strongly regular rings have been characterized by the properties of these soft union ideals. Based on these results, some further work can be done on the properties of soft union rings, which may be useful to characterize the classical rings, especially in the mean of regularity.

References