Filomat 32:6 (2018), 2139–2151
https://doi.org/10.2298/FIL1806139A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# The Essential Approximate Pseudospectrum and Related Results

**Aymen Ammar[a], Aref Jeribi[a], Kamel Mahfoudhi[a]**

[a]*Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Route de soukra Km 3.5, B.P. 1171, 3000, Sfax Tunisia*

**Abstract.** In this paper, we introduce and study the essential approximate pseudospectrum of closed, densely defined linear operators in the Banach space. We begin by the definition and we investigate the characterization, the stability by means of quasi-compact operators and some properties of these pseudospectrum.

## 1. Introduction

Instead of the spectra that is the traditional tool. The theory of the pseudospectra is a new method for studying matrices and linear operators . It reveals information on the behavior of normal matrices or operators. However, it is less informative as the matrix or the operator are non-normal. Pseudospectra have nevertheless proved to be an efficient tool to study them. They provide an analytical and graphical alternative to study this type of case. The definition of pseudospectra of a closed densely defined linear operator $T$, for every $\varepsilon > 0$, is given by

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

The pseudospectrum is defined, sometimes as

$$\Sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon}\right\}.$$

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$, (spectrum of $T$). For $\varepsilon > 0$, it can be shown that $\sigma_\varepsilon(T)$ is a larger set and is never empty. The pseudospectra of $T$ are a family of strictly nested closed sets, which grow to fill the whole complex plane as $\varepsilon \to \infty$ (see [3, 8, 18, 19]). From these definitions, it follows that the pseudospectra associated with various $\varepsilon$ are nested sets. Then for all $0 < \varepsilon_1 < \varepsilon_2$, we have

$$\begin{cases} \sigma(T) \subset \sigma_{\varepsilon_1}(T) \subset \sigma_{\varepsilon_2}(T), \text{ and} \\ \sigma(T) \subset \Sigma_{\varepsilon_1}(T) \subset \Sigma_{\varepsilon_2}(T), \end{cases}$$

2010 *Mathematics Subject Classification*. 47A53 - 47A55 - 54B35 - 47A13
*Keywords*. Approximate pseudospectrum, essential approximate pseudospectrum.
Received: 18 April 2016; Revised: 17 May 2018; Accepted: 12 August 2018
Communicated by Dragan S. Djordjević
*Email addresses:* ammar_aymen84@yahoo.fr (Aymen Ammar), Aref.Jeribi@fss.rnu.tn (Aref Jeribi), kamelmahfoudhi72@yahoo.com (Kamel Mahfoudhi)

and that the intersections of all the pseudospectra are the spectra,

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(T) = \sigma(T) \text{ and } \bigcap_{\varepsilon>0} \Sigma_\varepsilon(T) = \Sigma(T).$$

In Refs [1, 2, 4, 5, 10, 11], A. Ammar and A. Jeribi introduced the definition of Ammar-Jeribi pseudospectrum of a closed densely defined linear operator on a Banach space $X$ by

$$\begin{aligned}\sigma_{w,\varepsilon}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_\varepsilon(T+K),\\ &:= \bigcup_{\|D\|<\varepsilon} \sigma_w(T+D) \qquad \text{(see [1, Theorem 2.3])},\end{aligned}$$

where

$$\sigma_w(T) := \bigcap_{K\in\mathcal{K}(X)} \sigma(T+K),$$

and $\mathcal{K}(X)$ is the subspace of compact operators from $X$ into $X$.

One impetus for writing this paper is the issue of approximate pseudospectrum introduced by M. P. H. Wolff in (2001). The latter study motivates us to investigate the essential approximate pseudospectrum of closed, densely defined linear operators on a Banach space. We survey the historical development of this subject. In 1967, J. M. Varah [22] introduced the first idea of pseudospectra. In 1986, J. H. Wilkinson [23] came with the modern interpretation of pseudospectrum where he defined it for an arbitrary matrix norm induced by a vector norm. Throughout the 1990$s$, L. N. Trefethen [17–19, 21] not only initiated the study of pseudospectrum for matrices and operators, but also he talked of approximate eigenvalues and pseudospectrum and used this notion to study interesting problems in mathematical physics. By the same token, several authors worked on this field. For example, we may refer to E. B. Davies [7], A. Harrabi [8] and M. P. H. Wolff [24] who has introduced the term approximate pseudospectrum for linear operators.

In this paper, the notion of essential approximate pseudospectrum can be extending our studies of this process from the essential spectrum to the essential approximate spectrum. For $\varepsilon > 0$ and closed densely defined operator $T$, we define

$$\begin{aligned}\sigma_{eap,\varepsilon}(T) &= \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+K), \qquad (1)\\ \Sigma_{eap,\varepsilon}(T) &= \bigcap_{K\in\mathcal{K}(X)} \Sigma_{ap,\varepsilon}(T+K),\end{aligned}$$

where

$$\begin{aligned}\sigma_{ap,\varepsilon}(T) &:= \sigma_{ap}(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| < \varepsilon\right\},\\ \Sigma_{ap,\varepsilon}(T) &:= \sigma_{ap}(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| \le \varepsilon\right\},\end{aligned}$$

and

$$\sigma_{ap}(T) := \left\{\lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| = 0\right\}.$$

In the following, we measure the sensitivity of the set $\sigma_{ap}(T)$ with respect to additive perturbations of $T$ by an bounded operator $D$ of a norm less than $\varepsilon$. So we define the approximate pseudospectrum of $T$ by

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T+D), \qquad \text{(see Theorem 3.3)} \qquad (2)$$

and we characterize the essential approximate pseudospectrum.

The essential approximate pseudospectrum $\sigma_{eap,\varepsilon}(.)$ nicely blends these properties of the essential and the approximate pseudospectrum, and accordingly we are interested by the following essential approximate spectrum

$$\sigma_{eap}(T) \quad := \quad \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K). \tag{3}$$

We can now see that (1) inherits an $\varepsilon$-version from (3). We will also show that there is an essential version of (2), that is

$$\sigma_{eap,\varepsilon}(T) \quad = \quad \bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T+D) \qquad \text{(see Theorem 4.3).}$$

The paper is organized as follows. Section 2 contains preliminary and auxiliary properties that we will need in order to prove the main results of the other sections. The main aim of section 3 is to characterize the essential approximate pseudospectrum of closed, densely defined linear operators on a Banach space. Then we give different definitions of approximate pseudospectrum and we establish relations between approximate pseudospectrum and the union of the spectra approximate point of all perturbed operators with perturbations that have norms strictly less than $\varepsilon$. Finally, we will prove the invariance of the essential approximate pseudospectrum and establish some results of perturbation on the context of closed, densely defined linear operators on a Banach space.

## 2. Preliminaries

The goal of this section consists in establishing some preliminary results which will be needed in the sequel. Throughout the paper, we denote by $\mathcal{L}(X)$ (resp. $\mathcal{C}(X)$) the set of all bounded (resp. closed, densely defined) linear operators from $X$ into $X$. For $T \in \mathcal{C}(X)$, we denote by $\rho(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, the resolvent set, the null space and the range of $T$. The nullity of $T$, $\alpha(T)$, is defined as the dimension of $\mathcal{N}(T)$ and the deficiency of $T$, $\beta(T)$, is defined as the codimension of $\mathcal{R}(T)$ in $X$. The set of upper semi-Fredholm operators from $X$ into $X$ is defined by

$$\Phi_+(X) := \{T \in \mathcal{C}(X) : \ \alpha(T) < \infty, \ \ \mathcal{R}(T) \ \text{ is closed in } X\},$$

the set of all lower semi-Fredholm operators is defined by

$$\Phi_-(X) := \{T \in \mathcal{C}(X) : \ \beta(T) < \infty, \ \ \mathcal{R}(T) \ \text{ is closed in } X\}.$$

The set of all semi-Fredholm operators is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X), \ \text{ and}$$

the class $\Phi(X)$ of all Fredholm operators is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

The set of bounded Fredholm operators from $X$ into $X$ is defined by

$$\Phi^b(X) := \Phi(X) \cap \mathcal{L}(X).$$

The set of bounded upper (resp. lower ) semi-Fredholm operators from $X$ into $X$ is defined by

$$\Phi^b_+(X) := \Phi_+(X) \cap \mathcal{L}(X) \quad (\text{resp. } \Phi^b_-(X) := \Phi_-(X) \cap \mathcal{L}(X)).$$

The index of a semi-Fredholm operator $T$ is defined by $i(T) = \alpha(T) - \beta(T)$. Clearly, $i(T)$ is an integer or $\pm\infty$. If $T \in \Phi(X)$, then $i(T) < \infty$. If $T \in \Phi_+(X)\backslash\Phi(X)$ then $i(T) = -\infty$; and if $T \in \Phi_-(X)\backslash\Phi(X)$ then $i(T) = +\infty$.

**Definition 2.1.** *Let $X$ be a Banach space.*

($i$) *An operator $F \in \mathcal{L}(X)$ is called an upper semi-Fredholm perturbation, if $T + F \in \Phi_+(X)$, whenever, $T \in \Phi_+(X)$. The set of upper semi-Fredholm perturbations is denoted by $\mathcal{F}_+(X)$.*

($ii$) *An operator $T \in \mathcal{C}(X)$ is said to have a left Fredholm inverse if there are maps $R_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $I + K$ extends $R_l T$. The operator $R_l$ is called left Fredholm inverse of $T$.*

If we replace $\Phi_+(X)$ by $\Phi_+^b(X)$ in Definition 2.1, we obtain the set $\mathcal{F}_+^b(X)$.

**Theorem 2.2.** *Let $X$ a Banach space.*

($i$) [12, Lemma 2.1] *Let $T \in \mathcal{C}(X)$ and $K \in \mathcal{L}(X)$. Then,*

($i_1$) *If $T \in \Phi_+(X)$ and $K \in \mathcal{F}_+(X)$, then $T + K \in \Phi_+(X)$ and $i(T + K) = i(T)$.*

($i_2$) *If $T \in \Phi_+^b(X)$ and $K \in \mathcal{F}_+^b(X)$, then $T + K \in \Phi_+^b(X)$ and $i(T + K) = i(T)$.*

($ii$) [10, Theorem 6.3.1] *If the set $\Phi^b(X)$ is not empty, then*

($ii_1$) *$F \in \mathcal{F}_+^b(X)$ and $T \in \mathcal{L}(X)$ imply that $TF \in \mathcal{F}_+^b(X)$ and $FT \in \mathcal{F}_+^b(X)$.*

($iii$)[12, Theorem 3.9] *Let $T \in \Phi_+(X)$. Then the following statements are equivalent:*

($iii_1$) *$i(T) \leq 0$.*

($iii_2$) *$T$ can be expressed in the form $T = S + K$ where $K \in \mathcal{K}(X)$ and $S \in \mathcal{C}(X)$ is an operator with closed range and $\alpha(S) = 0$.*

**Definition 2.3.** *Let $X$ be a Banach space.*

($i$) *An operator $T \in \mathcal{L}(X)$ is called compact, written $T \in \mathcal{K}(X)$, if $T(B)$ is relatively compact in $X$ for every bounded subset $B \subset X$.*

($ii$) *An operator $B$ is called $T$-bounded, if $\mathcal{D}(T) \subset \mathcal{D}(B)$ and there exists nonnegative constant $c$ such that*

$$\|Bx\| \leq c(\|x\| + \|Tx\|).$$

($iii$) *An operator $T \in \mathcal{L}(X)$ is said to be quasi-compact operator, written $T \in \mathcal{QK}(X)$, if there exists a compact operator $K$ and an integer $m$ such that*

$$\|T^m - K\| < 1.$$

For $\varepsilon > 0$ and closed densely defined operator $T$, we define the sets

$$\mathcal{M}_\varepsilon(X) = \Big\{K \in \mathcal{L}(X) : \forall \lambda \in \rho(T + D + K),\ \ (\lambda - T - K - D)^{-1}K \in \mathcal{QK}(X) \\ \text{for all } D \in \mathcal{L}(X) \text{ such that } \|D\| < \varepsilon\Big\},$$

and

$$\mathcal{T}_\varepsilon(X) = \Big\{K \in \mathcal{L}(X) : \forall \lambda \in \rho(T + D + K),\ \ K(\lambda - T - K - D)^{-1} \in \mathcal{QK}(X) \\ \text{for all } D \in \mathcal{L}(X) \ \text{ such that } \|D\| < \varepsilon\Big\}.$$

The following Lemma is developed by P. H. Wolff in [24, Lemma 2.1].

**Lemma 2.4.** *If $T \in \mathcal{C}(X)$ and $\varepsilon > 0$, then $\Sigma_{ap,\varepsilon}(T)$ is closed.*

**Example 2.5.** *The following example introduced by P. H. Wolff* [24] *also shows that:* $\Sigma_{ap,\varepsilon}(T) \neq \Sigma_{\varepsilon}(T)$, *for* $\varepsilon > 0$. *Let*

$$l^2(\mathbb{N}) := \left\{ (x_j)_{j\geq 1} \text{ such that } x_j \in \mathbb{C} \text{ and } \sum_{j=1}^{+\infty} |x_j|^2 < \infty \right\}$$

*be equipped with the following norm*

$$\|x\| := \sum_{j=1}^{\infty} |x_j|^2$$

*and let T be the right shift on X given by*

$$\begin{cases} T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N}), \\ \quad x \longrightarrow Tx := (0, x_1, x_2, ..., x_n, ...). \end{cases}$$

*Since T is an isometry on X, it is easily checked that*

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| \geq 1 - |\lambda|$$

*holds for all* $\lambda$ *with* $0 \leq |\lambda| \leq 1$. *Moreover*

$$\Sigma_{ap}(T) = \left\{ \lambda \in \mathbb{C} \text{ such that } |\lambda| = 1 \right\}$$

*and*

$$\Sigma_{ap,\varepsilon}(T) = \left\{ \lambda \in \mathbb{C} \text{ such that } |\lambda| \geq 1 - \varepsilon \right\}.$$

*It is proved in* [24] *that for all* $0 < \varepsilon < 1$,

$$\left\{ \lambda \in \mathbb{C} \text{ such that } |\lambda| \leq \varepsilon \right\} \subset \Sigma_{\varepsilon}(T) \backslash \Sigma_{ap,\varepsilon}(T).$$

Now, we present the following simple and useful result:

**Proposition 2.6.** *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$.

($i$) $\sigma_{ap,\varepsilon}(T) \subset \sigma_{\varepsilon}(T)$.

($ii$) $\sigma_{ap}(T) = \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T)$.

($iii$) *If* $\varepsilon_1 < \varepsilon_2$, *then* $\sigma_{ap}(T) \subset \sigma_{ap,\varepsilon_1}(T) \subset \sigma_{ap,\varepsilon_2}(T)$.

($iv$) *If* $T \in \mathcal{L}(X)$ *and* $\lambda \in \sigma_{ap,\varepsilon}(T)$, *then* $|\lambda| < \varepsilon + \|T\|$.

($v$) *If* $\alpha \in \mathbb{C}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,\varepsilon}(T + \alpha I) = \alpha + \sigma_{ap,\varepsilon}(T)$.

($vi$) *If* $\alpha \in \mathbb{C}\backslash\{0\}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) = \alpha\sigma_{ap,\varepsilon}(T)$.

**Remark 2.7.** ($i$) *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$, *then the set* $\Sigma_{ap,\varepsilon}(T)$ *is obtained from the set* $\sigma_{ap,\varepsilon}(T)$ *by taking a non-strict inequality instead of a strict inequality. This set makes the approximate pseudospectrum an open set.*

($ii$) *It follows from the set* $\Sigma_{ap,\varepsilon}(T)$ *and the set* $\sigma_{ap,\varepsilon}(T)$ *that the boundary of* $\Sigma_{ap,\varepsilon}(T)$, $\partial\Sigma_{ap,\varepsilon}(T)$ *satisfies*

$$\partial\Sigma_{ap,\varepsilon}(T) = \left\{ \lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| = \varepsilon \right\},$$

*and* $\partial\Sigma_{ap,\varepsilon}(T)$ *depends continuously on* $\varepsilon$.

($iii$) $\sigma_{ap,\varepsilon}(T)$ *and* $\Sigma_{ap,\varepsilon}(T)$ *are related as follows*

$$\Sigma_{ap,\varepsilon}(T) = \sigma_{ap,\varepsilon}(T) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| = \varepsilon \right\}.$$

## 3. The approximate pseudospectrum.

In this section, we turn to the problem when the closure of $\sigma_{ap,\varepsilon}(T)$ is equal to $\Sigma_{ap,\varepsilon}(T)$ holds. We define the following condition for $T$ :

$$(\mathcal{H}) : \begin{cases} \text{There is no open set in } \rho_{ap}(T) := \mathbb{C}\backslash\sigma_{ap}(T) \text{ on which the} \\ \lambda \longrightarrow f(\lambda) = \inf\limits_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| \text{ is constant.} \end{cases}$$

Our first result is the following.

**Theorem 3.1.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then,* $\liminf\limits_{\varepsilon\to\varepsilon_0} \Sigma_{ap,\varepsilon}(T) = \Sigma_{ap,\varepsilon_0}(T)$.

*Proof.* The approximate pseudospectrum is a family increase in function $\varepsilon$. Then for $0 < \varepsilon_0 < \varepsilon$, we have $\sigma_{ap}(T) \subseteq \Sigma_{ap,\varepsilon_0}(T) \subseteq \Sigma_{ap,\varepsilon}(T)$. Hence

$$\liminf_{\varepsilon\to\varepsilon_0} \Sigma_{ap,\varepsilon}(T) = \bigcap_{\varepsilon>\varepsilon_0} \Sigma_{ap,\varepsilon}(T).$$

Proposition 2.6-(ii) justifies the equality $\bigcap\limits_{\varepsilon>\varepsilon_0} \Sigma_{ap,\varepsilon}(T) = \Sigma_{ap,\varepsilon_0}(T)$. □

**Theorem 3.2.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. If $(\mathcal{H})$ is satisfied, then* $\overline{\sigma_{ap,\varepsilon}(T)} = \Sigma_{ap,\varepsilon}(T)$.

*Proof.* Since $\sigma_{ap,\varepsilon}(T) \subset \Sigma_{ap,\varepsilon}(T)$ and $\Sigma_{ap,\varepsilon}(T)$ is closed, then

$$\overline{\sigma_{ap,\varepsilon}(T)} \subset \Sigma_{ap,\varepsilon}(T).$$

In order to prove the inverse inclusion, we take $\lambda \in \Sigma_{ap,\varepsilon}(T)$. We notice the existence of two cases:

$1^{st}$<u>case</u> : If $\lambda \in \sigma_{ap,\varepsilon}(T)$, then $\lambda \in \overline{\sigma_{ap,\varepsilon}(T)}$.

$2^{nd}$<u>case</u> : If $\lambda \in \Sigma_{ap,\varepsilon}(T) \setminus \sigma_{ap,\varepsilon}(T)$, then $\inf_{\|x\|=1,x\in\mathcal{D}(T)} \|(\lambda - T)x\| = \varepsilon$.
By using Hypothesis $(\mathcal{H})$, then there exists a sequence $\lambda_n \in \rho_{ap}(T)$ such that $\lambda_n \to \lambda$, and

$$\inf_{\|x\|=1,x\in\mathcal{D}(T)} \|(\lambda_n - T)x\| < \inf_{\|x\|=1,x\in\mathcal{D}(T)} \|(\lambda - T)x\| = \varepsilon.$$

We deduce that $\lambda_n \in \sigma_{ap,\varepsilon}(T)$ and also that $\lambda_n \to \lambda$, which implies that $\lambda \in \overline{\sigma_{ap,\varepsilon}(T)}$. Then,

$$\Sigma_{ap,\varepsilon}(T) \subset \overline{\sigma_{ap,\varepsilon}(T)}.$$

□

**Theorem 3.3.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. The following conditions are equivalent:*

$(i)$ $\lambda \in \sigma_{ap,\varepsilon}(T)$.

$(ii)$ *There exists a bounded operator $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{ap}(T + D)$.*

*Proof.* $(i) \Rightarrow (ii)$ Let $\lambda \in \sigma_{ap,\varepsilon}(T)$. There are two possible cases:
$1^{st}$<u>case</u> : If $\lambda \in \sigma_{ap}(T)$, then it is sufficient to take $D = 0$.

$2^{nd}$<u>case</u> : If $\lambda \notin \sigma_{ap}(T)$, then there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|(\lambda - T)x_0\| < \varepsilon$. By the Hahn Banach Theorem, (see [15]) there exists $x' \in X'$ (dual of X) such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$. Consider the operator $D$ defined by the formula

$$\begin{cases} D : X \longrightarrow X, \\ \quad x \longrightarrow Dx := x'(x)(\lambda - T)x_0, \end{cases}$$

then $D$ is a linear operator everywhere defined on $X$. It is bounded, since

$$\|Dx\| = \|x'(x)(\lambda - T)x_0\| \leq \|x'\|\|x\|\|(\lambda - T)x_0\|, \quad \text{for} \quad x \neq 0.$$

Therefore,

$$\frac{\|Dx\|}{\|x\|} \leq \|(\lambda - T)x_0\|.$$

Hence $\|D\| < \varepsilon$. We claim that

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - D)x\| = 0.$$

Let $x_0 \in X$, then

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - D)x\| \leq \|(\lambda - T - D)x_0\| \leq \|(\lambda - T)x_0 - x'(x_0)(\lambda - T)x_0\| = 0.$$

$(ii) \Rightarrow (i)$ We assume that there exists a bounded operator $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{ap}(T + D)$, which means that

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - D)x\| = 0.$$

In order to prove that

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| < \varepsilon,$$

we can write,

$$\|(\lambda - T)x_0\| = \|(\lambda - T - D + D)x_0\| \leq \|(\lambda - T - D)x_0\| + \|Dx_0\|.$$

Then,

$$\inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| < \varepsilon.$$

□

We can derive from Theorem 3.3 the following result:

**Corollary 3.4.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then,* $\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T + D)$.

**Theorem 3.5.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Let $E \in \mathcal{L}(X)$ such that $\|E\| < \varepsilon$. Then,*

$$\sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \sigma_{ap,\varepsilon}(T + E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(T).$$

*Proof.* Let $\lambda \in \sigma_{ap,\varepsilon-\|E\|}(T)$. Then by Theorem 3.3 there exists a bounded operator $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon - \|E\|$ such that

$$\lambda \in \sigma_{ap}(T + D) = \sigma_{ap}\Big((T + E) + (D - E)\Big).$$

The fact that $\|D - E\| \leq \|D\| + \|E\| < \varepsilon$ allows us to deduce that $\lambda \in \sigma_{ap,\varepsilon}(T + E)$. Using a similar reasoning to the first inclusion, we deduce that

$$\lambda \in \sigma_{ap,\varepsilon+\|E\|}(T).$$

□

The closure of $\sigma_{ap,\varepsilon}(T)$ is always contained in $\Sigma_{ap,\varepsilon}(T)$, but equality holds if, and only if, $T$ does not have constant infimum norm on any open set. The present part addresses the question on whether or not a similar equality holds in the case of non-strict inequalities:

$$\Sigma_{ap,\varepsilon}(T) \stackrel{?}{=} \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T + D).$$

**Theorem 3.6.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then,*

$$\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D) \quad \subset \quad \Sigma_{ap,\varepsilon}(T). \tag{4}$$

*Proof.* Let $\lambda \notin \Sigma_{ap,\varepsilon}(T)$, then $\inf\limits_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T)x\| > \varepsilon$. In order to prove that

$$\lambda \notin \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D),$$

which means that, $\inf\limits_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - D)x\| > 0$ for all $D \in \mathcal{L}(X)$ such that $\|D\| \leq \varepsilon$, we have

$$\begin{aligned} \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|(\lambda - T - D)x\| &\geq \inf_{\|x\|=1, x\in\mathcal{D}(T)} \Big| \|(\lambda - T)x\| - \|Dx\| \Big| \\ &\geq \inf_{\|x\|=1, x\in\mathcal{D}(T)} \Big| \|(\lambda - T)x\| - \varepsilon\|x\| \Big| > 0. \end{aligned}$$

□

We first consider the following example:

**Example 3.7.** *Let $l^1(\mathbb{N}) = \left\{ (x_j)_{j\geq 1} \text{ such that } x_j \in \mathbb{C} \text{ and } \sum\limits_{j=1}^{+\infty} |x_j| < \infty \right\}$ be equipped with the following norm*

$$\|x\| := \sum_{j=1}^{\infty} |x_j|$$

*and we define the operator $T$ by*

$$\begin{cases} \begin{cases} T : l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N}), \\ \qquad x \longrightarrow Tx, \end{cases} \\ \\ \text{where } \; Tx := \left( (1+2\varepsilon)x_1 - \sum\limits_{j=2}^{\infty} x_j, -\varepsilon_2 x_2, \ldots, -\varepsilon_n x_n, \ldots \right), \end{cases}$$

*$x = (x_1, x_2, \ldots, x_n, \ldots) \in l^1(\mathbb{N})$ and $\varepsilon_n$, where $n = 2, 3 \ldots$ is a sequence of positive numbers monotonically decreasing to $0$. It was proved by E. Shargorodsky in* [16], *that*

$$\begin{cases} \inf\limits_{\|x\|=1, x\in\mathcal{D}(T)} \|(2\varepsilon I - T)x\| = \varepsilon, \\ \text{and for all } D \in \mathcal{L}(X),\ \|D\| \leq \varepsilon, \text{ we have } 2\varepsilon \in \rho(T+D). \end{cases} \tag{5}$$

*It follows from (5) that $2\varepsilon \in \Sigma_{ap,\varepsilon}(T)$ and $2\varepsilon \notin \sigma_{ap}(T+D)$ for all $\|D\| \leq \varepsilon$. Then*

$$2\varepsilon \notin \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D).$$

*Hence*

$$\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D) \subsetneq \Sigma_{ap,\varepsilon}(T).$$

**Theorem 3.8.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. If $(\mathcal{H})$ is satisfied, then*

$$\Sigma_{ap,\varepsilon}(T) \quad = \quad \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D). \tag{6}$$

*Proof.* It follows from inclusion (4) and Theorem 3.3 that

$$\overline{\sigma_{ap,\varepsilon}(T)} \subseteq \overline{\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D)} \quad\subseteq\quad \Sigma_{ap,\varepsilon}(T).$$

If $(\mathcal{H})$ is satisfied, then $\overline{\sigma_{ap,\varepsilon}(T)} = \Sigma_{ap,\varepsilon}(T)$, hence

$$\overline{\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D)} \quad=\quad \Sigma_{ap,\varepsilon}(T).$$

It follows from inclusion (4) and Theorem 3.3 that (6) is an equality if, and only if, the level set

$$\left\{\lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1,\ x\in\mathcal{D}(T)} \|(\lambda - T)x\| = \varepsilon\right\}$$

is a subset of $\bigcup_{\|D\|=\varepsilon} \sigma_{ap}(T+D)$. □

## 4. Essential approximate pseudospectrum.

In this section, we have the following useful stability result for the essential approximate pseudospectrum.

**Definition 4.1.** *Let $T \in \mathcal{C}(X)$. We define the essential approximate spectrum of the operator $T$ by*

$$\sigma_{eap}(T) := \bigcap_{K\in\ \mathcal{K}(X)} \sigma_{ap}(T+K).$$

In what follows, we will bring a new definition of the essential approximate pseudospectrum.

**Definition 4.2.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. We define the essential approximate pseudospectrum of the operator $T$ by*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+K).$$

In what follows, Theorem 4.3 gives a characterization of the essential approximate pseudospectrum by means of semi-Fredholm operators.

**Theorem 4.3.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then the following properties are equivalent:*

*(i)* $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

*(ii) For all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have*

$$\lambda - T - D \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - D) \leq 0.$$

*Proof.* $(i) \Rightarrow (ii)$ Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. It follows that there exists a compact operator $K$ on $X$ such that

$$\lambda \notin \sigma_{ap,\varepsilon}(T+K).$$

By using Theorem 3.3, we notice that $\lambda \notin \sigma_{ap}(T+D+K)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. So,

$$\lambda - T - D - K \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - D - K) \leq 0,$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Using Theorem 2.2, we get, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$\lambda - T - D \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - D) \leq 0.$$

$(ii) \Rightarrow (i)$ We assume that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \leq 0.$$

Based on Lemma 2.2, $\lambda - T - D$ can be expressed in the form $\lambda - T - D = S + K$, where $K \in \mathcal{K}(X)$ and $S \in \mathcal{C}(X)$ is an operator with closed range and $\alpha(S) = 0$. So

$$\lambda - T - D - K = S \text{ and } \alpha(\lambda - T - D - K) = 0.$$

By using [15, Theorem 3.12] there exists a constant $c > 0$ such that

$$\|(\lambda - T - D - K)x\| \geq c\|x\|, \text{ for all } x \in \mathcal{D}(T).$$

This proves that $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D - K)x\| \geq c > 0$. Thus $\lambda \notin \sigma_{ap}(T + D + K)$, and therefore $\lambda \notin \sigma_{eap,\varepsilon}(T)$. $\square$

**Remark 4.4.** *It follows immediately from Theorem 4.3 that* $\lambda \notin \sigma_{eap,\varepsilon}(T)$ *if, and only if, for all* $D \in \mathcal{L}(X)$ *such that* $\|D\| < \varepsilon$ *we obtain*

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \leq 0.$$

*This is equivalent to*

$$\sigma_{eap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T + D).$$

**Proposition 4.5.** *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$. *Then,*

(*i*) $\bigcap\limits_{\varepsilon>0} \sigma_{eap,\varepsilon}(T) = \sigma_{eap}(T)$.

(*ii*) *If* $\varepsilon_1 < \varepsilon_2$, *then* $\sigma_{eap}(T) \subset \sigma_{eap,\varepsilon_1}(T) \subset \sigma_{eap,\varepsilon_2}(T)$.

(*iii*) $\sigma_{eap,\varepsilon}(T + F) = \sigma_{eap,\varepsilon}(T)$ *for all* $F \in \mathcal{K}(X)$.

*Proof.* (*i*) $\sigma_{eap}(T) \subset \sigma_{eap,\varepsilon}(T)$. Indeed, Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. Then, there exists $K \in \mathcal{K}(X)$, such that

$$\inf_{x\in\mathcal{D}(X),\|x\|=1} \|(\lambda - T - K)x\| > \varepsilon > 0.$$

Hence $\lambda \notin \sigma_{eap}(T)$, and so

$$\sigma_{eap}(T) \subset \bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(T).$$

Conversely, let $\lambda \in \bigcap\limits_{\varepsilon>0} \sigma_{eap,\varepsilon}(T)$. Hence, for all $\varepsilon > 0$, we have $\lambda \in \sigma_{eap,\varepsilon}(T)$. Then, for every $K \in \mathcal{K}(X)$ we obtain $\lambda \in \sigma_{ap,\varepsilon}(T + K)$. This implies that

$$\inf_{x\in\mathcal{D}(X),\ \|x\|=1} \|(\lambda - T - K)x\| < \varepsilon.$$

Taking limits as $\varepsilon \to 0^+$, we infer that $\lambda \in \sigma_{eap}(T)$.

(*ii*) Let $\lambda \in \sigma_{eap,\varepsilon_1}(T)$, then there exists $K \in \mathcal{K}(X)$, such that

$$\inf_{x\in\mathcal{D}(X),\ \|x\|=1} \|(\lambda - T - K)x\| < \varepsilon_1 < \varepsilon_2.$$

So, $\lambda \in \sigma_{eap,\varepsilon_2}(T)$.

(*iii*) It is clear from Definition of the essential approximate pseudospectrum . $\square$

**Theorem 4.6.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then, for all $E \in \mathcal{L}(X)$, we have*

*(i)* $\sigma_{eap,\varepsilon-\|E\|}(T) \subseteq \sigma_{eap,\varepsilon}(T+E) \subseteq \sigma_{eap,\varepsilon+\|E\|}(T)$.

*(ii) For every $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$*

$$\sigma_{eap,\varepsilon}(\alpha I + \beta T) = \alpha + \beta\sigma_{eap,\varepsilon|\beta|}(T).$$

*Proof.* The proof of this theorem is inspired from the proof of Theorem 3.5 and Propositions 2.6 . □

The following result gives the essential approximate spectrum of the operator $T$ in terms of quasi-compact operators.

**Theorem 4.7.** *Let $T \in \mathcal{C}(X)$ with nonempty resolvent set. Then,*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K\in \mathcal{M}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K).$$

*Proof.* Let $\lambda \notin \bigcap_{K\in \mathcal{M}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K)$, then there exists $K \in \mathcal{M}_\varepsilon(X)$ such that for every $\|D\| < \varepsilon$ and $\lambda \in \rho(T+D+K)$, we have $(\lambda - T - D - K)^{-1}K \in \mathcal{QK}(X)$ and $\lambda \notin \sigma_{ap,\varepsilon}(T+K)$. Using [6, Theorem 1.6] we obtain that

$$I + (\lambda - T - D - K)^{-1}K \in \Phi(X) \text{ and } i(I + (\lambda - T - D - K)^{-1}K) = 0.$$

Since we can write

$$\lambda - T - D = (\lambda - T - D - K)(I + (\lambda - T - D - K)^{-1}K).$$

We conclude that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) = i(\lambda - T - D - M) \leq 0.$$

By using Theorem 4.3, we obtain $\lambda \notin \sigma_{eap,\varepsilon}(T)$. The opposite inclusion follows from $\mathcal{K}(X) \subseteq \mathcal{M}_\varepsilon(X)$. Then,

$$\bigcap_{K\in \mathcal{M}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K) \subseteq \bigcap_{K\in \mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+K).$$

□

**Corollary 4.8.** *Let $T \in \mathcal{L}(X)$ with nonempty resolvent set. Then,*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K\in \mathcal{T}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K).$$

It follows immediately, from Theorem 4.7, that

**Remark 4.9.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*

*(i) Using Theorem 4.7, we infer that $\sigma_{eap,\varepsilon}(T+F) = \sigma_{eap,\varepsilon}(T)$ for all $F \in \mathcal{M}_\varepsilon(X)$.*

*(ii) Let $\mathcal{I}(X)$ be a subset of $\mathcal{L}(X)$. If $\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{M}_\varepsilon(X)$, then*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{M\in \mathcal{I}(X)} \sigma_{ap,\varepsilon}(T+M).$$

*We have $\sigma_{eap,\varepsilon}(T+J) = \sigma_{eap,\varepsilon}(T)$ for all $J \in \mathcal{I}(X)$.*

**Theorem 4.10.** *Let $T, B \in \mathcal{C}(X)$ and $\varepsilon > 0$ and $\lambda - T - D \in \Phi_+(X)$ for every $\|D\| < \varepsilon$. If, there exists a left Fredholm inverse $T_{\lambda,\varepsilon}$ of $\lambda - T - D$ such that $BT_{\lambda,\varepsilon}$ is a quasi-compact operator, then $\sigma_{eap,\varepsilon}(T+B) \subseteq \sigma_{eap,\varepsilon}(T)$.*

*Proof.* ($i$) Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \leq 0.$$

Let $T_{\lambda,\varepsilon}$ be the left Fredholm inverse of $\lambda - T - D$, then there exists $K \in \mathcal{K}(X)$ such that

$$T_{\lambda,\varepsilon}(\lambda - T - D) = I - K \text{ on } X. \tag{7}$$

We infer from Eq. (7) that the operator $\lambda - T - B - D$ can be written in the form

$$\begin{aligned} \lambda - T - B - D &= \lambda - T - D - (BT_{\lambda,\varepsilon}(\lambda - T - D) + BK) \\ &= (I - BT_{\lambda,\varepsilon})(\lambda - T - D) - BK. \end{aligned} \tag{8}$$

Since $BT_{\lambda,\varepsilon} \in \mathcal{QK}(X)$ and applying [6, Theorem 1.6] we obtain that

$$I - BT_{\lambda,\varepsilon} \in \Phi(X) \text{ and } i(I - BT_{\lambda,\varepsilon}) = 0.$$

Consequently, $I - BT_{\lambda,\varepsilon} \in \Phi_+(X)$. It follows from Eq. (8) and [10, Lemma 2.1]

$$\begin{aligned} (I - BT_{\lambda,\varepsilon})(\lambda - T - D) &\in \Phi_+(X), \\ \text{and, } i((I - BT_{\lambda,\varepsilon})(\lambda - T - D)) &= i(I - BT_{\lambda,\varepsilon}) + i(\lambda - T - D) \\ &= i(\lambda - T - D) \leq 0. \end{aligned}$$

Hence,

$$\lambda - T - B - D \in \Phi_+(X) \text{ and } i(\lambda - T - B - D) \leq 0.$$

Then, $\lambda \notin \sigma_{eap,\varepsilon}(T + B)$. Thus, $\sigma_{eap,\varepsilon}(T + B) \subseteq \sigma_{eap,\varepsilon}(T)$. □

By using a similar reasoning as Theorem 4.7, we will give a fine characterization of $\sigma_{eap,\varepsilon}(.)$ by means of $T + D$-bounded perturbations. For this way we define the set

$$\begin{aligned} \mathcal{R}_\varepsilon(X) = \Big\{ K \in \mathcal{C}(X) : &\text{ for all } D \in \mathcal{L}(X) \text{ such that } \|D\| < \varepsilon, \ K \text{ is } (T + D)\text{-bounded} \\ &\text{and } K(\lambda - T - D - K)^{-1} \in \mathcal{QK}(X) \text{ for some } \lambda \in \rho(T + D + K) \Big\}. \end{aligned}$$

**Theorem 4.11.** *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \mathcal{R}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T + K).$$

*Proof.* ($i$) Because $\mathcal{K}(X) \subseteq \mathcal{R}_\varepsilon(X)$, then

$$\bigcap_{K \in \mathcal{R}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T + K) \subseteq \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap,\varepsilon}(T + K) := \sigma_{eap,\varepsilon}(T).$$

Conversely, let $\lambda \notin \bigcap_{K \in \mathcal{R}_\varepsilon(X)} \sigma_{ap,\varepsilon}(T + K)$ then there exists $K \in \mathcal{R}_\varepsilon(X)$ such that $\lambda \notin \sigma_{ap,\varepsilon}(T + K)$, which means that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $\lambda - T - D - K$ is injective. Using [6, Theorem 1.6] we obtain that

$$I + (\lambda - T - D - K)^{-1}K \in \Phi(X) \text{ and } i(I + (\lambda - T - D - K)^{-1}K) = 0.$$

In the same manner, we can write $\lambda - T - D$ in the form

$$\lambda - T - D = (I + K(\lambda - T - D - K)^{-1})(\lambda - T - D - K).$$

And also by using the Atkinson's theorem we obtain that

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) = i(\lambda - T - D - M) \leq 0.$$

This means that $\lambda \notin \sigma_{eap,\varepsilon}(T)$. □

**Remark 4.12.** *Let $T \in \mathcal{C}(X)$, $\varepsilon > 0$. and let $\Gamma(X)$ be a subset of $X$ containing $\mathcal{QK}(X)$. If $\Gamma(X) \subseteq \mathcal{R}_\varepsilon(X)$, then*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \Gamma(X)} \sigma_{ap,\varepsilon}(T+K).$$

**Acknowledgments**:

The authors wish to warmly thank the referee for all his suggestions and for his meticulousness.

## References

[1] A. Ammar and A. Jeribi, *A characterization of the essential pseudospectra on a Banach space, Arab. J. Math.* 2, 139–145 (2013).
[2] A. Ammar and A. Jeribi, *A characterization of the essential pseudospectra and application to a transport equation, Extra Math.* 28, 95–112 (2013).
[3] A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the condition pseudospectrum on Banach space*, Funct. Anal. Approx. Comput. **10** (2), 13–21 (2018).
[4] A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the essential approximation pseudospectrum on a Banach space.* Filomath, 3599–3610 (2017).
[5] A. Ammar, A. Jeribi and K. Mahfoudhi, *Measure of noncompactness, essential approximation and defect pseudospectrum.* to appear Methods of Funct. Anal. and Topology (2018).
[6] A. Brunel and D. Revuz, *Quelques applications probabilistes de la quasi-compacité. Ann. Inst. Henri et Poincaré,* **10**(3)(1974) , 301–337 .
[7] E. B. Davies, *Linear Operators and their spectra,Cambridge Studies in Advanced Mathematics, 106. Camb. Univ. Press, Cambridge* (2007).
[8] A. Harrabi, *Pseudospectre d'une suite d'opérateurs bornés, RAIRO Modél. Math. Anal. Numér.* 32, 671–680 (1998).
[9] R. Hagger, M. Lindner and M. Seidel, *Essential pseudospectra and essential norms of band-dominated operators, J. Math. Anal. Appl.* 437, no. 1, 255–291 (2016).
[10] A. Jeribi, *Spectral theory and applications of linear operators and block operator matrices, Springer-Verlag, New-York,* (2015).
[11] A. Jeribi, *Linear operators and their essential pseudospectra, CRC Press, Boca Raton* (2018).
[12] A. Jeribi and N. Moalla, *A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation, J. Math. Anal. Appl.* 358, 434–444 (2009).
[13] T. Kato, *Perturbation theory for linear operators,Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New-York,* (1966).
[14] M. Schechter, *Spectra of partial differential operators*, North-Holland, Amsterdam, New-York (1986).
[15] M. Schechter, *Principles of functional analysis, Second edition. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI,* 36, (2002).
[16] E. Shargorodsky, *On the definition of pseudospectra,Bull. London Math. Soc.* 41 524–534 (2009).
[17] L. N. Trefethen, *Approximation theory and numerical linear algebra, Algorithms for Approximation II, JC Mason and MG Cox, eds., Chapman and Hall, London* , 336–360 (1990).
[18] L. N. Trefethen, *Pseudospectra of matrices. numerical analysis*, (1991). *In Pitman Research notes in Mathematics Series, Longman Science and Technology, Harlow* 260, 234–266 (1992).
[19] L. N. Trefethen, *Pseudospectra of linear operators, SAIM rev.* 39 (3) 383–406, (1997).
[20] L. N. Trefethen and M. Embry, *Spectra and pseudospectra: The behavior of nonnormal matrices and operators,Prin. Univ. Press, Princeton and Oxford,* (2005).
[21] S. C. Reddy and L. N. Trefethen, *Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues, Comput. Meth. Appl. Mech. and Engrg.* 80 , no. 1, 147–164 (1990).
[22] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrix operator,Ph.D. thesis, stanford university ProQuest LLC, Ann Arbor,* (1967).
[23] J. H. Wilkinson, *Sensitivity of eigenvalues II, Util. Math.* 30 , 243–286 (1986).
[24] M. P. H. Wolff, *Discrete approximation of unbounded operators and approximation of their spectra, J. Approx. Theo.* 113, 229–244 (2001).