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Approximation by (p,q)-Analogue of Balázs-Szabados Operators

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Abstract. In the present paper, we introduce a generalization of Balázs-Szabados operators by means of (p,q)-calculus. We give the rate of convergence of Balázs-Szabados operators on based (p,q)-integrers by using Lipschitz class function and the Peetre's K-functional. We give the degree of asymptotic approximation by means of Voronoskaja type theorem. Further, we give some comparisons associated the convergence of Balázs-Szabados , q- Balázs-Szabados and (p,q)- Balázs-Szabados operators to certain functions by illustrations. Moreover, we investigate the properties of the weighted approximation for these operators.

1. Introduction

In this paper, we introduce a new generalization of q- Balázs-Szabados operators based on (p,q)-integers called (p,q)- Balázs-Szabados operators. K. Balázs [2] defined the Bernstein type rational functions and gave some convergence theorems for them. In [3], K.Balázs and J.Szabados obtained an estimate, which has several advantages respect to given in [2]. These estimates were obtained by usual modulus of continuity. In [5], the q-form of Balázs-Szabados operators was introduced and, the statistical approximation properties of these operators were investigated. The rational complex Balázs-Szabados operators was studied in [7]. The complex q-Balázs-Szabados operators was introduced attached to analytic functions on compact disks in [8]. In these works the order of convergence and Voronovskaja-type theorem with quantitative estimate of these operators and the exact degree of its approximation were given. In [16],[17] the approximation properties of q-Balázs-Szabados operators are studied.

Recently, Mursaleen et al [12] applied (p,q)-calculus in approximation theory and introduced (p,q)-analogue of Bernstein operators based on (p,q)-integers. Hence q-calculus is extented to (p,q)-calculus in approximation theory. In [11], [13], [14] and [15], (p,q)-analogue of some well-known operators and (p,q)-analogue of Lorentz polynomials on a compact disk are introduced and studied approximation properties.

Inspired by these works, we study (p,q)-analogue of Balázs-Szabados operators and investigated some direct and weighted approximation properties of these operators. Moreover, we give the degree of asymptotic approximation by Voronoskaja type theorem. We also show the convergence of the (p,q)- Balázs-Szabados operators to some functions by using graphics.

In order to introduce $(p,q)-$ analogue of Balázs-Szabados operators, we begin by recalling certain notation of $(p,q)-$ calculus. Let $0<q<p\leq 1$. For each nonnegative integer n,k, $n\geq k\geq 0$, the $(p,q)-$integer

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$[n]_{p,q}$, $(p,q)-$ factorial $[n]_{p,q}!$ and $(p,q)-$binomial are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p-q},$$

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}[n-1]_{p,q}...2.1, & n \geq 1 \\ 1, & n = 0 \end{cases},$$

and

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.$$

Note that if we take $p = 1$ in above notations, they reduce to $q-$analogues. Further,

$$(ax+by)_{p,q}^n := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a^{n-k} b^k x^{n-k} y^k, \tag{1}$$

$$(ax+by)_{p,q}^n = (ax+by)(pax+qby)\left(p^2ax+q^2by\right)...\left(p^{n-1}ax+q^{n-1}by\right).$$

2. Construction of Operators and Auxiliary Results

Considering (1) we set the basis function for $(p,q)-$analogue of Balazs-Szabados operators by

$$(1+a_n x)_{p,q}^n := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} (a_n x)^k, \tag{2}$$

where $n \in \mathbb{N}$, $0 < q < p \leq 1$, $a_n := [n]_{p,q}^{\beta-1}$ for $0 < \beta \leq \frac{2}{3}$.

We introduce $(p,q)-$analogue of Balazs-Szabados operators as

$$R_{n,p,q}(f;x) = \frac{1}{(1+a_n x)_{p,q}} \sum_{k=0}^{n} f\left(\frac{[k]_{p,q}}{q^{k-1} b_n}\right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} (a_n x)^k,$$

where $f : [0,\infty) \to \mathbb{R}$ is a continuous function, $0 < q < p \leq 1$, $n \in \mathbb{N}$, $x \in [0,\infty)$ and $a_n = [n]_{p,q}^{\beta-1}$, $b_n = [n]_{p,q}^{\beta}$ such that $0 < \beta \leq \frac{2}{3}$.

Lemma 2.1. *Let $0 < q < p \leq 1$ and $n \in \mathbb{N}$, we have*

$$R_{n,p,q}(1;x) = 1, \tag{3}$$

$$R_{n,p,q}(t;x) = \frac{x}{p^{n-1} + q^{n-1} a_n x}, \tag{4}$$

$$R_{n,p,q}\left(t^2;x\right) = \frac{x}{b_n\left(p^{n-1}+q^{n-1}a_n x\right)} + \frac{\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2}{\left(p^{n-1}+q^{n-1}a_n x\right)\left(p^{n-2}+q^{n-2}a_n x\right)}, \tag{5}$$

$$\begin{aligned} R_{n,p,q}\left(t^3;x\right) &= \frac{x}{b_n^2\left(p^{n-1}+q^{n-1}a_n x\right)} + \frac{\left(\frac{p^2}{q^2}+\frac{2p}{q}\right)\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2}{b_n\left(p^{n-1}+q^{n-1}a_n x\right)\left(p^{n-2}+q^{n-2}a_n x\right)} \\ &\quad + \frac{\frac{p^3}{q^3}\frac{[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2}x^3}{\left(p^{n-1}+q^{n-1}a_n x\right)\left(p^{n-2}+q^{n-2}a_n x\right)\left(p^{n-3}+q^{n-3}a_n x\right)}, \end{aligned} \tag{6}$$

$$R_{n,p,q}\left(t^4;x\right) = \frac{x}{b_n^3\left(p^{n-1}+q^{n-1}a_nx\right)} + \frac{\left(\frac{p^3}{q^3}+\frac{3p^2}{q^2}+\frac{3p}{q}\right)\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2}{b_n^2\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)}$$
$$+\frac{\left(\frac{3p^3}{q^3}+\frac{2p^4}{q^4}\right)\frac{[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2}x^3}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)\left(p^{n-3}+q^{n-3}a_nx\right)}$$
$$+\frac{\frac{p^5}{q^5}\frac{[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^3}x^4}{\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)\left(p^{n-3}+q^{n-3}a_nx\right)\left(p^{n-4}+q^{n-4}a_nx\right)}. \tag{7}$$

Proof. Considering (2), we get easily $R_{n,p,q}(1;x)=1$.

$$\begin{aligned} R_{n,p,q}(t;x) &= \frac{x}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-1)(k-2)}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-1} \\ &= \frac{x}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\prod_{s=0}^{n-2}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right) \\ &= \frac{x}{p^{n-1}+q^{n-1}a_nx}. \end{aligned}$$

Using identity $[k]_{p,q}=q^{k-1}+p[k-1]_{p,q}$, we obtain

$$\begin{aligned} R_{n,p,q}\left(t^2;x\right) &= \frac{1}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=1}^{n}\frac{q^{k-1}+p[k-1]_{p,q}}{q^{2k-2}[n]_{p,q}^{2\beta-1}}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k} \\ &= \frac{\frac{x}{[n]_{p,q}^{\beta}}}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-1)(k-2)}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-1} \\ &\quad+\frac{\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=2}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-2)(k-3)}{2}}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-2} \\ &= \frac{x}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)}+\frac{\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2}{\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)}, \end{aligned}$$

$$\begin{aligned} R_{n,p,q}\left(t^3;x\right) &= \frac{\frac{1}{[n]_{p,q}^{2\beta}}x}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-1)(k-2)}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-1} \\ &\quad+\frac{\left(\frac{p^2}{q^2}+\frac{2p}{q}\right)\frac{[n-1]_{p,q}}{[n]_{p,q}^{1+\beta}}x^2}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=2}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-2)(k-3)}{2}}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-2} \end{aligned}$$

$$+\frac{\frac{p^3}{q^3}\frac{[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2}x^3}{\prod_{s=0}^{n-1}\left(p^s+q^s[n]_{p,q}^{\beta-1}x\right)}\sum_{k=3}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-3)(k-4)}{2}}\left[\begin{array}{c}n-3\\k-3\end{array}\right]_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-3},$$

$$\begin{aligned}R_{n,p,q}\left(t^4;x\right) &= \frac{x}{[n]_{p,q}^{3\beta}\left(p^{n-1}+q^{n-1}a_nx\right)}\sum_{k=1}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-1)(k-2)}{2}}\left[\begin{array}{c}n-1\\k-1\end{array}\right]_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-1}\\
&+\frac{\left(\frac{p^3}{q^3}+\frac{3p^2}{q^2}+\frac{3p}{q}\right)\frac{[n-1]_{p,q}}{[n]_{p,q}^{1+2\beta}}x^2}{\prod_{j=1}^{2}\left(p^{n-j}+q^{n-j}a_nx\right)}\sum_{k=2}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-2)(k-3)}{2}}\left[\begin{array}{c}n-2\\k-2\end{array}\right]_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-2}\\
&+\frac{\left(\frac{3p^3}{q^3}+\frac{2p^4}{q^4}\right)\frac{[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^{2+\beta}}x^3}{\prod_{j=1}^{3}\left(p^{n-j}+q^{n-j}a_nx\right)}\sum_{k=3}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-3)(k-4)}{2}}\left[\begin{array}{c}n-3\\k-3\end{array}\right]_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-3}\\
&+\frac{\frac{p^5}{q^5}\frac{[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^{3}}x^4}{\prod_{j=1}^{4}\left(p^{n-j}+q^{n-j}a_nx\right)}\sum_{k=4}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{(k-3)(k-4)}{2}}\left[\begin{array}{c}n-4\\k-4\end{array}\right]_{p,q}\left([n]_{p,q}^{\beta-1}x\right)^{k-4}.\end{aligned}$$

Considering (2) in the last two equalities, the proof of lemma is completed. □

Corollary 2.2. *By simple applications of Lemma 2.1, we have the following central moments*

$$R_{n,p,q}(t-x;x)=\frac{\left(1-p^{n-1}\right)x}{p^{n-1}+q^{n-1}a_nx}-\frac{q^{n-1}a_n}{p^{n-1}+q^{n-1}a_nx}x^2, \tag{8}$$

$$\begin{aligned}R_{n,p,q}\left((t-x)^2;x\right) &= \frac{x}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)}+\left\{\frac{\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}}{\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)}\right.\\
&\left.-\frac{2}{p^{n-1}+q^{n-1}a_nx}+1\right\}x^2,\end{aligned} \tag{9}$$

$$\begin{aligned}R_{n,p,q}\left((t-x)^4;x\right) &= \frac{x}{b_n^3\left(p^{n-1}+q^{n-1}a_nx\right)}+\left\{\frac{\left(\frac{p^3}{q^3}+\frac{3p^2}{q^2}+\frac{3p}{q}\right)\frac{[n-1]_{p,q}}{[n]_{p,q}^{1+2\beta}}}{b_n^2\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)}\right.\\
&\left.-\frac{1}{b_n^2\left(p^{n-1}+q^{n-1}a_nx\right)}\right\}x^2\end{aligned} \tag{10}$$

$$+\left\{\frac{\left(\frac{2p^4}{q^4}+\frac{3p^3}{q^3}\right)\frac{[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2}}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)\left(p^{n-3}+q^{n-3}a_nx\right)}\right.$$
$$\left.-\frac{4\left(\frac{p^2}{q^2}+\frac{2p}{q}\right)\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)}+\frac{6}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)}\right\}x^3$$
$$+\left\{\frac{\frac{p^5}{q^5}\frac{[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^3}}{\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)\left(p^{n-3}+q^{n-3}a_nx\right)\left(p^{n-4}+q^{n-4}a_nx\right)}\right.$$
$$-\frac{4\frac{p^3}{q^3}\frac{[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2}}{\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)\left(p^{n-3}+q^{n-3}a_nx\right)}$$
$$\left.+\frac{6\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}}{\left(p^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)}-\frac{4}{p^{n-1}+q^{n-1}a_nx}+1\right\}x^4. \tag{11}$$

Proof. Using Lemma 2.1, we immediately have the central moments. □

Remark 2.3. *In order to obtain the order of convergence for the operators $R_{n,p,q}$, we take $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} (p_n)^n = \lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} (q_n)^n = c$ with $0 < c < 1$. Then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \frac{1}{b_n} = 0$. Such sequences can always be constructed. For example, we can take $p_n = 1 - \frac{1}{n^2}$ and $q_n = 1 - \frac{1}{n}$, clearly $\lim_{n\to\infty} (q_n)^n = e^{-1}$ and $\lim_{n\to\infty} (p_n)^n = 1$.*

Theorem 2.4. *Let (p_n) and (q_n) be the sequences defined as in Remark 2.3. For all $f \in C[0,\infty)$ we have $R_{n,p_n,q_n}(f;x)$ converges uniformly to f with respect to $x \in [0,a]$.*

Proof. For a fixed $a > 0$, we consider the lattice homomorphism $H_a : C[0,\infty) \to C[0,a]$ defined by $H_a(f) := f|_{[0,a]}$ for every $f \in C[0,\infty)$. From Lemma 2.1, for $m = 0,1,2$ $\lim_{n\to\infty} R_{n,p_n,q_n}(t^m;x) = x^m$ uniformly on $[0,a]$. By the universal Korovkin type property, we obtain, for all $f \in C[0,\infty)$, $\lim_{n\to\infty} R_{n,p_n,q_n}(f;x) = f(x)$ with respect to $x \in [0,a]$. □

3. Local Approximation

In this section, we give some local results for the operators $R_{n,p,q}(f;x)$.

Let $C_B[0,\infty)$ be the space of all real valued continous bounded functions defined on $[0,\infty)$. The norm on the space $C_B[0,\infty)$ is the supremum norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. Also, Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{g\in W^2}\left\{\|f-g\| + \delta\|g''\|\right\},$$

where $W^2 = \left\{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\right\}$. By [4] (p.177), there exist a positive constant $C > 0$ such that

$$K_2(f,\delta) \leq \omega_2\left(f,\delta^{1/2}\right), \delta > 0, \tag{12}$$

where

$$\omega_2\left(f,\delta^{1/2}\right) = \sup_{0<h<\delta^{1/2}, x\in[0,\infty)} \left|f(x+2h) - 2f(x+h) + f(x)\right|$$

is the second order modulus of continuity of function $f \in C_B[0,\infty)$.

Further, the usual modulus of continuity is defined by

$$\omega\left(f,\delta^{1/2}\right)=\sup_{0<h<\delta^{1/2},x\in[0,\infty)}\left|f(x+h)-f(x)\right|.$$

Now, we can give the following local theorem:

Theorem 3.1. *Let (p_n) and (q_n) be the sequences defined as in Remark 2.3 and $f\in C_B[0,\infty)$. Then for all $n\in\mathbb{N}$, there exist a positive constant $C>0$ such that*

$$\left|R_{n,p_n,q_n}(f;x)-f(x)\right|\leq C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)),$$

where $\delta_n^2(x)=R_{n,p_n,q_n}\left((t-x)^2;x\right)+\left(R_{n,p_n,q_n}(t-x;x)\right)^2$ and $\alpha_n(x)=$

$\left|R_{n,p_n,q_n}(t-x;x)\right|$. $R_{n,p_n,q_n}(t-x;x)$ and $R_{n,p_n,q_n}\left((t-x)^2;x\right)$ are given as in Corollary 2.2.

Proof. For $x\in[0,\infty)$, we introduce the auxiliary operator as follows:

$$R_n^*(f;x)=R_{n,p_n,q_n}(f;x)+f(x)-f(\xi_n(x)),$$

where $\xi_n(x)=x+\frac{\left(1-p_n^{n-1}\right)x-q_n^{n-1}a_nx^2}{p_n^{n-1}+q_n^{n-1}a_nx}$. Using Lemma 2.1, we obtain

$$\begin{aligned}R_n^*(t-x;x)&=R_{n,p_n,q_n}(t-x;x)-(\xi_n(x)-x)\\&=R_{n,p_n,q_n}(t;x)-xR_{n,p_n,q_n}(1;x)-\xi_n(x)+x\\&=0.\end{aligned}$$

Let $x\in[0,\infty)$ and $g\in W^2$. Using the Taylor formula

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)\,du.$$

Applying R_n^* to on both side of the above equation, we get

$$\begin{aligned}R_n^*(g(t);x)-g(x)&=R_n^*\left((t-x)g'(x);x\right)+R_n^*\left(\int_x^t(t-u)g''(u)\,du;x\right)\\&=g'(x)R_n^*(t-x;x)+R_{n,p_n,q_n}\left(\int_x^t(t-u)g''(u)\,du;x\right)\\&\quad-\int_x^{\xi_n(x)}(\xi_n(x)-u)g''(u)\,du\\&=R_{n,p_n,q_n}\left(\int_x^t(t-u)g''(u)\,du;x\right)-\int_x^{\xi_n(x)}(\xi_n(x)-u)g''(u)\,du.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\int_x^t(t-u)g''(u)\,du;x\right|&\leq\int_x^t|t-u|\left|g''(u)\right|du\\&\leq\left\|g''\right\|\int_x^t|t-u|\,du\leq(t-x)^2\left\|g''\right\|\end{aligned}$$

and

$$\begin{aligned} \left| \int_x^{\xi_n(x)} (\xi_n(x) - u) g''(u) du \right| &\leq (\xi_n(x) - x)^2 \left\| g'' \right\| \\ &= \left(R_{n,p_n,q_n}(t - x; x) \right)^2 \left\| g'' \right\|. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \left| R_n^*(g; x) - g(x) \right| &\leq \left| R_{n,p_n,q_n}\left(\int_x^t (t - u) g''(u) du; x \right) \right| + \left| \int_x^{\xi_n(x)} (\xi_n(x) - u) g''(u) du \right| \\ &\leq \left\{ R_{n,p_n,q_n}\left((t - x)^2; x \right) + \left(R_{n,p_n,q_n}(t - x; x) \right)^2 \right\} \left\| g'' \right\| \\ &= \delta_n^2(x) \left\| g'' \right\|. \end{aligned}$$

Considering Lemma 2.1, we have also

$$\begin{aligned} \left| R_n^*(f; x) \right| &\leq \left| R_{n,p_n,q_n}(f; x) \right| + \left| f(x) \right| + \left| f(\xi_n(x)) \right| \\ &\leq R_{n,p_n,q_n}\left(\left| f \right|; x \right) + 2 \left\| f \right\| \\ &\leq R_{n,p_n,q_n}(1; x) \left\| f \right\| + 2 \left\| f \right\| \\ &= 3 \left\| f \right\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| R_{n,p_n,q_n}(f; x) - f(x) \right| &\leq \left| R_n^*(f - g; x) - (f - g)(x) \right| + \left| f(\xi_n(x)) - f(x) \right| + \left| R_n^*(g; x) - g(x) \right| \\ &\leq \left| R_n^*(f - g; x) \right| + \left| (f - g)(x) \right| + \left| f(\xi_n(x)) - f(x) \right| + \left| R_n^*(g; x) - g(x) \right| \\ &\leq 4 \left\| f - g \right\| + \omega(f; \alpha_n(x)) + \delta_n^2(x) \left\| g'' \right\|. \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$\left| R_{n,p_n,q_n}(f; x) - f(x) \right| \leq 4K_2\left(f; \delta_n^2(x) \right) + \omega(f; \alpha_n(x)).$$

By the inequality (12), we obtain the desired result. □

Let E be any subset of $[0, \infty)$ and $\alpha \in (0, 1]$. Then $Lip_{M_f}(E, \alpha)$ denotes the space of functions $f \in C_B[0, \infty)$ satisfying the condition

$$\left| f(t) - f(x) \right| \leq M_f \left| t - x \right|^\alpha, \forall t \in \overline{E} \text{ and } x \in [0, \infty),$$

where M_f is a constant depending on f and $\overline{E}$ denotes the closure of E in $[0, \infty)$.

Theorem 3.2. *Let (p_n) and (q_n) be the sequences defined as in Remark 2.3 and $f \in C_B[0, \infty) \cap Lip_{M_f}(E, \alpha)$, $\alpha \in (0, 1]$ and E is a any bounded subset of $[0, \infty)$. Then, for each $x \in [0, \infty)$, we have*

$$\left| R_{n,p_n,q_n}(f; x) - f(x) \right| \leq M_f \left\{ \left(\mu_{n,p_n,q_n}(x) \right)^{\alpha/2} + 2 \left(d(x, E) \right)^\alpha \right\},$$

where $\mu_{n,p_n,q_n}(x) = R_{n,p_n,q_n}\left((t - x)^2; x \right)$ is given as in Corollary 2.2. Here M_f is a constant depending on f and $d(x, E)$ is a distance between point x and E that is

$$d(x, E) = \inf \left\{ |t - x| : t \in E \right\}.$$

Proof. Let $\bar{E}$ denote the closure of the set E. Then there exists a $x_0 \in \bar{E}$ such that $|x - x_0| = d(x, E)$, where $x \in [0, \infty)$. Thus we can write

$$\left|f(t) - f(x)\right| \leq \left|f(t) - f(x_0)\right| + \left|f(x_0) - f(x)\right|.$$

Since R_{n,p_n,q_n} is a positive linear operator, for $f \in Lip_{M_f}(E, \alpha)$, we get

$$\begin{aligned}\left|R_{n,p_n,q_n}(f; x) - f(x)\right| &\leq R_{n,p_n,q_n}\left(\left|f(t) - f(x_0)\right|; x\right) + R_{n,p_n,q_n}\left(\left|f(x_0) - f(x)\right|; x\right) \\ &\leq M_f\left(R_{n,p_n,q_n}\left(|t - x_0|^{\alpha}; x\right) + |x - x_0|^{\alpha}\right) \\ &\leq M_f\left(R_{n,p_n,q_n}\left(|t - x|^{\alpha}; x\right) + 2|x - x_0|^{\alpha}\right).\end{aligned}$$

In the last inequality, using the Hölder inequality with $p = 2/\alpha$ and $q = 2/(2 - \alpha)$, we have

$$\begin{aligned}\left|R_{n,p_n,q_n}(f; x) - f(x)\right| &\leq M_f\left\{R_{n,p_n,q_n}\left(|t - x|^{\alpha p}; x\right)\right\}^{1/p}\left(R_{n,p_n,q_n}(1; x)^{1/q} + 2(d(x, E))^{\alpha}\right) \\ &\leq M_f\left(\left(R_{n,p_n,q_n}\left((t - x)^2; x\right)\right)^{\alpha/2} + 2(d(x, E))^{\alpha}\right) \\ &\leq M_f\left(\mu_{n,p_n,q_n}(x)^{\alpha/2} + 2(d(x, E))^{\alpha}\right).\end{aligned}$$

This completes the proof of the theorem. □

4. Voronovskaya Type Theorem

Theorem 4.1. *Let $f \in C_B[0, \infty)$ be such that $f', f'' \in C_B[0, \infty)$ and the sequences (p_n) and (q_n) be defined as in Remark 2.3. Let*

$$\eta = \lim_{n\to\infty} b_n\left(1 - (p_n)^{n-1}\right)$$

and

$$\sigma = \lim_{n\to\infty} b_n\left(\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 2p_n^{n-2} + p_n^{2n-3}\right),$$

where $b_n = [n]_{p_n,q_n}^{\beta}$ for $0 < \beta < \frac{1}{2}$. Then we have

$$\lim_{n\to\infty} b_n\left(R_{n,p_n,q_n}(f; x) - f(x)\right) = \eta x f'(x) + \frac{1}{2}x(\sigma x + 1)f''(x)$$

uniformly on $[0, a]$ for any $a > 0$.

Proof. By the Taylor formula, we can write

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2}f''(x)(t - x)^2 + r(t, x)(t - x)^2, \tag{13}$$

where $r(t, x)$ is the remainder term and $\lim_{t\to x} r(t, x) = 0$.

Applying $R_{n,p_n,q_n}(f; x)$ to (13), we get

$$\begin{aligned}b_n\left(R_{n,p_n,q_n}(f; x) - f(x)\right) &= b_n R_{n,p_n,q_n}((t - x); x) f'(x) + \frac{1}{2}b_n R_{n,p_n,q_n}\left((t - x)^2; x\right)f''(x) \\ &\quad + b_n R_{n,p_n,q_n}\left(r(t, x)(t - x)^2; x\right).\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$R_{n,p_n,q_n}\left(r(t, x)(t - x)^2; x\right) \leq \sqrt{R_{n,p_n,q_n}\left(r^2(t, x); x\right)}\sqrt{R_{n,p_n,q_n}\left((t - x)^4; x\right)}. \tag{14}$$

It is clear that $r^2(x,x)=0$ and $r^2(.,x)\in C[0,\infty)$. In view of Theorem 2.4,

$$\lim_{n\to\infty} R_{n,p_n,q_n}\left(r^2(t,x);x\right)=r^2(x,x)=0 \tag{15}$$

uniformly on $[0,a]$.

Also, considering Corollary 2.2, $\lim_{n\to\infty} R_{n,p_n,q_n}\left((t-x)^4;x\right)=0$.

Now, from (14) and (15), we obtain

$$\lim_{n\to\infty} R_{n,p_n,q_n}\left(r(t,x)(t-x)^2;x\right)=0. \tag{16}$$

On the other hand, we compute the followings for $0<\beta<\frac{1}{2}$

$$\lim_{n\to\infty} b_n R_{n,p_n,q_n}((t-x);x)=\eta x, \tag{17}$$

and

$$\lim_{n\to\infty} R_{n,p_n,q_n}\left((t-x)^2;x\right)=x(\sigma x+1). \tag{18}$$

Finally, from (16), (17) and (18), we get the required result. This complete the proof of theorem. □

Remark 4.2. *We can find such sequences satisfying the condition of Theorem 2.4. For example, we take* $p_n=1-\frac{1}{n^2}$ *and* $q_n=1-\frac{1}{n}$, *so we can see that* $\eta=0=\sigma$.

5. Weighted Approximation

The weighted Korovkin type theorems was proved Gadzhiev [6].

Let $\rho(x)$ is a continuous and increasing function on $[0,\infty)$ satisfying $\rho(x)\geq 1$. $B_\rho[0,\infty)$ denotes the set of all functions f from to $\mathbb{R}$, satisfying $|f(x)|\leq M_f\rho(x)$, where M_f is a constant depending on f. $B_\rho[0,\infty)$ is a normed space with the norm $\|f\|_\rho=\sup_{x\in[0,\infty)}\frac{|f(x)|}{\rho(x)}$. $C^*[0,\infty)$ denotes the subspace of continuous functions in $B_\rho[0,\infty)$ for which $\lim_{x\to\infty}\frac{|f(x)|}{\rho(x)}$ exists finitely.

Now, we give the weighted approximation for the operators $R_{n,p_n,q_n}(f)$.

Theorem 5.1. *Let* (p_n) *and* (q_n) *be the sequences defined as in Remark 2.3. Then for* $f\in C^*[0,\infty)$, *we have* $\lim_{n\to\infty}\left\|R_{n,p_n,q_n}(f)-f\right\|_{\rho_1}=0$, *where* $\rho_1(x)=1+x^\lambda$, $\lambda\geq 4$.

Proof. *From (3) in Lemma 2.1, it is obvious that* $\lim_{n\to\infty}\left\|R_{n,p_n,q_n}(e_0)-e_0\right\|_{\rho_1}=0$. *Using(4), we see that*

$$\begin{aligned}\left|R_{n,p_n,q_n}(t;x)-x\right| &= \left|\frac{\left(1-p_n^{n-1}\right)x-a_nq_n^{n-1}x^2}{p_n^{n-1}+q_n^{n-1}a_nx}\right| \\ &\leq \frac{\left(1-p_n^{n-1}\right)x}{p_n^{n-1}+q_n^{n-1}a_nx}+\frac{a_nq_n^{n-1}x^2}{p_n^{n-1}+q_n^{n-1}a_nx} \\ &\leq \frac{1-p_n^{n-1}}{p_n^{n-1}}x+a_n\frac{q_n^{n-1}}{p_n^{n-1}}x^2,\end{aligned}$$

then we have

$$\left\|R_{n,p_n,q_n}(e_1)-e_1\right\|_{\rho_1}\leq \sup_{x\in[0,\infty)}\frac{x}{1+x^\lambda}\frac{1-p_n^{n-1}}{p_n^{n-1}}+\sup_{x\in[0,\infty)}\frac{x^2}{1+x^\lambda}a_n\frac{q_n^{n-1}}{p_n^{n-1}}.$$

Therefore,

$$\lim_{n\to\infty}\left\|R_{n,p_n,q_n}(e_1)-e_1\right\|_{\rho_1}=0.$$

Using (5), we have

$$\begin{aligned}R_{n,p_n,q_n}\left(t^2;x\right)-x^2 &= \frac{1}{b_n\left(p_n^{n-1}+q_n^{n-1}a_nx\right)}x+\frac{\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-p_n^{2n-3}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}x^2\\ &-\frac{\left(p_n^{n-1}q_n^{n-2}-p_n^{n-2}q_n^{n-1}\right)a_n}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}x^3\\ &-\frac{q_n^{2n-3}a_n^2}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}x^4.\end{aligned}$$

Applying triangle inequality, we get

$$\begin{aligned}\left|R_{n,p_n,q_n}\left(t^2;x\right)-x^2\right| &\leq \frac{1}{b_n\left(p_n^{n-1}+q_n^{n-1}a_nx\right)}x+\frac{\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-p_n^{2n-3}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}x^2\\ &+\frac{\left(p_n^{n-1}q_n^{n-2}-p_n^{n-2}q_n^{n-1}\right)a_n}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}x^3\\ &+\frac{q_n^{2n-3}a_n^2}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}x^4\\ &\leq \frac{1}{b_np_n^{n-1}}x+\frac{\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-p_n^{2n-3}}{p_n^{2n-3}}x^2+\frac{\left(p_n^{n-1}q_n^{n-2}-p_n^{n-2}q_n^{n-1}\right)a_n}{p_n^{2n-3}}x^3\\ &+\frac{q_n^{2n-3}a_n^2}{p_n^{2n-3}}x^4\\ &= \frac{1}{b_np_n^{n-1}}x+\left(\frac{1}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-1\right)x^2\\ &+\left(\frac{q_n^{n-2}}{p_n^{n-2}}+\frac{q_n^{n-1}}{p_n^{n-1}}\right)a_nx^3+\frac{q_n^{2n-3}}{p_n^{2n-3}}a_n^2x^4.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|R_{n,p_n,q_n}(e_2)-e_2\right\|_{\rho_1} &\leq \sup_{x\in[0,\infty)}\frac{x}{1+x^\lambda}\frac{1}{b_np_n^{n-1}}+\sup_{x\in[0,\infty)}\frac{x^2}{1+x^\lambda}\left(\frac{1}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-1\right)\\ &+\sup_{x\in[0,\infty)}\frac{x^3}{1+x^\lambda}\left(\frac{q_n^{n-2}}{p_n^{n-2}}+\frac{q_n^{n-1}}{p_n^{n-1}}\right)a_n+\sup_{x\in[0,\infty)}\frac{x^4}{1+x^\lambda}\frac{q_n^{2n-3}}{p_n^{2n-3}}a_n^2.\end{aligned}$$

Then we have

$$\lim_{n\to\infty}\left\|R_{n,p_n,q_n}(e_2)-e_2\right\|_{\rho_1}=0.$$

□

Now, we aim to estimate the weighted rate of convergence for the operators $R_{n,p_n,q_n}(f)$. For every $f \in C^*[0,\infty)$, we would like to consider a weighted modulus continuity $\Omega(f,\delta)$, which tends to zero as $\delta \to 0$. We consider the weighted modulus of continuity $\Omega(f,\delta)$ as

$$\Omega(f,\delta) = \sup_{0\leq h\leq\delta, x\geq 0} \frac{\left|f(x+h) - f(x)\right|}{(1+h^2)(1+x^2)},$$

for each every $f \in C^*[0,\infty)$.

Theorem 5.2. *Let (p_n) and (q_n) be the sequences defined as in Remark 2.3. Then for every $f \in C^*[0,\infty)$, we have the inequality*

$$\left\|R_{n,p_n,q_n}(f) - f\right\|_{\rho_2} \leq M(n,p_n,q_n)\,\Omega(f,\delta),$$

where $\rho_2(x) = 1 + x^\lambda$, $\lambda \geq 5$ and $M(n,p_n,q_n)$ is a positive real number depending on n, p_n and q_n.

Proof. From definition of $\Omega(f,\delta)$, we can write

$$\left|f(t) - f(x)\right| \leq \left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta},x\right)\Omega(f,\delta).$$

Applying R_{n,p_n,q_n} to the last inequality, we get

$$\begin{aligned} \left|R_{n,p_n,q_n}(f(t);x) - f(x)\right| \leq\ & \Omega(f,\delta)\left(1+x^2\right)\Big\{R_{n,p_n,q_n}\left(1+(t-x)^2;x\right) \\ & +R_{n,p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\Big\}, \end{aligned} \tag{19}$$

and also, applying the Cauchy-Schwarz inequality in the last term of inequality (19), we obtain

$$R_{n,p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right) \leq \left(R_{n,p_n,q_n}\left(\left(1+(t-x)^2\right)^2;x\right)\right)^{1/2}\left(R_{n,p_n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{1/2}. \tag{20}$$

Using (9) in Corollary 2.2, we can write

$$\begin{aligned} R_{n,p_n,q_n}\left(1+(t-x)^2;x\right) \leq\ & 1+\Bigg\{\frac{\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}+ \\ & \frac{2}{p_n^{n-1}+q_n^{n-1}a_nx}+1\Bigg\}x^2+\frac{x}{b_n\left(p_n^{n-1}+q_n^{n-1}a_nx\right)} \\ \leq\ & 1+\left(\frac{1}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{2}{p_n^{n-1}}+1\right)x^2+\frac{x}{b_n} \\ \leq\ & M_1(n,p_n,q_n)(x+1)^2. \end{aligned} \tag{21}$$

On the other hand, using (9) and (11)

$$R_{n,p_n,q_n}\left(\left(1+(t-x)^2\right)^2;x\right) = 1+2R_{n,p_n,q_n}\left((t-x)^2;x\right)+R_{n,p_n,q_n}\left((t-x)^4;x\right)$$

$$\begin{aligned} \leq\ & 1+2\left\{\frac{\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}+\frac{2}{p_n^{n-1}+q_n^{n-1}a_nx}+1\right\}x^2+\frac{2x}{b_n\left(p_n^{n-1}+q_n^{n-1}a_nx\right)} \\ & +\frac{x}{b_n^3\left(p_n^{n-1}+q_n^{n-1}a_nx\right)}+\left\{\frac{\left(\frac{p_n^3}{q^3}+\frac{3p_n^2}{q_n^2}+\frac{3p_n}{q_n}\right)\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}}{b_n^2\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}+\frac{1}{b_n^2\left(p_n^{n-1}+q_n^{n-1}a_nx\right)}\right\}x^2 \end{aligned}$$

$$+\left\{\frac{\left(\frac{2p_n^4}{q_n^4}+\frac{3p_n^3}{q_n^3}\right)\frac{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}}{[n]_{p_n,q_n}^2}}{b_n\left(p_n^{n-1}+q^{n-1}a_nx\right)\left(p^{n-2}+q^{n-2}a_nx\right)\left(p^{n-3}+q^{n-3}a_nx\right)}\right.$$
$$\left.+\frac{4\left(\frac{p^2}{q^2}+\frac{2p}{q}\right)\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}}{b_n\left(p^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}+\frac{6}{b_n\left(p_n^{n-1}+q_n^{n-1}a_nx\right)}\right\}x^3$$

$$+\left\{\frac{\frac{p_n^5}{q_n^5}\frac{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}{[n]_{p,q}^3}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)\left(p_n^{n-3}+q_n^{n-3}a_nx\right)\left(p_n^{n-4}+q_n^{n-4}a_nx\right)}\right.$$
$$+\frac{4\frac{p_n^3}{q_n^3}\frac{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}}{[n]_{p_n,q_n}^2}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)\left(p_n^{n-3}+q_n^{n-3}a_nx\right)}$$
$$\left.+\frac{6\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}}{\left(p_n^{n-1}+q_n^{n-1}a_nx\right)\left(p_n^{n-2}+q_n^{n-2}a_nx\right)}+\frac{4}{p_n^{n-1}+q_n^{n-1}a_nx}+1\right\}x^4.$$

$$\begin{aligned}
\leq\ & 1+\frac{2x}{b_np_n^{n-1}}+\frac{x}{b_n^3p_n^{n-1}}+\left\{2\frac{1}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{4}{p_n^{n-1}}+2\right.\\
&\left.+\frac{1}{b_n^2p_n^{2n-3}}\left(\frac{p_n^3}{q^3}+\frac{3p_n^2}{q_n^2}+\frac{3p_n}{q_n}\right)\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{1}{b_n^2p_n^{n-1}}\right\}x^2\\
&+\left\{\frac{1}{b_np_n^{3n-6}}\left(\frac{2p_n^4}{q_n^4}+\frac{3p_n^3}{q_n^3}\right)\frac{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}}{[n]_{p_n,q_n}^2}\right.\\
&\left.+\frac{4}{b_np^{2n-3}}\left(\frac{p_n^2}{q_n^2}+\frac{2p_n}{q_n}\right)\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{6}{b_np_n^{n-1}}\right\}x^3\\
&+\left\{\frac{1}{p_n^{4n-10}}\frac{p_n^5}{q_n^5}\frac{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}{[n]_{p,q}^3}\right.\\
&+\frac{4}{p_n^{3n-6}}\frac{p_n^3}{q_n^3}\frac{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}}{[n]_{p_n,q_n}^2}\\
&\left.+\frac{6}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{4}{p_n^{n-1}}+1\right\}x^4\\
\leq\ & M_2(n,p_n,q_n)\left(1+x+x^2+x^3+x^4\right)\\
\leq\ & M_2(n,p_n,q_n)(x+1)^4. \qquad (22)
\end{aligned}$$

And also,

$$\left(R_{n,p_n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{1/2}\leq\frac{1}{\delta}(x+1)\sqrt{\frac{1}{b_np_n^{n-1}}+\frac{1}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-\frac{2}{p_n^{n-1}}+1}. \qquad (23)$$

Applying (22) and (23) in (20), we obtain

$$R_{n,p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\leq\frac{1}{\delta}\sqrt{M_2(n,p_n,q_n)}(x+1)^3\sqrt{\frac{1}{b_np_n^{n-1}}+\frac{1}{p_n^{2n-3}}\frac{p_n}{q_n}\frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-\frac{2}{p_n^{n-1}}+1}. \quad (24)$$

Choosing $M(n,p_n,q_n) = \left(M_1(n,p_n,q_n) + \sqrt{M_2(n,p_n,q_n)}\right)M_3$, where

$M_3 = \sup_{x\in[0,\infty)} \frac{(1+x^2)(x+1)^3}{1+x^\lambda}$, $\lambda \geq 5$ and $\delta = \sqrt{\frac{1}{b_n p_n^{n-1}} + \frac{1}{p_n^{2n-3}} \frac{p_n}{q_n} \frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - \frac{2}{p_n^{n-1}} + 1}$ and using (21) and (24) in (19), we obtain

$$\left\| R_{n,p_n,q_n}(f) - f \right\|_{\rho_2} \leq M(n,p_n,q_n)\,\Omega(f,\delta),$$

which completes the proof of the theorem. □

We give some illustrative examples which show the rate of convergence of the operators R_{n,p_n,q_n} to certain functions in the following examples:

Example 5.3. *Let $q_n = (n-1)/n, p_n = (n^2-1)/n^2, n = 50$ and $n = 150$. In case of $\beta = 2/3$ and $\beta = 1/4$, the convergence of the operators $R_{n,p_n,q_n}(f,x)$ to $f(x) = x\sin(2x)$ is illustrated in Figures 1 and 2. It is clearly that,increasing the values of n, the degree of approximation become better. We also observe that the convergence of the operators to function is better for $\beta = 2/3$.*

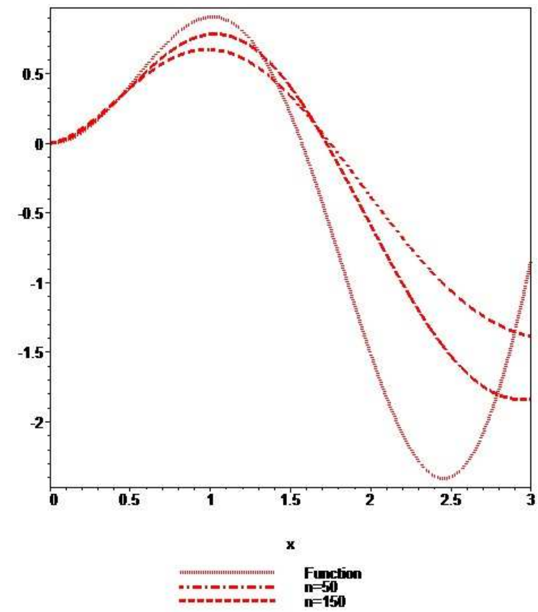

Figure 1: The convergence of R_{n,p_n,q_n} for $\beta = 2/3$.

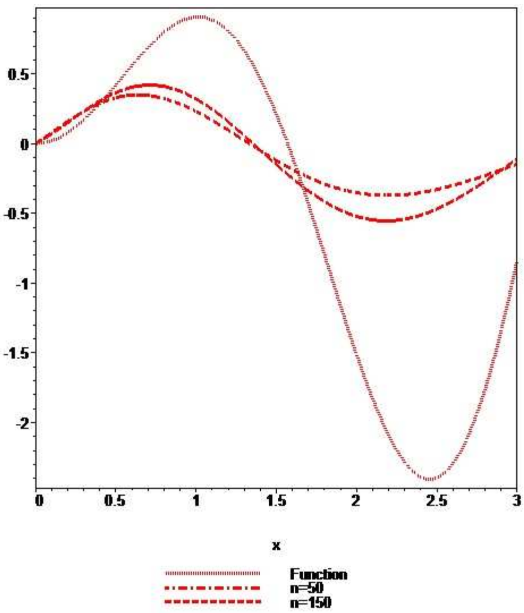

Figure 2: The convergence of R_{n,p_n,q_n} for $\beta = 1/4$.

Example 5.4. *We will examine a comparation of the convergence of Balász-Szasz operators $R_n(f;x)$, q- Balász-Szasz operators $R_{n,q}(f;x)$ and (p,q)-Balász-Szasz operators $R_{n,p,q}(f;x)$ to certain functions. For $n = 5$, $q = 0.80, p = 0.90$ and $n = 15$, $q = 0.90, p = 0.95$ with $\beta = 2/3$, convergence of the operators the above-mentioned to $f(x) = x + \sin(3x)$ and $f(x) = 1 + 5\sin(3x)$ are illustrated, respectively, in Figures 3 and 4.*

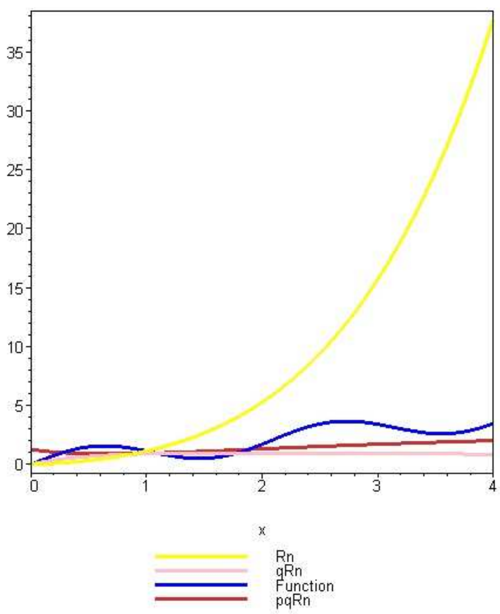

Figure 3: Comparation of convergence of the operators $R_n, R_{n,q}$ and $R_{n,p,q}$ to f for $n = 5, q = 0.80$ and $p = 0.90$.

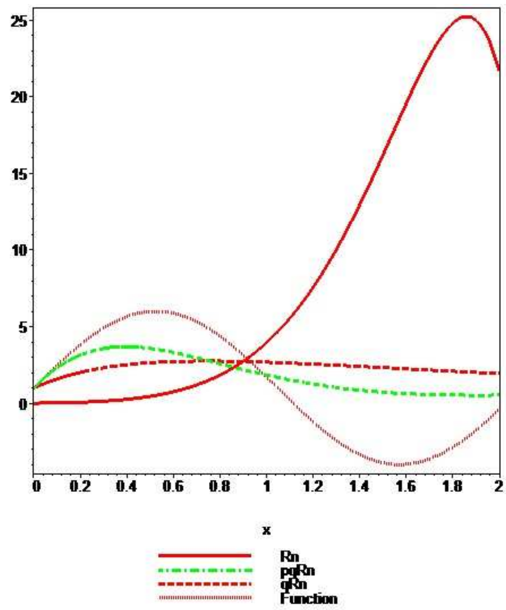

Figure 4: Comparation of convergence of the operators $R_n, R_{n,q}$ and $R_{n,p,q}$ to f for $n = 15, q = 0.90$ and $p = 0.95$.

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