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Fredholm Generalized Composition Operators on Weighted Hardy Spaces

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Abstract. The main purpose of this paper is to study Fredholm generalized composition operators on weighted Hardy spaces.

1. Introduction

Let f be an analytic function on the open unit disk Ω in a complex plane $\mathbb{C}$ given by $f(z) = \sum_{n=0}^{\infty} f_n z^n$, where $\{f_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. Let $\{\beta_n\}$ be a sequence of positive real numbers with $\beta(0) = 1$. For $p \in [1, \infty)$, let $H^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} |f_n|^p \beta_n^p < \infty\}$ be the space of formal series. Then $H^p(\beta)$ is a Banach space under the norm $\|f\|_\beta^p = \sum_{n=0}^{\infty} |f_n|^p \beta_n^p$. For p = 2, the space $H^2(\beta)$ is a Hilbert space under the inner product defined as $\langle f, g\rangle = \sum_{n=0}^{\infty} f_n \overline{g}_n \beta_n^2$, where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$. The weighted Hardy space is denoted by $H^2(\beta)$. Let $e_k(z) = z^k$ and $\hat{e_k}(z)=\frac{z^k}{\beta_k}$, clearly $\{e_k\}_{k=0}^{\infty}$ is an orthogonal basis for $H^2(\beta)$. If $\phi : \Omega \to \Omega$ is a mapping such that the transformation $C_\phi : H^2(\beta) \to H^2(\beta)$ defined by $C_\phi f = f o \phi$, for every $f \in H^2(\beta)$, is continuous , we shall call it a composition operator induced by ϕ. A generalized composition operator $C_\phi^d : H^2(\beta) \to H^2(\beta)$ is defined by $C_\phi^d f = f' o \phi$, where f' is the derivative of f. By the anti-differential operatorD_a we shall mean the operator $D_a : H^2(\beta) \to H^2(\beta)$ defined by

$$D_a(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} \frac{f_n z^{n+1}}{n+1}$$

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Also the Differential operator D on $H^2(\beta)$ is defined by

$$D(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} n f_n z^{n-1}$$

Composition operators on the spaces of analytic functions were studied by Cowen[1], Ryff[4], Schwartz[5]and Singh[8]. Properties of generalized composition operators on weighted Hardy spaces were mentioned in the papers of Sharma[6]-[7], further Fredholm composition and weighted composition operators can be seen in the papers of Kumar[2], Maccluer[3] and Takagi[9]. In this paper we initiate the study of Fredholm generalized composition operators on weighted Hardy spaces. The symbol B(H) denote the Banach algebra of all bounded linear operators on H into itself and N_o denote the set $\{0, 1, 2, 3, ...\}$.

2. Fredholm generalized composition operators on weighted Hardy spaces

The necessary and sufficient condition for generalized composition operators to be Fredholm is investigated in this section.

Theorem 2.1. *Suppose $\phi : \Omega \to \Omega$ is a mapping such that $\{\phi^n : n \in N_0\}$ is an orthogonal family in $H^2(\beta)$. Then $kerC_\phi^d = span\{e_0\}$, where $\phi^n(z) = (\phi(z))^n$.*

Proof. If $f = \alpha_0 e_0$, then clearly $C_\phi^d f = 0$, therefore $f \in kerC_\phi^d$

Next, if $C_\phi^d f = 0$ then for $f = \sum_{n=0}^{\infty} f_n e_n$

We have

$$C_\phi^d f = \sum_{n=1}^{\infty} n f_n \phi^{n-1} = 0$$

this implies that

$$\|C_\phi^d f\|^2 = \sum_{n=1}^{\infty} |f_n|^2 n^2 \beta_n^2 \|\phi^{n-1}\|^2 = 0$$

so that

$$|f_n| = 0 \qquad \text{for every} \qquad n \in \mathbb{N}$$

Hence

$$f = f_0 e_0.$$

□

Theorem 2.2. *Suppose $\phi : \Omega \to \Omega$ is a mapping such that $\{\phi^n : n \in N_0\}$ is an orthogonal family in $H^2(\beta)$. Then C_ϕ^d has closed range if and only if there exists $\epsilon > 0$ such that $n\|\phi^{n-1}\| \geq \epsilon \beta_n$ for all $n \in \mathbb{N}$.*

Proof. We first assume that C_ϕ^d has closed range. Then C_ϕ^d is bounded away from zero on $(kerC_\phi^d)^\perp$, therefore there exists $\epsilon > 0$ such that

$$\|C_\phi^d e_n\| \geq \epsilon \|e_n\| \text{ for all } n \in \mathbb{N}$$

which implies that

$$n\|\phi^{n-1}\| \geq \epsilon \beta_n \text{ for all } n \in \mathbb{N}$$

Conversely suppose that the conditions is true. Then for $f \in (kerC^d_\phi)^\perp$ we have

$$\|C^d_\phi f\|^2 = \|\sum_{n=1}^{\infty} f_n C^d_\phi e_n\|^2 = \sum_{n=1}^{\infty} |f_n|^2 n^2 \|\phi^{n-1}\|^2 \geq \epsilon^2 \sum_{n=1}^{\infty} |f_n|^2 \beta_n^2 = \epsilon^2 \|f\|^2 \text{ for every } f \in (kerC^d_\phi)^\perp$$

Then C^d_ϕ is bounded away from zero on $(kerC^d_\phi)^\perp$. Consequently C^d_ϕ has closed range. □

Theorem 2.3. *Let $\phi : \Omega \to \Omega$ be such that $\{\phi^n : n \in N_0\}$ is an orthogonal family in $H^2(\beta)$. Then C^d_ϕ is Fredholm if and only if there exists $\in > 0$ such that*

$$\frac{n\|\phi^{n-1}\|}{\beta_n} \geq \in \text{ for every } n \in \mathbb{N}.$$

Proof. Suppose the condition is true. Then in view of the theorem (2.2) C^d_ϕ has closed range. Also in view of theorem (2.1), $kerC^d_\phi$ is a finite dimensional.
We show that $kerC^{d^*}_\phi$ is zero dimensional. Let $g \in kerC^{d^*}_\phi$, then $C^{d^*}_\phi g = 0$.
Therefore, for $n \in N_0$ we have

$$\begin{aligned} 0 = \langle C^{d^*}_\phi g, e_n \rangle &= \langle g, C^d_\phi e_n \rangle \\ &= n\langle g, \phi^{n-1} \rangle. \end{aligned}$$

Hence g = 0, thus $kerC^{d^*}_\phi = \{0\}$. Hence C^d_ϕ is Fredholm.
The converse is easy to prove in view of theorem (2.1) and theorem (2.2). □

Example 2.4. *Let $\phi : \Omega \to \Omega$ be defined by $\phi(z) = z$, let $\beta_n = n!$, then $\frac{n\|\phi^{n-1}\|}{\beta_n} = \frac{n\beta_{n-1}}{\beta_n} = 1$. Therefore C^d_ϕ has closed range. Now $kerC^d_\phi = span\{e_0\}$ and $kerC^{d^*}_\phi = \{0\}$.*
Hence C^d_ϕ is Fredholm.

3. Fredholm Differential and Anti-Differential operators on weighted Hardy spaces

In this section we obtain adjoint of anti-differential operator on weighted Hardy spaces. The condition for anti-differential operator to be Fredholm is also investigated in this section.

Theorem 3.1. *Let $f \in H^2(\beta)$. Then*

$$D^*_a f = \sum_{n=0}^{\infty} \frac{f_{n+1}}{(n+1)} \left(\frac{\beta_{n+1}}{\beta_n}\right)^2 z^n$$

where D^*_a is the adjoint of D_a.

Proof. For any $n \in N_0$
Consider

$$\langle D^*_a e_{n+1}, f \rangle = \langle e_{n+1}, D_a f \rangle = \frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n}\right)^2 \langle e_n, f \rangle \text{ for every } f \in H^2(\beta).$$

Therefore,

$$D^*_a e_{n+1} = \frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n}\right)^2 e_n \text{ and } D^*_a e_0 = 0.$$

Now for $f = \sum_{n=0}^{\infty} f_n e_n$

$$D_a^* f = \sum_{n=0}^{\infty} f_n D_a^* e_n = \sum_{n=0}^{\infty} f_{n+1} \frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n}\right)^2 e_n$$

□

Theorem 3.2. *Let $D_a \in B(H^2(\beta))$. Then D_a is Fredholm operator if and only if $\frac{\beta_n}{n\beta_{n-1}} \geq \in \quad \forall\, n \geq 1$.*

Proof. Clearly, for $n \geq 1$, $D_a^* e_n = \frac{1}{n}(\frac{\beta_n}{\beta_{n-1}})^2 e_{n-1}$.
Since

$$D_a^* e_0 = 0, \ \ so \ \ e_0 \in ker D_a^*.$$

We shall show that $ker D_a^* = span\{e_0\}$
Let $f \in ker D_a^*$, then

$$D_a^* f = D_a^* \sum_{n=0}^{\infty} f_n e_n = \sum_{n=1}^{\infty} f_n \frac{1}{n} \left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1} = 0$$

which implies that $f_n = 0, \forall \ n \geq 1$.
Hence $\quad f = f_0 e_0$
Thus $\quad ker D_a^* = span\{e_0\} = M$
Next we will see that D_a^* is bounded away from zero on $(ker D_a^*)^\perp$ if and only if $\frac{\beta_n}{n\beta_{n-1}} \geq \in \ \ \forall \ n \geq 1$
Let $f \in (ker D_a^*)^\perp = M^\perp$
Consider

$$\|D_a^* f\|^2 = \|\sum_{n=1}^{\infty} f_n D_a^* e_n\|^2 = \sum_{n=1}^{\infty} (\frac{1}{n} \frac{\beta_n}{\beta_{n-1}})^2 |f_n|^2 \beta_n^2 \geq \epsilon^2 \sum_{n=1}^{\infty} |f_n|^2 \beta_n^2 = \epsilon^2 \|f\|^2$$

This is true for every $f \in (ker D_a^*)^\perp$
Hence D_a^* has closed range. Also $ker D_a = \{0\}$. For if we have $D_a f = 0$,
then $\sum_{n=0}^{\infty} f_n D_a e_n = 0$ implies that $\sum_{n=0}^{\infty} f_n \frac{e_{n+1}}{n+1} = 0$ or $\frac{f_n}{n+1} = 0 \ for\ all\ n \in N_0$
This implies that $f = 0$.
Thus $ker D_a = \{0\}$. Hence D_a is Fredholm. The converse follows by reversing the arguments. □

In the next theorem we characterize Fredholm differential operator.

Theorem 3.3. *Let $D \in B(H^2(\beta))$. Then D is Fredholm operator if and only if $\frac{n\beta_{n-1}}{\beta_n} \geq \epsilon$ for every $n \geq 1$.*

Proof. We first note that $ker D = span\{e_0\}$.
For if we suppose that $Df = 0$ for $f \in H^2(\beta)$,
then for $f = \sum_{n=0}^{\infty} f_n e_n$ we have

$$Df = \sum_{n=1}^{\infty} f_n n e_{n-1} = 0$$

which implies that

$$\sum_{n=1}^{\infty} n^2 |f_n|^2 \beta_{n-1}^2 = 0$$

which further implies that $f_n = 0$ for all $n = 1, 2, ...$
Hence $f = f_0 e_0$ so that $f \in span\{e_0\}$.
Next we shall see that $kerD^* = \{0\}$. Suppose $f \in kerD^*$.
Then $D^* f = 0$
or

$$D^*(\sum_{n=0}^{\infty} f_n e_n) = \sum_{n=0}^{\infty} f_n (n+1)(\frac{\beta_n}{\beta_{n+1}})^2 e_{n+1} = 0$$

which implies that $f_n = 0$ for all n = 0,1,... . Thus $f = 0$.
Finally we can show that if the given condition is satisfied, then D has closed range.
Let $f \in (kerD)^{\perp}$ and $f = \sum_{n=1}^{\infty} f_n e_n$.
Then

$$\|Df\|^2 = \|\sum_{n=1}^{\infty} f_n n e_{n-1}\|^2 = \sum_{n=0}^{\infty} |f_{n+1}|^2 (n+1)^2 \beta_n^2 = \sum_{n=0}^{\infty} |f_{n+1}|^2 (n+1)^2 \frac{\beta_n^2}{\beta_{n+1}^2} .\beta_{n+1}^2 \geq \epsilon^2 \sum_{n=0}^{\infty} |f_{n+1}|^2 \beta_{n+1}^2 = \epsilon^2 \|f\|^2$$

Thus D is bounded away from zero on $(kerD)^{\perp}$ which proves that D has closed range. We can conclude that D is Fredholm.

Conversely suppose D is Fredholm. Then D has closed range. Therefore D is bounded away from zero on $(kerD)^{\perp}$.
We can find $\epsilon > 0$ such that

$$\|De_n\| \geq \epsilon \|e_n\| \quad \forall \; n = 1, 2, ...$$

or

$$\frac{n \, \beta_{n-1}}{\beta_n} \geq \epsilon \quad \forall \; n = 1, 2, ...$$

This complete the proof of the theorem. □

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