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Some Unifying Inequalities for Starlike Functions in a Half-plane and a Sector

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Abstract. Sharp coefficient inequalities are given for f normalised and analytic in $z \in \mathbb{D} = \{z : |z| < 1\}$, and satisfying $\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}$ $(z \in \mathbb{D})$ for $\alpha \in [0, 1)$ and $\beta \in (0, 1]$. The results generalise and unify known inequalities for starlike functions in a half-plane, and strongly starlike functions.

1. Introduction and definitions

Let $\mathcal{S}$ be the class of analytic normalised univalent functions f, defined for $z \in \mathbb{D} = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Denote by $\mathcal{S}^*$ the subset of functions f, starlike with respect to the origin, so that $f \in \mathcal{S}^*$ if, and only if,

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad (z \in \mathbb{D}).$$

The subclasses of starlike functions $\mathcal{S}^*(\alpha)$ in a half-plane, and strongly starlike functions $\mathcal{SS}^*(\beta)$ defined in a sector, have been widely studied, see e.g. [1, 2, 3, 4, 13]. Thus $f \in \mathcal{S}^*(\alpha)$ if, and only if, for $\alpha \in [0, 1)$,

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > \alpha \qquad (z \in \mathbb{D}),$$

and $f \in \mathcal{SS}^*(\beta)$ if, and only if, for $\beta \in (0, 1]$,

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\beta}{2} \qquad (z \in \mathbb{D}).$$

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The object of this paper is to study a combination of these two subclasses by defining a set of functions $\mathcal{SS}^*(\alpha, \beta)$ by the relationship

$f \in \mathcal{SS}^*(\alpha, \beta)$ if, and only if, for $\alpha \in [0, 1)$ and $\beta \in (0, 1]$,

$$\left|\arg\left[\frac{zf'(z)}{f(z)} - \alpha\right]\right| < \frac{\pi\beta}{2} \qquad (z \in \mathbb{D}). \tag{1}$$

Functions defined by (1), and referred to as strongly starlike of order β and type α, where considered in [12], and some inclusion results were obtained.

In this paper we give some coefficient inequalities for functions in $\mathcal{SS}^*(\alpha, \beta)$, which generalise and unify known results for $S^*(\alpha)$ (see e.g. [4], [13]) and $\mathcal{SS}^*(\beta)$ [1–3, 15].

2. Necessary lemmas

Denote by $\mathcal{P}$, the class of functions p satisfying $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, with coefficients p_n given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

We shall use the following lemmas [1, 2, 8, 9], the first one of which was originally proved by Ma and Minda in [9], with a simpler proof given by Ali [1].

Lemma 2.1. *If $p \in \mathcal{P}$, then $|p_n| \le 2$ for $n \ge 1$, and*

$$\left|p_2 - \frac{\mu}{2}p_1^2\right| \le \max\{2, 2|\mu - 1|\} = \begin{cases} 2, & 0 \le \mu \le 2, \\ 2|\mu - 1|, & \textit{elsewhere.} \end{cases}$$

Also

$$\left|p_2 - \frac{1}{2}p_1^2\right| \le 2 - \frac{1}{2}\left|p_1^2\right|.$$

Lemma 2.2 (Lemma 3, [1]). *Let $p \in \mathcal{P}$. If $0 \le B \le 1$ and $B(2B - 1) \le D \le B$, then*

$$\left|p_3 - 2Bp_1p_2 + Dp_1^3\right| \le 2.$$

Lemma 2.3 (Corollary 1, [1]). *If $p \in \mathcal{P}$, and $0 \le B \le 1$, then*

$$\left|p_3 - 2Bp_1p_2 + Bp_1^3\right| \le 2.$$

Lemma 2.4 (Lemma 4, [1]). *If $p \in \mathcal{P}$, then*

$$\left|p_3 - (1 + \mu)p_1p_2 + \mu p_1^3\right| \le \max\{2, 2|2\mu - 1|\} = \begin{cases} 2, & 0 \le \mu \le 1 \\ 2|2\mu - 1|, & \textit{elsewhere} \end{cases}.$$

Lemma 2.5 ([8]). *If $p \in \mathcal{P}$, then for some complex valued x with $|x| \le 1$, and some complex valued ζ with $|\zeta| \le 1$*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2),\\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\zeta. \end{aligned}$$

Lemma 2.6 ([14]). *Let $f(z)$ be subordinate to $g(z)$, with*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

If $g(z)$ is univalent for $z \in \mathbb{D}$ and $g(\mathbb{D})$ is convex, then

$$|a_n| \le |b_1|.$$

3. Initial coefficients

First note that if $f \in \mathcal{SS}^*(\alpha, \beta)$, then from (1) we can write

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)p(z)^{\beta} \tag{2}$$

for $p \in \mathcal{P}$. Equating coefficients in (2) then gives

$$\begin{aligned} a_2 &= (1-\alpha)\beta p_1,\\ a_3 &= \frac{1}{2}(1-\alpha)\beta\left[p_2 - \frac{1}{2}(1 + (2\alpha - 3)\beta)p_1^2\right],\\ a_4 &= \frac{1}{3}(1-\alpha)\beta\Big\{p_3 - \frac{1}{2}\left[2 + (3\alpha - 5)\beta\right]p_1p_2 + \frac{1}{12}\left[4 + 3(3\alpha - 5)\beta\right.\\ &\quad \left. + (17 - 21\alpha + 6\alpha^2)\beta^2\right]p_1^3\Big\}. \end{aligned} \tag{3}$$

We now give sharp inequalities for these coefficients as follows.

Theorem 3.1. *Let $f \in S^*(\alpha, \beta)$, then $|a_2| \le 2\beta(1-\alpha)$.*

If $\frac{1}{3} < \beta \le 1$ and $0 \le \alpha < \frac{3\beta - 1}{2\beta}$, then

$$|a_3| \le (1-\alpha)(3 - 2\alpha)\beta^2,$$

and

$$|a_3| \le (1-\alpha)\beta,$$

otherwise.

Also

$$|a_4| \le \frac{2}{9}(1-\alpha)\beta\left[1 + (17 - 21\alpha + 6\alpha^2)\beta^2\right], \tag{4}$$

when

$$\frac{7}{4}-\frac{\sqrt{16+11\beta^2}}{4\beta\sqrt{3}} \le \alpha < 1 \quad \text{and} \quad \sqrt{\frac{2}{17}} < \beta < 1,$$

and

$$|a_4| \le \frac{2}{3}(1-\alpha)\beta, \tag{5}$$

otherwise.

All the estimates for $|a_2|$, $|a_3|$ and $|a_4|$ are sharp.

Proof. Since $|p_1| \le 2$, the inequality for $|a_2|$ is trivial.

For a_3 we apply Lemma 2.1 in (3) with $\mu = 1 + (2\alpha - 3)\beta$, so that $\mu \in [0,2]$ when $0 < \beta \le \frac{1}{3}$ and $0 \le \alpha < 1$, or when $\frac{1}{3} < \beta < 1$ and $\frac{3\beta-1}{2\beta} \le \alpha < 1$. This gives the first two inequalities for $|a_3|$.

When $\frac{1}{3} < \beta \le 1$ and $0 \le \alpha < \frac{3\beta-1}{2\beta}$ it follows that $\mu \le 0$, and Lemma 2.1 also gives the third inequality.

Next, in order to prove (4), note that in (3) the coefficient of p_1p_2 is positive when $\frac{2}{5} < \beta \le 1$, and $0 \le \alpha < \frac{5\beta-2}{3\beta}$, and the coefficient of p_1^3 is positive when $0 < \beta \le 1$ and $0 \le \alpha < 1$. Since $|p_n| \le 2$ when $n = 1, 2, 3$, the second inequality is therefore satisfied when $\frac{2}{5} < \beta \le 1$, and $0 \le \alpha < \frac{5\beta-2}{3\beta}$.

For the remaining intervals we use Lemma 2.3 with $B = \frac{1}{4}[2 + (3\alpha - 5)\beta]$ and $D = \frac{1}{12}[4 + 3(3\alpha - 5)\beta + (17 - 21\alpha + 6\alpha^2)\beta^2]$, and write

$$p_3 - 2Bp_1p_2 + Dp_1^3 = p_3 - 2Bp_1p_2 + Bp_1^3 + (D - B)p_1^3.$$

Then since $0 \le B \le 1$ and $D \ge B$ provided $\sqrt{\frac{2}{17}} < \beta \le \frac{2}{5}$ and $0 \le \alpha \le \frac{7}{4} - \frac{\sqrt{16+11\beta^2}}{4\beta\sqrt{3}}$, or $\frac{2}{5} < \beta < 1$ and $\frac{5\beta-2}{3\beta} \le \alpha \le \frac{7}{4} - \frac{\sqrt{16+11\beta^2}}{4\beta\sqrt{3}}$, we obtain, using $|p_1| \le 2$,

$$\begin{aligned} |a_4| &\le \frac{1}{3}(1-\alpha)\beta\left\{2 + \left[\frac{2}{3}\left(-2 + (17 - 21\alpha + 6\alpha^2)\beta^2\right)\right]\right\} \\ &= \frac{2}{9}(1-\alpha)\beta\left[1 + (17 - 21\alpha + 6\alpha^2)\beta^2\right]. \end{aligned}$$

To prove (5), we first use Lemma 2.2 in (3), so that $0 \le B \le 1$ and $B(2B-1) \le D \le B$ are satisfied when

$$0 < \beta \le \sqrt{\frac{2}{17}} \quad \text{and} \quad 0 \le \alpha < 1,$$

or when

$$\sqrt{\frac{2}{17}} < \beta < 1 \quad \text{and} \quad \frac{7}{4} - \frac{\sqrt{16+11\beta^2}}{4\beta\sqrt{3}} \le \alpha < 1.$$

This establishes the inequality (5), and completes the proof of Theorem 3.1.

Choosing $p_1 = 2$ in (3) shows that the inequality for $|a_2|$ is sharp. Choosing $p_1 = 0$ and $p_2 = 2$ shows that the first two inequalities for $|a_3|$ are sharp, and $p_1 = 2$ and $p_2 = 2$ that the second inequality for $|a_3|$ is sharp. Finally choosing $p_1 = 0$, $p_2 = 0$ and $p_3 = 2$ shows that the first two inequalities for $|a_4|$ are sharp, and choosing $p_1 = 2$, $p_2 = 2$ and $p_3 = 2$ shows that the third inequality for $|a_4|$ is sharp. □

We note that when $\beta = 0$, we obtain the classical inequalities for $f \in S^*(\alpha)$, see e.g. [4], and when $\alpha = 0$, the results in [2, 3].

4. Inverse coefficients

We first note that since $f \in \mathcal{S}^*(\alpha,\beta)$ is univalent, f^{-1} exists in some disc $|\omega| < r_0(f)$.

Let

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots .$$

Since $f(f^{-1}(\omega)) = \omega$, equating coefficients gives

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \\ A_4 &= -5a_2^3 + 5a_2a_3 - a_4. \end{aligned} \tag{6}$$

We now give sharp inequalities for these coefficients as follows.

Theorem 4.1. *Let $f \in \mathcal{SS}^*(\alpha,\beta)$, then $|A_2| \le 2\beta(1-\alpha)$.*

If $\frac{1}{5} < \beta \le 1$ and $0 \le \alpha < \frac{5\beta-1}{6\beta}$, then

$$|A_3| \le (5-6\alpha)(1-\alpha)\beta^2,$$

and

$$|A_3| \le (1-\alpha)\beta,$$

otherwise.

Also

$$|A_4| \le \frac{2}{3}(1-\alpha)\beta,$$

when

$$0 \le \alpha < 1 \quad \text{and} \quad 0 < \beta \le \frac{1}{\sqrt{31}}, \tag{7}$$

or when

$$\frac{13}{16} - \frac{\sqrt{16+11\beta^2}}{16\beta\sqrt{3}} \le \alpha < 1 \quad \text{and} \quad \frac{1}{\sqrt{31}} < \beta \le \frac{1}{2}, \tag{8}$$

or when

$$\frac{17}{20} - \frac{\sqrt{40 - 13\beta^2}}{20\beta\sqrt{3}} \leq \alpha < 1 \textit{ and } \frac{1}{2} < \beta \leq 1. \tag{9}$$

Further

$$|A_4| \leq \frac{2}{9}(1-\alpha)\beta\left[1 + 2(31 - 78\alpha + 48\alpha^2)\beta^2\right], \tag{10}$$

when

$$0 \leq \alpha < \frac{13}{16} - \frac{\sqrt{16 + 11\beta^2}}{16\beta\sqrt{3}} \quad \textit{and} \quad \frac{1}{\sqrt{31}} < \beta \leq \frac{1}{2}, \tag{11}$$

or when

$$0 \leq \alpha \leq \frac{13\beta - 2}{16\beta} - \frac{\sqrt{11\beta^2 + 4\beta - 4}}{16\beta\sqrt{3}} \quad \textit{and} \quad \frac{1}{2} < \beta \leq 1. \tag{12}$$

Also

$$|A_4| \leq \frac{2}{9}(1-\alpha)\beta\left[5 - 2(31 - 78\alpha + 48\alpha^2)\beta^2\right], \tag{13}$$

when

$$\frac{13\beta + 2}{16\beta} - \frac{\sqrt{11\beta^2 - 4\beta + 60}}{16\beta\sqrt{3}} \leq \alpha < \frac{17}{20} - \frac{\sqrt{40 - 13\beta^2}}{20\beta\sqrt{3}} \textit{ and } \frac{1}{2} < \beta \leq 1. \tag{14}$$

The inequalities for $|A_2|$, $|A_3|$ and $|A_4|$ are sharp.

Proof. The inequality for $|A_2|$ follows at once from (6) and Theorem 3.1.

For A_3 we use (3) and (6) to obtain

$$A_3 = \frac{1}{2}(1-\alpha)\beta\left\{p_2 - \frac{1}{2}\left[1 - (6\alpha - 5)\beta\right]p_1^2\right\}.$$

We now apply Lemma 2.1 with $\mu = 1 - (6\alpha - 5)\beta$, so that $\mu \in [0, 2]$ when

$$0 < \beta \leq \frac{1}{5} \quad \text{and} \quad 0 \leq \alpha < 1,$$

or when

$$\frac{1}{5} < \beta \leq 1 \quad \text{and} \quad \frac{5\beta - 1}{6\beta} \leq \alpha < 1.$$

This gives the first two inequalities for $|A_3|$.

When μ is outside $[0, 2]$, Lemma 2.1 also gives $|A_3| \leq (5 - 6\alpha)(1 - a)\beta^2$ when

$$\frac{1}{5} < \beta \leq 1 \quad \text{and} \quad 0 \leq \alpha < \frac{5\beta - 1}{6\beta},$$

which proves the third inequality for $|A_3|$.

For A_4 we use (3) and (6) to obtain

$$\begin{aligned}|A_4| &= \frac{1}{3}(1-\alpha)\beta\Big|p_3 + [-1+(-5+6\alpha)\beta]p_1p_2 \\ &\quad + \frac{1}{6}\left[2-3(-5+6\alpha)\beta + (31-78\alpha+48\alpha^2)\beta^2\right]p_1^3\Big| \\ &= \frac{1}{3}(1-\alpha)\beta\left|p_3 - 2Bp_1p_2 + Dp_1^3\right|,\end{aligned}$$

with $B = \frac{1}{2}[1-(6\alpha-5)\beta]$ and $D = \frac{1}{6}[2-3(-5+6\alpha)\beta + (31-78\alpha+48\alpha^2)\beta^2]$.

We first use Lemma 2.2, so that $0 \le B \le 1$ and $B(2B-1) \le D \le B$, are equivalent to the conditions (7) or (8) or (9). This gives the inequality $|A_4| \le \frac{2}{3}(1-\alpha)\beta$.

Now, note that if conditions (11) and (12) hold, then $D \ge B$ and one of the following:

(i) $0 \le B \le 1$ and ($D < B(2B-1)$ or $D > B$),

(ii) $B > 1$ and ($D \le 1$ or $D \ge 2B-1$),

(iii) $B > 1$ and $1 < D < 2B-1$ and $3|D-B| \ge 2(B-1)$.

Similarly, if condition (14) holds, then $D < B$ and one of (i), (ii) or (iii) holds.

If (i) holds (regardless of whether $D \ge B$ or not), then using Lemma 2.3 we have

$$\begin{aligned}|A_4| &= \frac{1}{3}(1-\alpha)\beta\left|p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3\right| \\ &\le \frac{2}{3}(1-\alpha)\beta\,(1+4|D-B|) \\ &= \begin{cases} \frac{2}{9}(1-\alpha)\beta\left[1+2(31-78\alpha+48\alpha^2)\beta^2\right], & D \ge B \\ \frac{2}{9}(1-\alpha)\beta\left[5-2(31-78\alpha+48\alpha^2)\beta^2\right], & D < B \end{cases}.\end{aligned}$$

If (ii) or (iii) holds (regardless of whether $D \ge B$ or not), we write

$$\begin{aligned}&p_3 - 2Bp_1p_2 + Dp_1^3 \\ &= p_3 - 2p_1p_2 + p_1^3 + 2(1-B)p_1p_2 + (D-1)p_1^3 \\ &= p_3 - 2p_1p_2 + p_1^3 + 2(1-B)p_1\left[p_2 + \frac{D-1}{2(1-B)}\cdot p_1^2\right] \\ &= p_3 - 2p_1p_2 + p_1^3 + 2(1-B)p_1\left[p_2 - \frac{p_1^2}{2} + \frac{D-B}{2(1-B)}\cdot p_1^2\right],\end{aligned}$$

and using Lemma 2.3 obtain

$$\begin{aligned}|A_4| &\le \frac{1}{3}(1-\alpha)\beta\left[2 + 2\cdot|1-B|\cdot|p_1|\left(2-\frac{1}{2}\cdot|p_1|^2 + \frac{1}{2}\cdot\left|\frac{D-B}{1-B}\right|\cdot|p_1|^2\right)\right] \\ &= \frac{1}{3}(1-\alpha)\beta\left\{2 + |p_1|\cdot\left[4\cdot(B-1) + (|D-B|-(B-1))\cdot|p_1|^2\right]\right\} := h(|p_1|).\end{aligned}$$

Next note that

$$h'(|p_1|) = \frac{1}{3}(1-\alpha)\beta\left[4\cdot(B-1) + 3(|D-B|-(B-1))\cdot|p_1|^2\right]$$

so if (ii) holds, then $|D-B|-(B-1)\geq 0$, and so $h'(|p_1|)\geq 0$ on $(0,2)$.

If (iii) holds, then $h'(|p_1|)=0$ has only one positive solution

$$p_* = 2\sqrt{\frac{B-1}{3(B-1-|D-B|)}} \geq 2$$

and so again $h'(|p_1|)\geq 0$ for $|p_1|\in(0,2)$.

Thus, if (ii) or (iii) holds, then $h(|p_1|)$ increases on $(0,2)$ and

$$|A_4| \leq h(2) = \begin{cases} \frac{2}{9}(1-\alpha)\beta\left[1+2(31-78\alpha+48\alpha^2)\beta^2\right], & D\geq B\\ \frac{2}{9}(1-\alpha)\beta\left[5-2(31-78\alpha+48\alpha^2)\beta^2\right], & D<B \end{cases}.$$

Thus (10) and (13) are established, and so all the inequalities for $|A_4|$ are proved.

Choosing $p_1=2$ in (6) shows that the inequality for $|A_2|$ is sharp. Choosing $p_1=0$ and $p_2=2$ shows that the first two inequalities for $|A_3|$ are sharp, and $p_1=2$ and $p_2=2$ that the second inequality for $|A_3|$ is sharp. Finally choosing $p_1=0$, $p_2=0$ and $p_3=2$ shows that the first inequality for $|A_4|$ is sharp, choosing $p_1=2$, $p_2=2$ and $p_3=2$ shows that the second inequality for $|A_4|$ is sharp and choosing $p_1=-2$, $p_2=2$ and $p_3=2$ shows that the third inequality for $|A_4|$ is sharp. □

We note finally that when $\beta=1$, Theorem 2 gives the initial inverse coefficients of $f\in S^*(\alpha)$ in [7, 13], and when $\alpha=0$, the corresponding results found in [1].

5. Logarithmic coefficients

The logarithmic coefficients of f are defined in D by

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n. \tag{15}$$

They play a central role in the theory of univalent functions, and were used by de Branges in his celebrated proof of the Bieberbach conjecture. We prove the following.

Theorem 5.1. *Let $f\in\mathcal{SS}^*(\alpha,\beta)$, then for $n\geq 1$*

$$|\gamma_n| \leq \frac{\beta(1-\alpha)}{n}. \tag{16}$$

The inequalities are sharp.

Proof. From (2) and (15), we have

$$z\left\{\log\frac{f(z)}{z}\right\}' = \frac{zf'(z)}{f(z)} - 1 = \alpha - 1 + (1-\alpha)p(z)^\beta$$

and so

$$z\left\{\log\frac{f(z)}{z}\right\}' \prec \alpha - 1 + (1-\alpha)\Big(\frac{1+z}{1-z}\Big)^\beta = 2(1-a)\beta z + ...$$

Applying Lemma 2.6 gives (16) at once. The inequality is sharp when $p_n=2$ for $n\geq 1$. □

We note that when $f\in S^*(\alpha)$, the above result is a trivial consequence of differentiating (15) and using (2), and when $f\in\mathcal{SS}^*(\beta)$ for $\beta\in(0,1]$, the result was proved in [15].

6. Second Hankel determinant

The qth Hankel determinant $H_q(n)$ of a function f is defined for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1}\ldots & a_{n+q+1} \\ a_{n+1} & \ldots & \vdots \\ \vdots & & \\ a_{n+q-1} & \ldots & a_{n+2q-2} \end{vmatrix}.$$

In recent years a great deal of attention has been devoted to finding estimates of Hankel determinants whose elements are the coefficients of univalent (and multivalent) functions. For $f \in S$, growth results have been established for the general Hankel determinant $H_q(n)$, [11]. The second Hankel determinant $H_2(2) = |a_2a_4 - a_3^2|$ has received more attention, with significant results being obtained for $f \in S$ in [5, 10].

For starlike functions, the sharp inequality $H_2(2) \leq 1$ was found in [6], and many subsequent results have been obtained for $H_2(2)$ for a variety of subclasses of S, most of which are subclasses of S^*. Relevant to this paper are the sharp results in [16] that $H_2(2) \leq \frac{1}{3}(1-\alpha)^2|(3-2\alpha)(2\alpha-1)|$ for $f \in S^*(\alpha)$, and in [15] that $H_2(2) \leq \beta^2$ when $f \in \mathcal{SS}^*(\beta)$.

We prove the following.

Theorem 6.1. *If $f \in \mathcal{SS}^*(\alpha, \beta)$, then*

$$H_2(2) \leq (1-\alpha)^2\beta^2.$$

The inequality is sharp.

Proof. From (3) we have

$$\begin{aligned} H_2(2) &= |a_2a_4 - a_3^2| \\ &= \left|\frac{1}{144}(1-\alpha)^2\beta^2\Big[\Big(7 - 6\beta - (13 - 24\alpha + 12\alpha^2)\beta^2\Big)p_1^4 \right. \\ &\quad \left. - 12(1-\beta)p_1^2p_2 - 36p_2^2 + 48p_1p_3\Big]\right|. \end{aligned} \tag{17}$$

We now use Lemma 2.5 to express p_2 and p_3 in terms of p_1, and since without loss in generality we may normalise the coefficient p_1 to assume that $p_1 = p$, where $p \in [0, 2]$, we obtain after simplification

$$H_2(2) = \frac{1}{144}(1-\alpha)^2\beta^2\left|\Big[4 - (13 - 24\alpha + 12\alpha^2)\beta^2\Big]p^4 + 24pVX + 6\beta p^2xX - 12p^2x^2X - 9x^2X^2\right|,$$

where for simplicity we have written $X = 4 - p^2$ and $V = (1 - |x|^2)\zeta$.

We now use the triangle inequality to obtain

$$\begin{aligned} H_2(2) \leq \frac{1}{144}(1-\alpha)^2\beta^2\Big[&6\beta p^2(4-p^2)|x| + 12p^2(4-p^2)|x|^2 \\ &+ 9(4-p^2)^2|x|^2 + 24p(4-p^2)(1-|x|^2) \\ &+ \left|4 - (13 - 24\alpha + 12\alpha^2)\beta^2\right|p^4\Big] := \phi(|x|). \end{aligned}$$

Since

$$\phi'(|x|) = \frac{1}{24}(1-\alpha)^2\beta^2\left(4-p^2\right)\left[\beta p^2 + (6-p)(2-p)|x|\right],$$

it follows that $\phi'(|x|) \geq 0$ for $|x| \in [0,1]$. Thus $\phi(|x|) \leq \phi(1)$ and so

$$\begin{aligned} H_2(2) \leq \frac{1}{144}(1-\alpha)^2\beta^2\Big\{&3(4-p^2)[12+(1+2\beta)p^2] \\ &+ \left|4-(13-24\alpha+12\alpha^2)\beta^2\right| p^4\Big\}. \end{aligned} \tag{18}$$

The only critical point of the above expression is a minimum point when $p = 0$. Noting that $p(0) = (1-\alpha)^2\beta^2$, and that $p(0) \geq p(2)$, when $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, the required estimate for $H_2(2)$ follows.

Choosing $p_1 = 0$, $p_2 = 2$ and $p_3 = 0$ in (17) shows that the inequality is sharp. □

Setting $\beta = 1$, we obtain the following known sharp estimate for functions in $\mathcal{S}^*(\alpha)$ (see e.g [16]).

Corollary 6.2. *Let $f \in \mathcal{S}^*(\alpha)$ for $0 \leq \alpha < 1$. Then*

$$H_2(2) \leq (1-\alpha)^2.$$

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