Univalence Conditions for an Integral Operator Defined by a Generalization of the Srivastava-Attiya Operator

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Abstract. The main object of this paper is to introduce and study systematically the univalence criteria of a new family of integral operators by using a substantially general form of the widely-investigated Srivastava-Attiya operator. In particular, we derive several new sufficient conditions of univalence for this generalized Srivastava-Attiya operator. Relevant connections with other related earlier works are also pointed out.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic and univalent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$ 

If the function $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

\textsuperscript{2010 Mathematics Subject Classification. Primary 30C45, 33C60; Secondary 11M35, 30C50}

Keywords. Analytic functions; Univalent functions; Hadamard product (or convolution); $\lambda$-Generalized Hurwitz-Lerch function; Series representations; Fox’s $H$-function; Mellin-Barnes contour integral; Integral operators; Srivastava-Attiya Operator

Received: 22 April 2017; Accepted: 24 July 2017

Communicated by Dragan S. Djordjević

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then the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is defined by (see also [27])

\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).
\]

(3)

In the year 2007, Srivastava and Attiya (see [21]) defined the operator \( J_{s,a} \) by

\[
J_{s,a}(f)(z) = z + \sum_{n=2}^{\infty} \left( \frac{1+n}{n+a} \right)^{s} a_n z^n
\]

(4)

\((z \in \mathbb{U}; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mathbb{Z}_0^- = \{0, 1, 2, \cdots \}; \ s \in \mathbb{C})\).

In fact, in terms of the Hadamard product (or convolution), the linear Srivastava-Attiya operator \( J_{s,a}(f) \) defined by (4) can be written as follows (see also the recent works [8], [25] and [28]):

\[ J_{s,a} f(z) = G_{s,a} (z) * f(z), \]

where \( G_{s,a}(z) \) is given by

\[ G_{s,a}(z) = (1 + a)^{s} [\Phi(z, s, a) - a^{-s}] \quad (z \in \mathbb{U}) \]

(5)

and the function \( \Phi(z, s, a) \) involved in the right-hand side of (5) is the well-known Hurwitz-Lerch zeta function defined by (see [22])

\[ \Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \]

(6)

\((z \in \mathbb{U}; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ s \in \mathbb{C} \quad \text{when} \ |z| < 1; \ \Re(s) > 1 \quad \text{when} \ |z| > 1)\).

Recently, a new family of \( \lambda \)-generalized Hurwitz-Lerch zeta functions was investigated by Srivastava (see [20]) who introduced this \( \lambda \)-generalized Hurwitz-Lerch zeta function

\[
\Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q}(z, s, a; b, \lambda)
\]

as well as gave the following explicit series representation for it (see [20, p. 1489, Eq. (2.1)]):

\[
\Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q}(z, s, a; b, \lambda) = \frac{1}{\lambda \Gamma(s)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{np_j}}{(a+n)^s \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \left[ (a+n)b^{\lambda} \right] (s, 1), (0, \lambda^+) \cdot \frac{z^n}{n!} \quad (\lambda > 0)
\]

(7)

\(
\lambda > 0; \ \lambda_j \in \mathbb{C} \quad (j = 1, \cdots, p); \ \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \cdots, q);
\)

\[
\rho_j > 0 \quad (j = 1, \cdots, p); \ \sigma_j > 0 \quad (j = 1, \cdots, q)
\]

\[
1 + \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \geq 0; \ \min \{\Re(a), \Re(b)\} > 0.
\]
where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nu := \left(\prod_{j=1}^{p} \rho_j^{-\nu_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\nu_j}\right).$$

$$(\lambda)_c$$. $(\lambda, \nu \in \mathbb{C})$ denotes the general Pochhammer symbol (or the shifted factorial), occurring in (7), is defined, in terms of the familiar Gamma function, by

$$\frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the above $\Gamma$-quotient exists. Moreover, the $H$-function involved in the right-hand side of (7) is the well-known Fox’s $H$-function which is defined by (see, for example, [26, Chapter 2] and [12, pp. 58 et seq.])

$$H_{m,n}^{\mu} z = H_{p,q}^{\mu} \left[ \begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p) \\ b_q, B_q \end{array} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \Xi(s) z^{-s} ds,$$

where

$$\Xi(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + \beta_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - \alpha_j s\right)}{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \beta_j s\right) \prod_{j=n+1}^{p} \Gamma\left(a_j + \alpha_j s\right)}.$$  

Here

$$z \in \mathbb{C} \setminus \{0\} \quad \text{with} \quad |\arg(z)| < \pi,$$

an empty product is interpreted as 1, $m, n, p$ and $q$ are integers such that $1 \leq m \leq q$ and $0 \leq n \leq p$,

$$A_j > 0 \quad (j = 1, \cdots, p) \quad \text{and} \quad B_j > 0 \quad (j = 1, \cdots, q),$$

$$\alpha_j \in \mathbb{C} \quad (j = 1, \cdots, p) \quad \text{and} \quad \beta_j \in \mathbb{C} \quad (j = 1, \cdots, q),$$

and $\mathcal{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\left\{\Gamma(b_j + \beta_j s)\right\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\left\{\Gamma(1 - a_j - \alpha_j s)\right\}_{j=1}^{n}.\$$

If, in the series representation (7), we make use of the following limit formula (see [20, p. 1496, Eq. (4.12)])

$$\lim_{b \to 0} \left\{ H_{n/2}^{2,1} \left[ (a + n) b^{1\over 2} \right] \left( s, 1, \left( 0, \frac{1}{\lambda} \right) \right) \right\} = \lambda \Gamma(s) \quad (\lambda > 0),$$
we find for the extended Hurwitz-Lerch zeta function

\[ \Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{(p,\cdots,q)}(z, s, a) \]

that (see [29, p. 503, Eq. (6.2)])

\[ \Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{(p,\cdots,q)}(z, s, a) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)^{\eta_{\nu_j}}}{n! \prod_{j=1}^{q} (\mu_j)^{\eta_{\nu_j}}} \frac{z^n}{(n+a)^s} \tag{11} \]

\[
\left( p, q \in \mathbb{N}_0; \lambda_j \in \mathbb{C} \ (j = 1, \cdots, p); a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ (j = 1, \cdots, q); \right.
\]

\[
\rho_j, \sigma_k \in \mathbb{R}^+ \ (j = 1, \cdots, p); k = 1, \cdots, q); \Delta > -1 \text{ when } s, z \in \mathbb{C};
\]

\[
\Delta = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \sqrt{\nu};
\]

\[
\Delta = -1 \text{ and } \Re(\nu) > \frac{1}{2} \text{ when } |z| = \sqrt{\nu}.
\]

which was defined by Srivastava et al. (see [20, p. 1496, Eq. (4.12)]). In fact, the function

\[ \Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{(p,\cdots,q)}(z, s, a) \]

in (11), which was introduced by Srivastava et al. [29], is a multiparameter extension and generalization of the classical Hurwitz-Lerch zeta function \( \Phi(z, s, a) \) defined by (6).

By applying Srivastava’s \( \lambda \)-generalized Hurwitz-Lerch zeta function

\[ \Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{(p,\cdots,q)}(z, s, a; b, \lambda) \]

occurring on the left-hand side of (7), Srivastava and Gaboury [24] introduced the following linear operator:

\[ J_{(\lambda_1), (\mu_1), b}^{(s, \lambda)}(f) : \mathcal{A} \to \mathcal{A}, \]

which they defined by

\[ J_{(\lambda_1), (\mu_1), b}^{(s, \lambda)}(f)(z) = C_{(\lambda_1), (\mu_1), b}^{(s, \lambda)}(z) * f(z), \tag{12} \]

where

\[
C_{(\lambda_1), (\mu_1), b}^{(s, \lambda)}(z) = \frac{\lambda \Gamma(s) \prod_{j=1}^{p} \left( \mu_j \right)^{(a+1)^s}}{\prod_{j=1}^{p} \left( \lambda_j \right)^{(a+1)^s}} \left[ \Lambda(a+1, b, s, \lambda) \right]^{-1}
\]

\[
\cdot \Phi_{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q}^{(p,\cdots,q)}(z, s, a; b, \lambda) - \frac{a^{-s}}{\lambda \Gamma(s)} \Lambda(a, b, s, \lambda)
\]

\[
= z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \left( \lambda_j + 1 \right)^{(a+1)^s}}{\prod_{j=1}^{q} \left( \mu_j + 1 \right)^{(a+1)^s}} \left( \frac{a+1}{a+n} \right)^{s} \Lambda(a+n, b, s, \lambda) \frac{z^n}{(a+n)^{s(n-1)}} \tag{13}
\]
with
\[
\Lambda(a, b, s, \lambda) := H_{0,2}^{2,0}\left(\left(a + n\right)b^{2}\right)_{(s, 1), \left(0, \frac{1}{\lambda}\right)}. 
\] (14)

Now, from (12) and (13), we have
\[
\mathcal{J}^{\alpha, \lambda}_{(l_{0}), (p_{1}), b} f(z) = z + \sum_{n=2}^{\infty} \frac{\left(\lambda_{1} + 1\right)_{n-1}}{\left(\mu_{1} + 1\right)_{n-1}} \left(\frac{a + 1}{a + n}\right) \Lambda(a + n, b, s, \lambda) \frac{z^{n}}{n!} 
\] (15)

\[
\left(\lambda_{j} \in \mathbb{C}\right) \quad (j = 1, \cdots, p); \quad \mu_{j} \in \mathbb{C} \setminus \mathbb{Z}_{0} \quad (j = 1, \cdots, q); \quad p \leq q + 1; \quad z \in \mathbb{U}; \quad \min \{|\mathcal{R}(a), \mathcal{R}(s)| > 0; \quad \lambda > 0 \quad \text{when} \quad \mathcal{R}(b) > 0 \quad \text{and} \quad s \in \mathbb{C}; \quad a \in \mathbb{C} \setminus \mathbb{Z}_{0} \quad \text{when} \quad b = 0\). \]

It is easy to see from the definition (15) that
\[
z \left(\mathcal{J}^{\alpha, \lambda}_{(l_{0}), (p_{1}), b} f(z)\right)' = (\lambda_{1} + 1) \mathcal{J}^{\alpha, \lambda}_{(l_{1} + 1, \lambda_{2}, \cdots, \lambda_{p}), (p_{1}), b} f(z) - \lambda_{1} \mathcal{J}^{\alpha, \lambda}_{(l_{0}), (p_{1}), b} f(z) \] (16)

**Definition 1.** Let \(\Psi\) be the set of complex-valued functions \(\psi(u, v, w)\) given by
\[
\psi(u, v, w) : \mathbb{C}^{3} \rightarrow \mathbb{C}
\]
such that
(i) \(\psi(u, v, w)\) is continuous in a domain \(D \subset \mathbb{C}^{3}\);
(ii) \((0, 0, 0) \in D\) and \(|\psi(0, 0, 0)| < 1\);
(iii) The following inequality holds true:
\[
\left|\psi\left(\left(e^{\theta}, \frac{\lambda_{1} + t}{\lambda_{1} + 1}\right) e^{i\theta}, \frac{1}{\lambda_{1} + 1} \left[\lambda_{1} + 2t + \frac{L}{\lambda_{1} + 1}\right] e^{i\theta}\right)\right| \geq 1
\]
when \(\lambda_{1} \notin \mathbb{Z}_{0}\) and
\[
\left(\left(e^{\theta}, \frac{\lambda_{1} + t}{\lambda_{1} + 1}\right) e^{i\theta}, \frac{1}{\lambda_{1} + 1} \left[\lambda_{1} + 2t + \frac{L}{\lambda_{1} + 1}\right] e^{i\theta}\right) \in D
\]
with \(\mathcal{R}(l) \geq t(l - 1)\) for real \(\theta \in \mathbb{R}\) and \(l \geq 1\).

By using the generalization of the Srivastava-Attiya operator defined by (15), we now introduce the following integral operator:
\[
\widetilde{\omega}^{\beta, \alpha, \lambda}_{(l_{0}), (p_{1}), b} (\gamma_{1}, \cdots, \gamma_{k}; z) : \mathcal{A}^{n} \rightarrow \mathcal{A}.
\]

**Definition 2.** For \(\beta, \gamma_{1}, \gamma_{2}, \cdots, \gamma_{k} \in \mathbb{C}\) with
\[
\mathcal{R}(\beta) > 0 \quad \text{and} \quad \mathcal{R}(\gamma_{m}) > 0 \quad (m \in \{1, \cdots, k\}),
\]
we define the integral operator:
\[
\widetilde{\omega}^{\beta, \alpha, \lambda}_{(l_{0}), (p_{1}), b} (\gamma_{1}, \cdots, \gamma_{k}; z) : \mathcal{A}^{n} \rightarrow \mathcal{A}
\]
where the operator

\[ \frac{F_{\lambda_1,\mu_1}}{\lambda_1,\mu_1} \left( \gamma_1, \ldots, \gamma_n \right) = \left( \beta \right) \int_0^\infty \int_0^t \cdots \int_0^t \left( \frac{t^{\gamma_1} \cdots t^{\gamma_n}}{\gamma_1 \cdots \gamma_n} \right) \left( \frac{F_{\lambda_1,\mu_1}(t)}{t^\lambda} \right) \left( \frac{f_m(t)}{t^\mu} \right)^{\gamma_1} dt \right) \]

By suitably specializing Definition 2, we are led to the following integral operators:

\[ \frac{F_{(\lambda_1,\mu_1)}}{\lambda_1,\mu_1} \left( \gamma_1, \ldots, \gamma_n \right) = \left( \beta \right) \int_0^\infty \int_0^t \cdots \int_0^t \left( \frac{t^{\gamma_1} \cdots t^{\gamma_n}}{\gamma_1 \cdots \gamma_n} \right) \left( \frac{F_{\lambda_1,\mu_1}(t)}{t^\lambda} \right) \left( \frac{f_m(t)}{t^\mu} \right)^{\gamma_1} dt \right) \]

where the operator \( F_{\alpha}(\lambda_1,\mu_1; z) \) was investigated by Selvaraj and Karthikeyan [19];

\[ \frac{F_{(\lambda_1,\mu_1)}}{\lambda_1,\mu_1} \left( \gamma_1, \ldots, \gamma_n \right) = \left( \beta \right) \int_0^\infty \int_0^t \cdots \int_0^t \left( \frac{t^{\gamma_1} \cdots t^{\gamma_n}}{\gamma_1 \cdots \gamma_n} \right) \left( \frac{F_{\lambda_1,\mu_1}(t)}{t^\lambda} \right) \left( \frac{f_m(t)}{t^\mu} \right)^{\gamma_1} dt \right) \]

where the operator \( F_{\alpha}(\lambda_1,\mu_1; z) \) was investigated by Breaz et al. (see [1], [3] and [5]);

\[ \frac{F_{(\lambda_1,\mu_1)}}{\lambda_1,\mu_1} \left( \gamma_1, \ldots, \gamma_n \right) = \left( \beta \right) \int_0^\infty \int_0^t \cdots \int_0^t \left( \frac{t^{\gamma_1} \cdots t^{\gamma_n}}{\gamma_1 \cdots \gamma_n} \right) \left( \frac{F_{\lambda_1,\mu_1}(t)}{t^\lambda} \right) \left( \frac{f_m(t)}{t^\mu} \right)^{\gamma_1} dt \right) \]

where the operator \( F_{\alpha}(\lambda_1,\mu_1; z) \) was investigated by Breaz et al. (see also Stanciu et al. [30]);

\[ \frac{F_{(\lambda_1,\mu_1)}}{\lambda_1,\mu_1} \left( \gamma_1, \ldots, \gamma_n \right) = \left( \beta \right) \int_0^\infty \int_0^t \cdots \int_0^t \left( \frac{t^{\gamma_1} \cdots t^{\gamma_n}}{\gamma_1 \cdots \gamma_n} \right) \left( \frac{F_{\lambda_1,\mu_1}(t)}{t^\lambda} \right) \left( \frac{f_m(t)}{t^\mu} \right)^{\gamma_1} dt \right) \]

where the operator \( F_{\alpha}(\lambda_1,\mu_1; z) \) was investigated by Seenivasagan and Breaz [18] (see also [6]);

\[ \frac{F_{(\lambda_1,\mu_1)}}{\lambda_1,\mu_1} \left( \gamma_1, \ldots, \gamma_n \right) = \left( \beta \right) \int_0^\infty \int_0^t \cdots \int_0^t \left( \frac{t^{\gamma_1} \cdots t^{\gamma_n}}{\gamma_1 \cdots \gamma_n} \right) \left( \frac{F_{\lambda_1,\mu_1}(t)}{t^\lambda} \right) \left( \frac{f_m(t)}{t^\mu} \right)^{\gamma_1} dt \right) \]

where the operator \( F_{\alpha}(\lambda_1,\mu_1; z) \) was investigated by Breaz and Breaz [2];
where the operator $F_a(z)$ was investigated by Selvaraj and Karthikeyan [19];

$$
\delta^1_{(2,1),(1,0),0}\left(\frac{1}{\alpha_1}, \cdots, \frac{1}{\alpha_k}; z\right) = F_{a,\cdot,\cdot,\cdot}(z) \\
= \int_0^\infty \left| f_1'(t) \right|^{a_1-1} \cdots \left| f_k'(t) \right|^{a_k-1} dt,
$$

(24)

where the operator $F_a(z)$ was investigated by Breaz et al. [7];

$$
\delta^1_{(\lambda,1),(1,0),0}\left(\frac{1}{\alpha-1}, \cdots, \frac{1}{\alpha-1}; z\right) = F_a(z) = \left( \frac{1}{a} \int_0^\infty |f(t)|^{a-1} \right)^{\frac{1}{a}} dt,
$$

(25)

where the operator $F_a(z)$ was investigated by Pescar [17].

By making use of the integral operator defined in (15), we have the following definition.

**Definition 3.** A function $f_m \in \mathcal{A}$ ($m \in \{1, \cdots, k\}$) is said to be in the class $S_{(\lambda,\beta),(\mu,\delta);(\gamma,\rho),\mu,b}^{\alpha,\lambda} \in \mathcal{A}$ if it satisfy the following condition:

$$
\left| \frac{z^2\left(f_{(\lambda,\beta),(\mu,\delta);(\gamma,\rho),\mu,b}^{\alpha,\lambda} f_m(t)\right)'}{\left(f_{(\lambda,\beta),(\mu,\delta);(\gamma,\rho),\mu,b}^{\alpha,\lambda} f_m(t)\right)^2} - 1 \right| < 1 \quad (z \in \mathcal{U}; \ m \in \{1, \cdots, k\}).
$$

(26)

In our investigation of the function class $S_{(\lambda,\beta),(\mu,\delta);(\gamma,\rho),\mu,b}^{\alpha,\lambda}$ given by Definition 3, we shall need the univalence criteria and other results asserted by the following lemmas.

**Lemma 1.** (see [14]) Let the function $f$ be analytic in the disk

$$
\mathcal{U}_R = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < R\}
$$

with $|f(z)| < M$ for some fixed $M > 0$. If $f(z)$ has one zero with multiplicity order bigger that $m$ for $z = 0$, then

$$
|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathcal{U}_R).
$$

(27)

The equality holds true in (27) only if

$$
f(z) = e^{\theta} \frac{M}{R^m} z^m \quad (z \in \mathcal{U}_R),
$$

where $\theta$ is real constant.

**Lemma 2.** (see [15] and [16]) Let $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$. If the function $f(z) \in \mathcal{A}$ is constrained by

$$
\frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{z^m f''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathcal{U}),
$$

then the function $F_{\beta}(z)$ given in terms of the following integral operator:

$$
F_{\beta}(z) = \left( \beta \int_0^\infty t^{\beta-1} f'(t) dt \right)^{\frac{1}{\beta}}
$$

$$
= z + \frac{2a_2}{\beta + 1} z^2 + \left( \frac{3a_3}{\beta + 2} - \frac{26(1 - \beta) m^2}{(\beta + 1)^2} \right) z^3 + \cdots
$$

(28)

is in the class $S$ of normalized analytic and univalent functions in $\mathcal{U}$. 


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Lemma 3. (see [17]) Let $\beta \in \mathbb{C}$ with

$$\Re (\beta) > 0 \quad \text{and} \quad c \in \mathbb{C} \quad |c| \leq 1.$$ 

If the function $f(z) \in A$ is constrained by

$$\left| c |z|^{\beta} + (1 - |z|^{\beta}) \frac{z f''(z)}{\beta f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}),$$

then the function $F_\beta(z)$ given in terms of the following integral operator:

$$F_\beta(z) = \left( \beta \int_0^z t^{\beta-1} f'(t) \, dt \right)^\frac{1}{\beta}$$

(29)

is in the class $S$ of normalized analytic and univalent functions in $\mathbb{U}$.

Lemma 4. (see [13]) Let the function $w(z)$ given by

$$w(z) = a + \omega_r z^r + \omega_{r+1} z^{r+1} + \cdots$$

be analytic in $\mathbb{U}$ with

$$\omega(z) \neq a \quad \text{and} \quad r \in \mathbb{N}.$$ 

If

$$z_0 = r_0 e^{i\theta} \quad (0 < r_0 < 1) \quad \text{and} \quad |\omega(z_0)| = \max_{|z| \leq r_0} |\omega(z)|,$$

then

$$\frac{z_0 \omega'(z_0)}{\omega(z_0)} = \tau \quad \text{and} \quad \Re \left( 1 + \frac{z_0 \omega''(z_0)}{\omega'(z_0)} \right) \geq \tau,$$ 

(30)

where $\tau$ is a real number and

$$\tau \geq r \left( \frac{|\omega(z_0) - a|^2}{|\omega(z_0)|^2 - |a|^2} \right) \geq r \left( \frac{|\omega(z_0) - a|}{|\omega(z_0)| + |a|} \right).$$

2. Main Results and Their Corollaries

We begin by proving Theorem 1 below.

Theorem 1. Let the functions $f_m(z) \in A \quad (m = 1, \cdots, k)$. Suppose that $\beta, \gamma_m \in \mathbb{C} \quad (m = 1, \cdots, k)$ with

$$\Re (\beta) > 0 \quad \text{and} \quad M_m > 0 \quad (m = 1, \cdots, k).$$

Also let

$$\sum_{m=1}^k \frac{2M_m + 1}{|\gamma_m|} \leq \Re (\beta).$$

(31)

If, for all $m \in \{1, \cdots, k\}$,

$$f_m(z) \in S_{(\lambda_p),(\mu_q),b}^{(\alpha,\gamma)}(z)$$

and

$$\left| J_{(\lambda_p),(\mu_q),b} f_m(z) \right| \leq M_m \quad (z \in \mathbb{U}),$$

(32)

then the general integral operator defined by (17) is analytic and univalent in $\mathbb{U}$.
Proof. It is easy to verify that
\[ \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)}{z} \neq 0. \]

Hence, for \( z = 0 \), we find that
\[ \left( \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{1}(z)}{z} \right)^{\frac{1}{n}} \cdots \left( \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)}{z} \right)^{\frac{1}{n}} = 1. \]

Let us define the function \( g(z) \) as follows:
\[ g(z) = \int_{0}^{z} \prod_{m=1}^{k} \left( \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(t)}{t} \right)^{\frac{1}{n}} \, dt. \]  
(33)

Then we have
\[ \frac{z g''(z)}{g'(z)} = \sum_{m=1}^{k} \frac{1}{\gamma_{m}} \frac{z}{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)} - 1, \]
so that
\[ \left| \frac{z g''(z)}{g'(z)} \right| \leq \sum_{m=1}^{k} \frac{1}{\gamma_{m}} \left| \frac{z}{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)} - 1 \right|. \]

Therefore, we get
\[ \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{z g''(z)}{g'(z)} \right| \leq \frac{1}{\Re(\beta)} \sum_{m=1}^{k} \frac{1}{\gamma_{m}} \left| \frac{z}{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)} \right| + 1 \]
\[ \leq \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{m=1}^{k} \frac{1}{\gamma_{m}} \left| \frac{z}{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)} \right| \left( \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)}{z} \right) + 1 \]
\[ \leq \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{m=1}^{k} \frac{1}{\gamma_{m}} \left| \frac{z}{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)} \right| \left( \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)}{z} \right) - 1 + 1 \cdot \left| \frac{\mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z)}{z} \right| + 1 \]
\[ \leq \frac{1}{\Re(\beta)} \sum_{m=1}^{k} \frac{2M_{m} + 1}{\gamma_{m}}. \]

By using the Schwarz lemma, we have
\[ \left| \mathcal{J}^{a,b}_{(\lambda_{1},\mu_{1}),b} f_{m}(z) \right| \leq M_{m} |z| \quad (z \in \mathcal{U}). \]

Now, from (31), we obtain
\[ \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{z g''(z)}{g'(z)} \right| \leq 1. \]
Finally, by applying Lemma 2 for the function \( g(z) \), we obtain the required result asserted by Theorem 1.

\( \square \)

**Remark 1.** If, in Theorem 1, we set
\[
\lambda_1 = \alpha_1 - 1, \ldots, \lambda_p = \alpha_p - 1, \quad \mu_1 = \beta_1 - 1, \ldots, \mu_q = \beta_q - 1, \\
\gamma_1 = \frac{1}{\alpha - 1}, \ldots, \gamma_k = \frac{1}{\alpha - 1} \quad \text{and} \quad M_m = 1 \quad (1 \leq m \leq k),
\]
we obtain a known result proven in [19].

**Corollary 1.** Let the functions \( f_m(z) \in A \) \((m \in \{1, \ldots, k\})\). Also let \( \alpha \in \mathbb{C} \) with
\[
\Re(\alpha) > 0 \quad \text{and} \quad |\alpha - 1| \leq \frac{\Re(\alpha)}{3k}.
\]
If
\[
\left| \frac{z^2(H_p^q(\alpha_1, \beta_1) f_m(t))'}{[H_p^q(\alpha_1, \beta_1) f_m(t)]^2} - 1 \right| < 1
\]
and
\[
|H_p^q(\alpha_1, \beta_1) f_m(t)| \leq M_m \quad (m = 1, \ldots, k; \ z \in \mathbb{U}),
\]
then the general integral operator defined by (18) is analytic and univalent in \( \mathbb{U} \).

**Remark 2.** Putting
\[
p = 2, \ q = 1, \ \lambda_1 = \lambda, \ \lambda_2 = 1, \ \mu_1 = \lambda, \ \gamma_j = \frac{1}{\alpha - 1} \quad (j = 1, \ldots, k)
\]
and
\[
M_m = 1 \quad (1 \leq m \leq k)
\]
in Theorem 1, we obtain another known result given in [4].

**Corollary 2.** Let the functions \( f_m(z) \in A \) \((m \in \{1, \ldots, k\})\). Also let \( \alpha \in \mathbb{C} \) with
\[
\Re(\alpha) > 0 \quad \text{and} \quad |\alpha - 1| \leq \frac{\Re(\alpha)}{3k}.
\]
If
\[
\left| \frac{z^2f_m(t)}{f_m(t)} - 1 \right| < 1 \quad \text{and} \quad |f_m(t)| \leq 1 \quad (m = 1, \ldots, k; \ z \in \mathbb{U}),
\]
then the general integral operator defined by (19) is analytic and univalent in \( \mathbb{U} \).

We now prove another result asserted by Theorem 2 below.

**Theorem 2.** Let the functions \( f_m(z) \in A \) \((m = 1, \ldots, k)\). Suppose that
\[
c, \beta \in \mathbb{C} \quad \text{and} \quad M_m > 0 \quad (m = 1, \ldots, k).
\]
Proof. From Theorem 1, we have
\[
\frac{zg''(z)}{g'(z)} = \sum_{m=1}^{k} \frac{1}{\gamma_m} \left[ \frac{z \left( J_{(l_p),(\mu_1),b} f_m(z) \right)'}{J_{(l_p),(\mu_1),b} f_m(z)} - 1 \right],
\]
so that
\[
\left| c |z|^{2\beta} \cdot \frac{zg''(z)}{g'(z)} \right| = \left| c |z|^{2\beta} \frac{1}{\beta g'(z)} \sum_{m=1}^{k} \frac{1}{\gamma_m} \left[ \frac{z \left( J_{(l_p),(\mu_1),b} f_m(z) \right)'}{J_{(l_p),(\mu_1),b} f_m(z)} - 1 \right] \right|
\]
\[
\leq |c| \left| \frac{1}{\beta} \sum_{m=1}^{k} \frac{1}{\gamma_m} \left| \frac{z \left( J_{(l_p),(\mu_1),b} f_m(z) \right)'}{J_{(l_p),(\mu_1),b} f_m(z)} - 1 \right| \right|
\]
\[
\leq |c| \left| \frac{1}{\beta} \sum_{m=1}^{k} \frac{1}{\gamma_m} \left| \frac{z \left( J_{(l_p),(\mu_1),b} f_m(z) \right)'}{J_{(l_p),(\mu_1),b} f_m(z)} - 1 \right| + 1 \right|
\]
\[
\leq |c| \left| \frac{1}{\beta} \sum_{m=1}^{k} \frac{2M_m + 1}{\gamma_m} \right|
\]
\[
\leq |c| \left| \frac{1}{\beta} \sum_{m=1}^{k} \frac{2M_m + 1}{\gamma_m} \right|
\]
\[
\leq |c| \left| \frac{1}{\beta} \sum_{m=1}^{k} \frac{2M_m + 1}{\gamma_m} \right|
\]
\[
\leq |c| \left| \frac{1}{\beta} \sum_{m=1}^{k} \frac{(2M_m + 1)k}{\gamma_m} \right|
\]
Now, by making use of (34), we obtain
\[
\left| c |z|^{2\beta} \cdot \frac{zg''(z)}{g'(z)} \right| \leq 1.
\]
Finally, if we apply Lemma 3 for the function \( g(z) \), we obtain the result asserted by Theorem 2. \( \square \)
Remark 3. If we set
\[ p = 2, \quad q = 1, \quad \lambda_1 = \lambda, \quad \lambda_2 = 1, \quad \mu_1 = \lambda \quad \text{and} \quad \gamma_j = \frac{1}{\alpha_j} \quad (j = 1, \cdots, k) \]
in Theorem 2, we obtain a known result (see [31]).

Corollary 3. Let the functions \( f_m(z) \in A \) \((m = 1, \cdots, k)\). Suppose that
\[ c, \beta \in \mathbb{C} \quad \text{and} \quad M_m \geq 1 \quad (m = 1, \cdots, k). \]
Also let
\[ \alpha_m \in \left[ 1, \max_{1 \leq m \leq k} \left\{ \frac{(2M_m + 1)k}{(2M_m + 1)k - 1} \right\} \right] \quad (m = 1, \cdots, k) \]
and
\[ |\gamma| \leq 1 - \frac{k}{\mathbb{R}(\beta) \max_{1 \leq m \leq k} (2M_m + 1)|\alpha_m|}. \]
If
\[ |f_m(z)| \leq M_m \quad \text{and} \quad \left| \frac{z f_m'(z)}{f_m^2(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}; \ m = 1, \cdots, k), \]
then the general integral operator defined by (20) is analytic and univalent in \( \mathbb{U} \).

Finally, we state and prove Theorem 3 below.

Theorem 3. Let \( \lambda_1 \notin \mathbb{Z}_0^+ \). Suppose that \( \psi(u, v, w) \in \Psi \) and that
\[
\left( J_{(\lambda_1), (\mu_1), b}^{\alpha_1, A} f(z), J_{(\lambda_1 + 1, \lambda_2, \cdots, \lambda_p), (\mu_1), b}^{\alpha_1, A} f(z), \right. \\
\left. J_{(\lambda_1 + 2, \lambda_2, \cdots, \lambda_p), (\mu_1), b}^{\alpha_1, A} f(z) \right) \in \mathbb{C}^3. \tag{36}
\]
If
\[
\left| \psi \left( J_{(\lambda_1), (\mu_1), b}^{\alpha_1, A} f(z), J_{(\lambda_1 + 1, \lambda_2, \cdots, \lambda_p), (\mu_1), b}^{\alpha_1, A} f(z), \right. \\
\left. J_{(\lambda_1 + 2, \lambda_2, \cdots, \lambda_p), (\mu_1), b}^{\alpha_1, A} f(z) \right) \right| < 1 \quad (z \in \mathbb{U}), \tag{37}
\]
then
\[
\left| \left( J_{(\lambda_1), (\mu_1), b}^{\alpha_1, A} f(z) \right) \right| < 1 \quad (z \in \mathbb{U}). \]

Proof. Let
\[
J_{(\lambda_1), (\mu_1), b}^{\alpha_1, A} f(z) = \omega(z) \quad (z \in \mathbb{U}). \tag{38}
\]
Thus, clearly, it follows that \( \omega(z) \) is analytic in \( \mathbb{U} \),
\[ \omega(0) = 1 \quad \text{and} \quad \omega(z) \neq 1 \quad (z \in \mathbb{U}). \]
Thus, by letting

\[ (\lambda_1 + 1) \left( \mathcal{J}^{s,a,b}_{(\lambda_1,\lambda_2,\ldots,\lambda_p)} f(z) \right) = z\alpha'(z) + \lambda_1\alpha(z). \]  

(39)

Moreover, by differentiating (39) with respect to \( z \) and using the following identity:

\[ z \left( \mathcal{J}^{s,a,b}_{(\lambda_1+1,\lambda_2,\ldots,\lambda_p)} f(z) \right)' = (\lambda_1 + 2) \mathcal{J}^{s,a,b}_{(\lambda_1+2,\lambda_2,\ldots,\lambda_p)} f(z) - (\lambda_1 + 1) \mathcal{J}^{s,a,b}_{(\lambda_1+1,\lambda_2,\ldots,\lambda_p)} f(z), \]

which is a consequence of the identity (16), we obtain

\[ (\lambda_1 + 2) \left( \mathcal{J}^{s,a,b}_{(\lambda_1+1,\lambda_2,\ldots,\lambda_p)} f(z) \right) = \lambda_1\alpha(z) + 2z\alpha'(z) + \frac{1}{\lambda_1 + 1}z^2\alpha''(z) \quad (z \in \mathbb{U}). \]  

(40)

We now claim that

\[ |\alpha(z)| < 1 \quad (z \in \mathbb{U}). \]

Otherwise, there exists a point \( z_0 \in \mathbb{U} \) such that

\[ \max_{|z| \leq |z_0|} |\alpha(z)| = |\alpha(z_0)| = 1. \]  

(41)

Thus, by letting \( \alpha(z_0) = e^{i\theta} \) and using Lemma 4 with \( a = 1 \) and \( r = 1 \), we see that

\[ \mathcal{J}^{s,a,b}_{(\lambda_1,\lambda_2,\ldots,\lambda_p)} f(z) = e^{i\theta}, \]

and

\[ \mathcal{J}^{s,a,b}_{(\lambda_1+1,\lambda_2,\ldots,\lambda_p)} f(z) = \frac{1}{\lambda_1 + 1} (\lambda_1 + \tau) e^{i\theta} \]

and

\[ \mathcal{J}^{s,a,b}_{(\lambda_1+2,\lambda_2,\ldots,\lambda_p)} f(z) = \frac{1}{\lambda_1 + 2} \left( \lambda_1 + 2\tau + \frac{L}{\lambda_1 + 1} \right) e^{i\theta}, \]

where

\[ L = \frac{z_0^2\alpha''(z_0)}{\alpha(z_0)} \quad \text{and} \quad \tau \geq 1. \]

Furthermore, by an application of (30) in Lemma 4, we get

\[ \Re(L) \geq \tau(\tau - 1). \]

Since \( \psi(u,v,w) \in \Psi \), we have

\[ \left| \psi \left( e^{i\theta}, \left[ \frac{\lambda_1 + \tau}{\lambda_1 + 1} \right] e^{i\theta}, \frac{1}{\lambda_1 + 1} \left[ \lambda_1 + 2\tau + \frac{L}{\lambda_1 + 1} \right] e^{i\theta} \right) \right| \geq 1, \]  

(42)

which contradicts the condition (37) of Theorem 3. Therefore, we conclude that

\[ \left| \mathcal{J}^{s,a,b}_{(\lambda_1,\lambda_2,\ldots,\lambda_p)} f(z) \right| < 1 \quad (z \in \mathbb{U}), \]

which evidently completes the proof of Theorem 3. \( \square \)
3. Concluding Remarks and Observations

In our present investigation, we have introduced and studied systematically the univalence criteria of a new family of integral operators by using a substantially general form of the widely-investigated Srivastava-Attiya operator. In particular, we have derived new sufficient conditions of univalence for this generalized Srivastava-Attiya operator. Our main results are contained in Theorems 1, 2 and 3. By suitably specializing these main results, we have deduced several corollaries and consequences which were derived in a number related earlier works on the subject of investigation here (see also the recent works [9], [10], [11] and [23]).

References