Two-Geodesic-Transitive Graphs Which are Neighbor Cubic or Neighbor Tetravalent

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Abstract. A vertex triple \((u, v, w)\) with \(v\) adjacent to both \(u\) and \(w\) is called a 2-geodesic if \(u \neq w\) and \(u, w\) are not adjacent. A graph \(\Gamma\) is said to be 2-geodesic-transitive if its automorphism group is transitive on both arcs and 2-geodesics. In this paper, a complete classification of 2-geodesic-transitive graphs is given which are neighbor cubic or neighbor tetravalent.

1. Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph \(\Gamma\), we use \(V(\Gamma)\) and \(\text{Aut}(\Gamma)\) to denote its vertex set and automorphism group, respectively. For the group theoretic terminology not defined here we refer the reader to [2, 8, 22]. In a non-complete graph \(\Gamma\), a vertex triple \((u, v, w)\) with \(v\) adjacent to both \(u\) and \(w\) is called a 2-geodesic if \(u \neq w\) and in addition \(u, w\) are not adjacent. An arc is an ordered pair of adjacent vertices. The graph \(\Gamma\) is said to be 2-geodesic-transitive if its automorphism group \(\text{Aut}(\Gamma)\) is transitive on both arcs and 2-geodesics. The family of 2-geodesic-transitive graphs is closely related to the well-known family of 2-arc-transitive graphs. A vertex triple \((u, v, w)\) with \(v\) adjacent to both \(u\) and \(w\) is called a 2-arc if \(u \neq w\) and \(u, w\) are not adjacent. The graph \(\Gamma\) is said to be 2-arc-transitive if its automorphism group \(\text{Aut}(\Gamma)\) is transitive on both arcs and 2-arcs. Clearly, each 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If \(\Gamma\) has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. For instance, the complete multipartite graph \(K_{3[3]}\) is 2-geodesic-transitive neighbor cubic but not 2-arc-transitive. Thus the family of non-complete 2-arc-transitive graphs is properly contained in the family of 2-geodesic-transitive graphs.

The local structure of the family of 2-geodesic-transitive graphs was determined in [4]. In [5], Devillers, Li, Praeger and the author classified 2-geodesic-transitive graphs of valency 4. Later, in [6], a reduction theorem for the family of normal 2-geodesic-transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified. The family of 2-geodesic-transitive but not 2-arc-transitive graphs with prime valency was precisely determined in [7].
For a subset $U$ of the vertex set of $\Gamma$, we denote by $[U]$ the subgraph of $\Gamma$ induced by $U$, and $[\Gamma(u)]$ is the subgraph induced by the neighborhood of the vertex $u$. Devillers, Li, Praeger and the author in [4, Theorem 1] proved that if $\Gamma$ is a 2-geodesic-transitive graph of valency at least 2, then for each vertex $u$, either

1. $[\Gamma(u)]$ is connected of diameter 2; or
2. $[\Gamma(u)] \cong nK_r$, for some integers $m \geq 2, r \geq 1$.

Further, Theorem 1.4 of [4] shows that there is a bijection between the family of graphs in case (2) and a certain family of partial linear spaces. In particular, if $r = 1$, then $\Gamma$ is 2-arc-transitive. The first remarkable result about 2-arc-transitive graphs comes from Tutte [19, 20], and this family of graphs has been studied extensively, see [1, 9, 10, 12, 15–18, 21]. The graphs in case (1) were investigated in [14]; and in [13], the author completely determined such graphs with valency twice a prime. In this paper, we continue the investigation of the graphs in case (1).

A connected graph is said to be neighbor cubic or neighbor tetravalent if its local subgraph is connected of valency 3 or 4, respectively. For a graph $\Gamma$, its complement $\bar{\Gamma}$ is the graph with vertex set $V(\Gamma)$, and two vertices are adjacent if and only if they are not adjacent in $\Gamma$.

Let $\Gamma$ be a 2-geodesic-transitive graph. Let $u \in V(\Gamma)$. Suppose that $[\Gamma(u)]$ is connected of valency 2. Then $\Gamma$ is either the octahedron or the icosahedron, see [5, Corollary 1.4]. Thus the next natural problem is to classify the family of 2-geodesic-transitive graphs whose neighbor subgraph is connected of valency 3. Our first theorem precisely determines such graphs.

**Theorem 1.1.** Let $\Gamma$ be a 2-geodesic-transitive neighbor cubic graph. Then $\Gamma$ is one of the following graphs: $K_{3[3]}$, $J(5, 2)$, complement of the triangular graph $T(7)$, the Conway-Smith graph, or the Hall graph.

We denote by $K_{m[n]}$ the complete multipartite graph with $m$ parts, and each part has $b$ vertices where $m \geq 3, b \geq 2$. Let $\Omega = [1, n]$ where $n \geq 3$, and let $1 \leq k \leq \lfloor \frac{n}{3} \rfloor$ where $\lfloor \frac{n}{3} \rfloor$ is the integer part of $\frac{n}{3}$. Then the Johnson graph $J(n, k)$ is the graph whose vertex set is the set of all $k$-subsets of $\Omega$, and two $k$-subsets $u$ and $v$ are adjacent if and only if $|u \cap v| = k - 1$.

The second theorem determines the family of 2-geodesic-transitive graphs whose neighbor subgraph is connected of valency 4.

**Theorem 1.2.** Let $\Gamma$ be a 2-geodesic-transitive neighbor tetravalent graph. Then $\Gamma$ is one of the following three graphs: $J(6, 2)$, $J(6, 3)$ or $K_{4[2]}$.

### 2. Proof of Theorem 1.1

The first lemma determines the family of 2-geodesic-transitive neighbor cubic graphs whose local subgraph is symmetric.

**Lemma 2.1.** Let $\Gamma$ be a 2-geodesic-transitive neighbor cubic graph. Suppose that $[\Gamma(u)]$ is arc-transitive for some $u \in V(\Gamma)$. Then $\Gamma$ is one of the following graphs: $K_{3[3]}$, complement of the triangular graph $T(7)$, the Conway-Smith graph, or the Hall graph.

**Proof.** Let $(u, v)$ be an arc and $A = \text{Aut}(\Gamma)$. Since $\Sigma := [\Gamma(u)]$ is not an empty graph, $\Gamma$ has girth 3. Further, the graph $\Gamma$ is 2-geodesic-transitive, so it follows from Theorem 1.1 (1) of [4] that $\Sigma$ has diameter 2 and the arc stabilizer $A_{uv}$ is transitive on $\Sigma_2(v)$. Since $\Sigma$ is arc-transitive, $\Sigma$ is distance-transitive, and it is listed in [3, p.221-223]. Thus by inspecting the candidates, $\Sigma$ is either the complete bipartite graph $K_{3,3}$ or the Petersen graph $O_3$. If $\Sigma = K_{3,3}$, then $\Gamma$ is $K_{3[3]}$. If $\Sigma = O_3$, then by [11, Theorem 1.1], $\Gamma$ is the Conway-Smith graph, the Hall graph, or the complement of the triangular graph $T(7)$. \square

Let $u, v \in V(\Gamma)$. Then the distance between $u, v$ in $\Gamma$ is denoted by $d_\Gamma(u, v)$. A graph $\Gamma$ is said to be 2-distance-transitive if, for $i = 1, 2$ and for any two vertex pairs $(u_1, v_1)$ and $(u_2, v_2)$ with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in \text{Aut}(\Gamma)$ such that $(u_1, v_1)^g = (u_2, v_2)$. By the definition, every 2-geodesic-transitive graph is 2-distance-transitive.
Lemma 2.2. Let $\Gamma$ be a 2-distance-transitive graph. If $[\Gamma(u)] \cong C_n$ for some $u \in V(\Gamma)$ and $n \geq 5$, then $\Gamma$ is either $I(5,2)$ or the icoshedron.

Proof. Let $\Sigma := [\Gamma(u)]$. Suppose $\Sigma \cong C_n$ where $n \geq 5$. If $\Sigma$ is arc-transitive, then $n = 5$ and $\Sigma \cong C_5$. By [5, Corollary 1.4], $\Gamma$ is the icoshedron. In the remaining of this proof, we assume that $\Sigma$ is not arc-transitive. Hence $n \geq 6$ and $\Gamma$ has diameter 2.

Let $v \in V(\Sigma)$. Then for any $v' \in \Sigma_2(v)$, $|\Sigma(v) \cap \Sigma(v')| = n - 4$. Since $u \in \Gamma(v) \cap \Gamma(v')$, it follows that $n - 3 \leq |\Gamma(v) \cap \Gamma(v')|$. Since $|\Sigma_2(v) \cap \Sigma(v')| = 1$, it follows that $|\Gamma(v) \cap \Gamma(v')| \leq n - 1$, so $|\Gamma(v) \cap \Gamma(v')| = n - 3, n - 2$ or $n - 1$.

As $\Sigma \cong C_n$, the valency of $\Sigma$ is $n - 3$, so $|\Gamma(u) \cap \Gamma(v)| = n - 3$. Thus $|\Gamma(v) \cap \Gamma(v')| = n - 2$, so $|\Gamma_2(v) \cap \Gamma(u)| = 2$, as $\Gamma$ is arc-transitive. Hence there are 2$n$ edges between $\Gamma(v)$ and $\Gamma_2(v)$. By the assumption $\Gamma$ is 2-distance-transitive, the value $|\Gamma(v) \cap \Gamma(v')|$ divides $2n$. Since $|\Gamma(v) \cap \Gamma(v')| < n$, it follows that $2|\Gamma(v) \cap \Gamma(v')| < 2n$, so $3|\Gamma(v) \cap \Gamma(v')| \leq 2n$. If $|\Gamma(v) \cap \Gamma(v')| = n - 3$, then $n = 6$ or 9. If $|\Gamma(v) \cap \Gamma(v')| = n - 2$, then $n = 6$. If $|\Gamma(v) \cap \Gamma(v')| = n - 1$, then $n = 5$. By [3, p.224], $n \neq 9$. Thus $n = 6$.

Set $\Gamma(v) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$. Let $(v_1,v_2,v_3)$ and $(v_4,v_5,v_6)$ be two triangles and let $(v_1,v_6),(v_2,v_3)$ and $(v_5,v_4)$ be three arcs. Then $|\Gamma_2(u) \cap \Gamma_2(v)| = 2$, say $\Gamma_2(u) \cap \Gamma_2(v) = \{w_1,w_2\}$. Hence $\Gamma_2(v) = \{u,v_2,v_3,v_6,w_1,w_2\}$.

Since $|\Gamma(v)| \cong C_6$, $(u,v_2,v_3)$ is a triangle and neither $v_2$ nor $v_3$ is adjacent to $v_6$, it follows that $(v_6,w_1,w_2)$ is a triangle, $v_2$ is adjacent to exactly one of $w_1,w_2$, say $w_1$, and $v_3$ is adjacent to $w_2$. Set $\Gamma_2(u) \cap \Gamma_2(v) = \{w_1,w_2\}$. Then $\Gamma(v_2) = \{u,v_1,v_3,v_5,w_1,w_2\}$.

Since $|\Gamma(v)| \cong C_6$, it follows that $(v_2,w_1,w_2)$ is a triangle. Thus $\Gamma(u) \cap \Gamma(v_2) = \{v_1,v_2,v_5,v_6\}$ and $\Gamma_2(u) \cap \Gamma_2(v_2) = \{w_2,w_3\}$. Since $\Gamma$ is 2-distance-transitive, $\Gamma$ has diameter 2 and is distance-transitive. By inspecting the graphs in [3, p.223], $\Gamma \cong J(5,2)$.

Lemma 2.3. Let $\Gamma$ be a 2-geodesic-transitive neighbor cubic graph. If $[\Gamma(u)]$ is not arc-transitive for some $u \in V(\Gamma)$, then $\Gamma$ is $J(5,2)$.

Proof. Suppose that $\Sigma := [\Gamma(u)]$ is not arc-transitive. Let $(u,v)$ be an arc and $A = \text{Aut}(\Gamma)$. Since $\Sigma$ is not an empty graph, $\Gamma$ has girth 3. Since $\Gamma$ is 2-geodesic-transitive, it follows from Theorem 1.1 (1) of [4] that $\Sigma$ has diameter 2 and $A_{\text{arc}}$ is transitive on $\Sigma_2(v)$.

If $\Sigma$ has girth at least 5, then for any $x,y \in \Sigma(v)$, $(\Sigma_2(v) \cap \Sigma(x)) \cap (\Sigma_2(v) \cap \Sigma(y)) = \emptyset$. Since $A_{\text{arc}}$ is transitive on $\Sigma(v)$, it follows that $A_{\text{arc}}$ is transitive on $\Sigma(v)$, contradicting that $\Sigma$ is not arc-transitive. Thus $\Sigma$ has girth 3 or 4.

Suppose $\Sigma$ has girth 4. Then there are 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$, as $\Sigma$ has valency 3. Further, for any $x \in \Sigma_2(v)$, $|\Sigma(v) \cap \Sigma(x)| = 2$ or 3. Suppose $|\Sigma(v) \cap \Sigma(x)| = 3$. Since $A_{\text{arc}}$ is transitive on $\Sigma_2(v)$ and $|\Sigma(v) \cap \Sigma(x)| = 3$, it follows that $\Sigma_2(v) \cap \Sigma_2(x) = \emptyset$. Any two vertices of $\Delta$ are non-adjacent, and every vertex of $\Delta$ is adjacent to all vertices which are not in $\Delta$, as $\Sigma$ has diameter 2. Thus $\Delta$ is a block of cardinality 3, and $\Sigma(v)$ is another block. Hence $\Sigma \cong K_{3,3}$, so $A_{\text{arc}}$ is transitive on $\Sigma(v)$, which is a contradiction. Suppose $|\Sigma(v) \cap \Sigma(x)| = 2$. Then there are at least two edges between $\Sigma(v)$ and $\Sigma_2(v)$ and $A_{\text{arc}}$ is transitive on $\Sigma(v)$, it follows that $\Sigma_2(v) \cap \Sigma_2(x) = \emptyset$. Set $\Sigma(v) = \{v_1,v_2,w_1\}$ and $\Sigma_2(v) = \{x_1,x_2,x_3\}$. Let $\Sigma_2(v) \cap \Sigma_1(x) = \{x_1,x_2\}$ and $\Sigma(v) \cap \Sigma(x) = \{w_1,w_2\}$. If $\Sigma_2(v) \cap \Sigma_1(x) = \{x_1,x_2\}$, then $\Sigma(v) \cap \Sigma(x) = \{w_1\}$, contradicting the fact that $|\Sigma(v) \cap \Sigma(x)| = 2$. Thus $\Sigma_2(v) \cap \Sigma_1(x) = \emptyset$. Hence $\Sigma_2(v)$ is adjacent to $x_3$.

In particular, $\Sigma_2(v) = \{v_1,v_2,(\Sigma_2(w_1) \cup \Sigma_2(w_2))\}$. Since $\Sigma$ has girth 4, it follows that $x_1$ is not adjacent to any vertex of $\{x_2,x_3\}$, so $\Sigma_2(v) \cap \Sigma_1(x) = \emptyset$, contradicting that $\Sigma$ has diameter 2. Thus $|\Sigma(v) \cap \Sigma(x)| = 2$, and so the girth of $\Sigma$ is not 4.

Therefore $\Sigma$ has girth 3. Set $\Sigma(v) = \{v_1,v_2,v_3\}$. Let $(v,v_1,w_1)$ be a 2-geodesic and let $(v_1,v_3,w_2)$ be a triangle. Set $\Sigma(v) = \{v_1,v_2,v_3\}$ and $(v_1,v_3,w_2)$ are not adjacent. Assume that $v_2,v_3$ are adjacent. Then $v_2$ is adjacent to both $v_1$ and $v_3$. Since $\Sigma$ is vertex-transitive, some vertex of $\Sigma(v_1)$ is adjacent to the remaining two vertices in $\Sigma(v)$. Since $(v,v_1,w_1)$ is a 2-geodesic, $v,v_1$ are not adjacent, it follows that $v_2$ is adjacent to both $v$ and $w_1$, so $\{v,v_1,v_2,w_1\} \not\subseteq \Sigma(v_1)$, contradicting the fact that $\Sigma$ has valency 3. Thus the arc $(v,v_2)$ is not in any triangle. Hence $|\Sigma_2(v) \cap \Sigma_2(v)| = 2$, and say $\Sigma_2(v) \cap \Sigma_2(v) = \{w,w'\}$. In particular, every vertex is in a unique triangle. Hence $w$ and $w'$ are adjacent.

Suppose that $|\Sigma(v) \cap \Sigma(w)| = 1$. Then $\Sigma(v) \cap \Sigma(w) = \Sigma(v) \cap \Sigma(w') = \{v_3\}$. Since $\Sigma$ has diameter 2, $|\Sigma_2(v) \cap \Sigma_2(w)| = 2$. Since $v_1$ is in a unique triangle, $w_1,w_2$ are not adjacent. Set $\Sigma_2(v) \cap \Sigma_2(v) = \{w_2\}$. As $v_2$ is
not adjacent to any one of \(|w, w'|, w_2 \notin \{w, w'\}\). Since \(A_w\) is transitive on \(\Sigma_2(v)\) and \(|\Sigma_2(v) \cap \Sigma(w)| = 2\), it follows that \(\Sigma_2(v)\) is a vertex-transitive graph of valency 2, so \(\Sigma_2(v) \cong C_4\). Hence the vertex \(w_1\) is not in any triangle, which is a contradiction. Thus \(|\Sigma(v) \cap \Sigma(w)| = 1\).

Hence \(|\Sigma(v) \cap \Sigma(w)| = 2\). Since \(A_w\) is transitive on \(\Sigma_2(v)\) and there are 4 edges between \(\Sigma(v)\) and \(\Sigma_2(v)\), it follows that \(\Sigma_2(v) = \{w, w'|\} \) and \(|\Sigma(v) \cap \Sigma(w')| = 2\). Thus \(\Sigma \cong \overline{C_6}\). It follows from Lemma 2.2 that \(\Gamma\) is \(J(5, 2)\) or the icosahedron. The icosahedron is not neighbor cubic, so \(\Gamma\) is \(J(5, 2)\) .

It follows from Lemmas 2.1 and 2.3 that Theorem 1.1 is true.

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by a series of lemmas.

**Lemma 3.1.** Let \(\Gamma\) be a tetravalent vertex-transitive graph. Let \(A := \text{Aut}(\Gamma)\) and \(u \in V(\Gamma)\). If \(\Gamma\) has girth 4 and \(A_u\) is transitive on \(\Gamma_2(u)\), then either \(\Gamma\) is symmetric or \(|\Gamma(u) \cap \Gamma(w)| \neq 3\) for any \(w \in \Gamma_2(u)\).

**Proof.** Suppose that \(\Gamma\) has girth 4 and \(A_u\) is transitive on \(\Gamma_2(u)\). Assume \(\Gamma\) is not a symmetric graph. Since \(\Gamma\) has both valency and girth 4, \(|\Gamma_2(u) \cap \Gamma(v)| = 3\) for each \(v \in \Gamma(u)\), so there are 12 edges between \(\Gamma(u)\) and \(\Gamma_2(u)\). Let \((u, v, w)\) be a 2-geodesic. Assume that \(|\Gamma(u) \cap \Gamma(w)| = 3\). By the assumption, \(A_u\) is transitive on \(\Gamma_2(u)\), so 12 \(3|\Gamma_2(u)|\), hence \(|\Gamma_2(u)| = 4\).

Set \(\Gamma(u) = \{v_1 = v, v_2, v_3, v_4\}\) and \(\Gamma_2(u) = \{w_1 = w, w_2, w_3, w_4\}\). Let \(\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}\) and \(\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}\). Let \(\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}\). Since \(\Gamma\) has girth 4, it follows that \(w_1\) is not adjacent to any vertex of \(\Gamma_2(u) \setminus \{w_1\}\), so \(|\Gamma_3(u) \cap \Gamma(v_1)| = 1\), say \(|\Gamma_3(u) \cap \Gamma(v_1) = \{z\}\). Then \((v_1, w_1, z)\) and \((v_2, w_1, z)\) are 2-geodesics. Thus \(|\Gamma(v_1) \cap \Gamma(z)| = 3 = |\Gamma(v_2) \cap \Gamma(z)|\), so \(\Gamma(z) = \Gamma_2(u)\). Hence \(\Gamma \cong K_{5.5} - K_2\), and \(A \cong S_2 \times S_5\). However \(A_u \cong S_3\) is transitive on \(\Gamma(u)\), contradicting the assumption that \(\Gamma\) is not a symmetric graph. Thus \(|\Gamma(u) \cap \Gamma(w)| \neq 3\).

A permutation group \(G\) on a set \(\Omega\) is said to be 2-homogeneous, if \(G\) is transitive on the set of 2-subsets of \(\Omega\).

**Lemma 3.2.** Let \(\Gamma\) be a tetravalent vertex-transitive but not arc-transitive graph. Let \(A := \text{Aut}(\Gamma)\) and \(u \in V(\Gamma)\). Suppose that \(A_u\) is transitive on \(\Gamma_2(u)\). Then \(\Gamma\) has girth 3.

**Proof.** Suppose \(\Gamma\) has girth at least 5. Then for any \(x, y \in \Gamma(u), (\Gamma_2(u) \cap \Gamma(x)) \cap (\Gamma_2(u) \cap \Gamma(y)) = \emptyset\). Since \(A_u\) is transitive on \(\Gamma_2(u)\), it follows that \(A_u\) is transitive on \(\Gamma(u)\), contradicting that \(\Gamma\) is not arc-transitive. Thus \(\Gamma\) has girth 3 or 4.

Assume that \(\Gamma\) has girth 4. Then \(|\Gamma(u) \cap \Gamma(w)| = 2, 3\) or 4, for any \(w \in \Gamma_2(u)\). By Lemma 3.1, \(|\Gamma(u) \cap \Gamma(w)| \neq 3\). Suppose that \(|\Gamma(u) \cap \Gamma(w)| = 2\). Since \(\Gamma\) is vertex-transitive and \(A_u\) is transitive on \(\Gamma_2(u)\), each 2-arc of \(\Gamma\) lies in a unique 4-cycle. Thus, there is a 1-1 mapping between the unordered vertex pairs in \(\Gamma(u)\) and vertices in \(\Gamma_2(u)\). Again since \(A_u\) is transitive on \(\Gamma_2(u)\), it follows that \(A_u\) is transitive on the set of unordered vertex pairs in \(\Gamma(u)\). Hence \(A_u\) is 2-homogeneous on \(\Gamma(u)\), so \(A_u\) is transitive on \(\Gamma(u)\), contradicting that \(\Gamma\) is not arc-transitive. Suppose \(|\Gamma(u) \cap \Gamma(w)| = 4\). Since \(\Gamma\) has girth 4, there are 12 edges between \(\Gamma(u)\) and \(\Gamma_2(u)\). As \(A_u\) is transitive on \(\Gamma_2(u)\), \(\Gamma\) has diameter 2 and \(|\Gamma_2(u)| = 3\). Let \(\Delta = \{u\} \cup \Gamma_2(u)\). Then any two vertices of \(\Delta\) are non-adjacent, and every vertex of \(\Delta\) is adjacent to all vertices which are not in \(\Delta\). By the structure of \(\Gamma\), \(\Delta\) is a block of cardinality 4, and \(\Gamma(u)\) is another block. Thus \(\Gamma \cong K_{4.4}\). Hence \(A_u\) is transitive on \(\Gamma(u)\), again a contradiction. Therefore \(\Gamma\) has girth 3.

**Lemma 3.3.** Let \(\Gamma\) be a tetravalent vertex-transitive but not arc-transitive graph of diameter 2. Let \(A := \text{Aut}(\Gamma)\) and \(u \in V(\Gamma)\). Suppose that \(A_u\) is transitive on \(\Gamma_2(u)\). Then \(|\Gamma(u) \cap \Gamma(w)| \neq 4\) for any \(w \in \Gamma_2(u)\).
Proof. Since $A_u$ is transitive on $\Gamma_2(u)$ but not on $\Gamma(u)$, it follows from Lemma 3.2 that $\Gamma$ has girth 3. If for any $v, v' \in \Gamma(u)$ we have $\Gamma_2(u) \cap \Gamma(v) \cap \Gamma(v') = \emptyset$, then as $A_2$ is transitive on $\Gamma_2(u)$, $A_u$ is transitive on $\Gamma(u)$, which is a contradiction. Thus, there exist $v, v' \in \Gamma(u)$ such that $\Gamma_2(u) \cap \Gamma(v) \cap \Gamma(v') \neq \emptyset$. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4\}$. Suppose that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \neq \emptyset$, and say $w_1 \in \Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_3)$. Then $\Gamma'(u) \cap \Gamma(w_i) = 2, 3$ or 4. Since $A_u$ is transitive on $\Gamma_2(u)$, for any $w \in \Gamma_2(u)$, $\Gamma(u) \cap \Gamma(w) = 2, 3$ or 4. In particular, $\Gamma'(u) \cap \Gamma(w)$ divides the number of edges between $\Gamma(u)$ and $\Gamma_2(u)$.

Assume that $\Gamma'(u) \cap \Gamma(w) = 4$. If $u$ lies in a unique triangle $(u, v_1, v_2)$, then there are 10 edges between $\Gamma(u)$ and $\Gamma_2(u)$, however 4 does not divide 10, which is a contradiction. Assume that $u$ is in two triangles. Then there are 8 edges between $\Gamma(u)$ and $\Gamma_1(u)$. Hence $\Gamma_2(u) = 2$ and $\Gamma$ has 7 vertices. Let $\Delta = \{u\} \cup \Gamma_2(u)$. Then any two vertices of $\Delta$ are non-adjacent, and every vertex of $\Delta$ is adjacent to all vertices which are not in $\Delta$. Since $\Gamma$ is vertex-transitive, $\Delta$ is a block of cardinality 3. However, 3 does not divide 7, so such a $\Delta$ does not exist. Assume that $u$ lies in more than two triangles. Then there are $x \leq 6$ edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since 4 divides $x$, $x = 4$, so $|\Gamma_2(u)| = 1$, $\Gamma$ has 6 vertices. Let $\Delta = \{u\} \cup \Gamma_2(u)$. Then any two vertices of $\Delta$ are non-adjacent, and every vertex of $\Delta$ is adjacent to all vertices which are not in $\Delta$. Since $\Gamma$ is vertex-transitive, $\Delta$ is a block of cardinality 2. In particular, $\Gamma(u)$ contains two such blocks $\Delta'$ and $\Delta''$, and $[\Delta' \cup \Delta''] \cong C_4$. Thus $\Gamma \cong K_{3,3}$. However $A_u$ is transitive on $\Gamma(u)$, contradicting that $\Gamma$ is not arc-transitive. Hence $\Gamma'(u) \cap \Gamma(w) = 4$.\] 

Let $\Gamma_1, \Gamma_2$ be two graphs. We use $\Gamma_1 \square \Gamma_2$ to denote the Cartesian product of $\Gamma_1$ and $\Gamma_2$, its vertex set is $V(\Gamma_1) \times V(\Gamma_2)$, two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if $u_2 = v_2$ and $u_1, v_1$ are adjacent in $\Gamma_1$ or $u_1 = v_1$ and $u_2, v_2$ are adjacent in $\Gamma_2$.

Now we can prove the second theorem.

**Proof of Theorem 1.2.** Let $\Gamma$ be a 2-geodesic-transitive neighbor tetravalent graph. Let $(u, v)$ be an arc and $A = \text{Aut}(\Gamma)$. Since $\Sigma := [\Gamma(u)]$ is not an empty graph, $\Gamma$ has girth 3. Since $\Gamma$ is 2-geodesic-transitive, it follows from Theorem 1.1 (1) of [4] that $\Sigma$ has diameter 2 and $A_{uv}$ is transitive on $\Sigma_2(v)$.

Suppose first that $\Sigma$ is arc-transitive. Then $\Sigma$ is distance-transitive, and it is listed in [3, p.221-223]. By inspecting the candidates, $\Sigma$ is either $K_{3,3}$ or $H(2,3)$. If $\Sigma$ is $K_{3,3}$, then $\Gamma$ is $K_{4,2}$. If $\Sigma$ is $H(2,3)$, then $\Gamma$ is $I(6,3)$.

In the remaining, we suppose that $\Sigma$ is not arc-transitive. Since $A_{uv}$ is transitive on $\Sigma_2(v)$, it follows from Lemma 3.2 that $\Sigma$ has girth 3. If for any $v', v'' \in \Sigma(v)$ we have $\Sigma_2(v) \cap \Sigma(v') \cap \Sigma(v'') = \emptyset$, then as $A_{uv}$ is transitive on $\Sigma_2(v)$, $A_{uv}$ is transitive on $\Sigma(v)$, which is a contradiction. Thus, there exist $v', v'' \in \Sigma(v)$ such that $\Sigma_2(v) \cap \Sigma(v') \cap \Sigma(v'') \neq \emptyset$. Set $\Sigma(v) = \{v_1, v_2, v_3, v_4\}$. Suppose that $\Sigma_2(v) \cap \Sigma(v_1) \cap \Sigma(v_3) \neq \emptyset$, and say $w_1 \in \Sigma_2(v) \cap \Sigma(v_1) \cap \Sigma(v_3)$. Then $\Sigma_2(v) \cap \Sigma(v_1) = 2, 3$ or 4. Since $A_{uv}$ is transitive on $\Sigma_2(v)$, for any $\delta \in \Sigma_2(v)$, $|\Sigma_2(v) \cap \Sigma(v) = 2, 3$ or 4. From Lemma 3.3 that $|\Sigma(v) \cap \Sigma(v) = 2, 3$ or 4. In particular, $|\Sigma(v) \cap \Sigma(v) = 2, 3$ or 4 divides the number of edges between $\Sigma_2(v)$ and $\Sigma_2(v)$.

Assume that $|\Sigma_2(v) \cap \Sigma(v) = 3$. If $v$ lies in one or two triangles, then there are 10 or 8 edges between $\Sigma(v)$ and $\Sigma_2(v)$, respectively. However 3 does not divide 8 or 10, which is a contradiction. Hence $v$ lies in more than two triangles. Then there are $x \leq 6$ edges between $\Sigma(v)$ and $\Sigma_2(v)$. Since $x$ divides $x$, $x = 3$ or 6, so $|\Sigma_2(v)| = 1$ or 2. Assume $|\Sigma_2(v)| = 1$, say $\Sigma_2(v) = \{w\}$. Then $\Sigma_3(v) \cap \Sigma(w) = 1$, contradicting the fact that $\Sigma$ has diameter 2. Hence $|\Sigma_2(v)| = 2$, say $\Sigma_2(v) = \{w, w'\}$. Since $\Sigma$ has diameter 2, $|\Sigma_2(v) \cap \Sigma(w) = 1$. Thus $\Sigma$ is a vertex-transitive graph of valency 2 with 7 vertices, so $\Sigma \cong C_7$. Thus $\Sigma \cong C_7$. By Lemma 2.2, $\Gamma$ does not exist.

Now assume that $|\Sigma(v) \cap \Sigma(v) = 2$. Since $\Sigma$ has diameter 2, it follows that $|\Sigma_2(v) \cap \Sigma(v) = 2$. Thus $|\Sigma_2(v)|$ is a vertex-transitive graph of valency 2. If $v$ lies in $r$ triangles for some $r \geq 1$, then there are $12 - 2r$ edges between $\Sigma(v)$ and $\Sigma_2(v)$. Since $A_{uv}$ is transitive on $\Sigma_2(v)$, $|\Sigma_2(v) \cap \Sigma(v) = 2$ divides $12 - 2r$. It follows that $r \leq 5$. Since $|\Sigma_2(v) \cap \Sigma(v) = 2$, it follows that $|\Sigma_2(v) = 3$, and so there are at least 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$. Hence $12 - 2r \geq 6$, so $r = 1, 2$ or 3.

If $r = 1$, then there are 10 edges between $\Sigma(v)$ and $\Sigma_2(v)$. Since $|\Sigma(v) \cap \Sigma(v) = 2$ for any $\delta \in \Sigma_2(v)$, one has $|\Sigma_2(v)| = 5$. Assume that $(v, v_1, v_2, v_3)$ is a triangle. Then $v_3$ is not adjacent to $v_1$. So, $A_{uv}$ fixes $\{v_1, v_2\}$ and $\{v_3, v_4\}$ setwise, respectively. Therefore, $A_{uv}$ fixes $\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2))$ setwise. As $|\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2)) \leq 4$, it follows that $|\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2)) \leq 4$. Thus $A_{uv}$ fixes $\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2))$ setwise, contradicting the fact that $A_{uv}$ is transitive on $\Sigma_2(v)$.\]
If \( r = 2 \), then there are 8 edges between \( \Sigma(v) \) and \( \Sigma_2(v) \). Further, \( |\Sigma_2(v)| = 4 \), so \( |\Sigma_2(v)| \cong C_4 \). Set \( \Sigma(v) = \{v_1, v_2, v_3, v_4\} \). Then \( |\Sigma(v) \cap \Sigma_2(v)| = 1 \) or 2. If \( |\Sigma(v) \cap \Sigma_2(v)| = 1 \) for each \( v_i \), then \( |\Sigma(v)| \cong 2K_2 \). Hence each arc lies in a unique triangle. Let \( (v_1, v_2) \) and \( (v_1, v_4) \) be two arcs. Let \( \Sigma_2(v) \cap \Sigma(v_1) = \{v_1, v_2\} \). Since \( \Sigma(v_1) \cong 2K_2 \), \( (v_1, v_2) \) is an arc and \( v_2 \) is not adjacent to any one of \( v_1, v_3 \). Set \( \Sigma_2(v) \cap \Sigma(v_2) = \{v_3, v_4\} \). Since \( |\Sigma_2(v)| = 2K_2 \), \( (v_3, v_4) \) is an arc. Since \( |\Sigma(v)| \cong C_4 \), it follows that \( (v_1, v_2, v_3, v_4) \) is a 4-cycle. Since \( |\Sigma(v) \cap \Sigma(w)| = 2 \) and each arc lies in a unique triangle, \( w_1 \) is adjacent to one of \( v_1, v_4 \), say \( v_1 \). Then \( \Sigma(v_1) = \{v_1, v_2, v_3, v_4\} \). Since \( |\Sigma(v_1)| \cong 2K_2 \), it follows that \( v_3, v_4 \) are adjacent. Hence \( v_1 \) is adjacent to both \( v_2 \) and \( v_3 \). Thus \( \Sigma \) is isomorphic to the Hamming graph \( H(2, 3) \). However \( H(2, 3) \) is arc-transitive, which is a contradiction. Thus there exists \( v_i \) such that \( |\Sigma(v) \cap \Sigma(v_i)| = 2 \). Without loss of generality, let \( v_1 = v_i \) and let \( \Sigma(v) \cap \Sigma(v_1) = \{v_2, v_3\} \). Then \( |\Sigma_2(v) \cap \Sigma(v_1)| = 1 \), and say \( \Sigma_2(v) \cap \Sigma(v_1) = \{w_1\} \). Since \( w_i \) lies in 2 triangles, \( v_i \) is the unique vertex of \( \Sigma(v) \) such that \( |\Sigma(v) \cap \Sigma(v_i)| = 2 \). Thus \( A_{uv} \) can not map \( v_i \) to other vertices of \( \Sigma_2(v) \), contradicting the fact that \( A_{uv} \) is transitive on \( \Sigma_2(v) \). Hence \( r \neq 2 \).

Finally, assume \( r = 3 \). Then there are 6 edges between \( \Sigma(v) \) and \( \Sigma_2(v) \). Further, \( |\Sigma_2(v)| = 3 \), so \( |\Sigma_2(v)| \cong C_3 \).

Set \( \Sigma(v) = \{v_1, v_2, v_3, v_4\} \). Then for any \( v_i \), \( |\Sigma(v) \cap \Sigma(v_i)| \leq 3 \). Since \( v_i \) is in 3 triangles, there exist at most one vertex \( v_i \) such that \( |\Sigma(v) \cap \Sigma(v_i)| = 1 \). Assume there exists a vertex, \( v_i \), such that \( |\Sigma(v) \cap \Sigma(v_i)| = 3 \). Then \( \Sigma(v_i) = \{v_1, v_2, v_3, v_4\} \) and vertices of \( \Sigma(v_1) \cup \Sigma(v_2) \cup \Sigma(v_3) \cup \Sigma(v_4) \) are pairwise non-adjacent. Hence \( \Sigma_2(v) = \{v_1, v_2, v_3, v_4\} \). Since there are no edges between sets \( \{v_1, v_2\} \) and \( \{v_3, v_4\} \), and there are \( 3 \) edges between \( \Sigma(v) \cap \Sigma(v_i) \), there exists \( v_i \) such that \( |\Sigma(v) \cap \Sigma(v_i)| = 1 \). Assume that there are \( x \) vertices in \( \Sigma(v) \) that are adjacent to exactly one vertex of \( \Sigma_2(v) \). Then counting the edges between \( \Sigma(v) \) and \( \Sigma_2(v) \), \( x + 2(4 - x) = 6 \), so \( x = 2 \). Suppose \( |\Sigma_2(v) \cap \Sigma(v_i)| = |\Sigma(v) \cap \Sigma_2(v)| = 1 \), say \( \Sigma(v) \cap \Sigma(v_1) = \{w_1\} \) and \( \Sigma(v) \cap \Sigma(v_2) = \{w_2\} \). Then \( A_{uv} \) can not map \( w_1 \) to any one of \( w_1, w_2 \), contradicting the fact that \( A_{uv} \) is transitive on \( \Sigma_2(v) \). Thus there exists a vertex \( v_i \) such that \( |\Sigma(v) \cap \Sigma(v_i)| = 0 \). Since \( v_i \) is in 3 triangles, \( |\Sigma_2(v) \cap \Sigma(v_i)| \cong C_3 \). Further \( \Sigma_2(v) \cap \Sigma(v_i) = \Sigma_2(v) \), and there are 3 edges between \( \Sigma(v) \setminus \{v_i\} \) and \( \Sigma_2(v) \). Hence for each \( v_j \in \Sigma(v) \setminus \{v_i\} \), \( \Sigma_2(v) \cap \Sigma(v_j) \) is 1. Therefore \( \Sigma \cong K_4 \square K_2 \). Then by [3, Theorem 9.1.3], \( \Gamma \) is \( J(6, 2) \).