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Regular Fuzzy Equivalences on Two - Mode Fuzzy Networks☆

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Abstract. The notion of social roles is a centerpiece of most sociological theoretical considerations. Regular equivalences were introduced by White and Reitz in [15] as the least restrictive among the most commonly used definitions of equivalence in social network analysis. In this paper we consider a generalization of this notion to a bipartite case. We define a pair of regular equivalences on a two-mode social network and we provide an algorithm for computing the greatest pair of regular equivalences.

1. Introduction

One of the main problems of the social network analysis is to find similarities between actors which indicate that they have the same role or position in a network. These similarities were formalized first by Lorrain and White [27], Breiger et al. [8] and Burt [9] by the concept of a structural equivalence. Two actors are considered to be structurally equivalent if they have identical links to the rest of the network. Structural equivalences are extensively studied in [1, 2, 4, 16–19, 21, 22]. In order to generalize the concept of structural equivalence, White and Reitz [32] introduced the notion of a regular equivalence. Two actors are said to be regularly equivalent if they are equally related to equivalent others [5, 20]. Afterwards, regular equivalences have been studied in numerous of papers (cf. [23, 24]).

The regular equivalence approach is important because it provides a method for identifying ”roles” from the patterns of ties present in a network. Rather than relying on attributes of actors to define social roles and to understand how social roles give rise to patterns of interaction, regular equivalence analysis seeks to identify social roles by identifying regularities in the patterns of network ties – whether or not the occupants of the roles have names for their positions. The regular equivalences enable the clustering of the set of actors only with respect to their relationship to each other. The aim of this paper is to introduce the generalization of the notion of regular equivalence which provides the clustering based on the actors relationship to some other group of actors (e.g. the group of students can be clustered by their interest in attending the certain group of exams).

We consider a two-mode network – an ordered triple (A, B, R), where A and B are non-empty sets and R is a relation between A and B, and we define the pair of regular equivalences (E, F), as the pair of equivalences (E, F), on A and B respectively, which satisfies $E \circ R = R \circ F$. Similar kind of relational equalities were extensively studied by Ćirić, Ignjatović et all in [10–15, 24–26], where the greatest solutions of the certain

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equalities were given. Based on general ideas of this study and of the well known Paige-Tarijan three partition refinement procedure [28], we provide an efficient procedure for computing the greatest pair of structural equivalences.

The paper is organized as follows. In Section 2 we recall some basic properties of relations in general, and of equivalence relations. In particular, we define the right and left residuals. In Section 3 we define pairs of regular equivalences on a two-mode network, and we examine their main properties. Section 4 contains our main results on the computation of the greatest pair of regular equivalences on a network. Specifically, we provide an algorithm for computing the greatest pair of regular equivalences on a network and we give an illustrative computational example.

2. Preliminaries

We will use complete residuated lattices as the structures of membership (truth) values.

A *residuated lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,

(L3) $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \rightarrow z. \tag{1}$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathcal{L}$ is called a *complete residuated lattice*.

In the further text $\mathcal{L}$ will be a complete residuated lattice. A *fuzzy subset* of a set A *over* $\mathcal{L}$, or simply a *fuzzy subset* of A, is any function from A into L. The *equality* of f and g is defined as the usual equality of functions, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. Endowed with this partial order the set $\mathcal{F}(A)$ of all fuzzy subsets of A forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are functions from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x), \qquad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x),$$

and the *product* $f \otimes g$ is a fuzzy subset defined by $(f \otimes g)(x) = f(x) \otimes g(x)$, for every $x \in A$.

Let A and B be non-empty sets. A *fuzzy relation between sets* A and B is any function from $A \times B$ into L, and the equality, inclusion (ordering), joins and meets of fuzzy relations are defined as for fuzzy sets. In particular, a *fuzzy relation on a set* A is any function from $A \times A$ into L. The set of all fuzzy relations from A to B will be denoted by $\mathcal{R}(A, B)$, and the set of all fuzzy relations on a set A will be denoted by $\mathcal{R}(A)$. The *converse* (in some sources called *inverse* or *transpose*) of a fuzzy relation $R \in \mathcal{R}(A, B)$ is a fuzzy relation $R^{-1} \in \mathcal{R}(B, A)$ defined by $R^{-1}(b, a) = R(a, b)$, for all $a \in A$ and $b \in B$.

A fuzzy relation R on A is said to be:

(R) *reflexive* (or *fuzzy reflexive*) if $R(a, a) = 1$, for every $a \in A$;

(S) *symmetric* (or *fuzzy symmetric*) if $R(a, b) = R(b, a)$, for all $a, b \in A$;

(T) *transitive* (or *fuzzy transitive*) if $R(a, b) \otimes R(b, c) \leq R(a, c)$, for all $a, b, c \in A$.

It can easily be shown, that $R \circ R = R$ holds for any reflexive and transitive relation R on A.

A reflexive and transitive fuzzy relation on A is called a *fuzzy quasi-order*. A reflexive, symmetric and transitive fuzzy relation on A is called a *fuzzy equivalence*. With the respect to the inclusion of fuzzy relations, the set $\mathcal{E}(A)$ of all fuzzy equivalences on A is a complete lattice.

For non-empty sets A and B and fuzzy subsets $\eta \in \mathcal{F}(A)$ and $\xi \in \mathcal{F}(B)$, fuzzy relations $\eta \to \xi \in \mathcal{R}(A,B)$ and $\eta \leftarrow \xi \in \mathcal{R}(A,B)$ are defined as follows

$$(\eta \to \xi)(a,b) = (\,\eta(a) \;\to\; \xi(b)\,), \tag{2}$$
$$(\eta \leftarrow \xi)(a,b) = (\,\xi(b) \;\to\; \eta(a)\,), \tag{3}$$

for arbitrary $a \in A$ and $b \in B$. Let us note that $\eta \leftarrow \xi = (\xi \to \eta)^{-1}$.

We have the following.

Lemma 2.1. *Let A and B be non-empty sets and let $\eta \in \mathcal{F}(A)$ and $\xi \in \mathcal{F}(B)$.*

(a) *The set of all solutions to the inequality $\eta \circ \chi \leq \xi$, where χ is an unknown fuzzy relation between A and B, is the principal ideal of $\mathcal{R}(A,B)$ generated by the fuzzy relation $\eta \to \xi$.*

(b) *The set of all solutions to the inequality $\chi \circ \xi \leq \eta$, where χ is an unknown fuzzy relation between A and B, is the principal ideal of $\mathcal{R}(A,B)$ generated by the fuzzy relation $\eta \leftarrow \xi$.*

Note that $(\eta \to \xi) \wedge (\eta \leftarrow \xi) = \eta \leftrightarrow \xi$, where $\eta \leftrightarrow \xi$ is a fuzzy relation between A and B defined by

$$(\eta \leftrightarrow \xi)(a,b) = (\,\eta(a) \;\leftrightarrow\; \xi(b)\,), \tag{4}$$

for arbitrary $a \in A$ and $b \in B$.

Next, let A and B be non-empty sets and let $\alpha \in \mathcal{R}(A)$, $\beta \in \mathcal{R}(B)$ and $\gamma \in \mathcal{R}(A,B)$. The *right residual* of γ by α is a fuzzy relation $\alpha\backslash\gamma \in \mathcal{R}(A,B)$ defined by

$$(\alpha\backslash\gamma)(a,b) = \bigwedge_{a'\in A} (\,\alpha(a',a) \to \gamma(a',b)\,), \tag{5}$$

for all $a \in A$ and $b \in B$, and the *left residual* of γ by β is a fuzzy relation $\gamma/\beta \in \mathcal{R}(A,B)$ defined by

$$(\gamma/\beta)(a,b) = \bigwedge_{b'\in B} (\,\beta(b,b') \to \gamma(a,b')\,), \tag{6}$$

for all $a \in A$ and $b \in B$. We think of the right residual $\alpha\backslash\gamma$ as what remains of γ on the right after "dividing" γ on the left by α, and of the left residual γ/β as what remains of γ on the left after "dividing" γ on the right by β. In other words,

$$\alpha \circ \gamma' \leqslant \gamma \iff \gamma' \leqslant \alpha\backslash\gamma, \qquad \gamma' \circ \beta \leqslant \gamma \iff \gamma' \leqslant \gamma/\beta, \tag{7}$$

for all $\alpha \in \mathcal{R}(A)$, $\beta \in \mathcal{R}(B)$ and $\gamma', \gamma \in \mathcal{R}(A,B)$. In the case when $A = B$, these two concepts become the well-known concepts of right and left residuals of fuzzy relations on a set (cf. [24]). In that case, for fuzzy relations $\delta, \gamma \in \mathcal{R}(A)$ we consider also the relation $\delta|\gamma \in \mathcal{R}(A)$ as:

$$\delta|\gamma = \delta/\gamma \wedge \gamma\backslash\delta.$$

We also have the following.

Lemma 2.2. *Let A and B be non-empty sets and let $\alpha \in \mathcal{R}(A)$, $\beta \in \mathcal{R}(B)$ and $\gamma \in \mathcal{R}(A,B)$.*

(a) *The set of all solutions to the inequality $\alpha \circ \chi \leq \gamma$, where χ is an unknown fuzzy relation between A and B, is the principal ideal of $\mathcal{R}(A,B)$ generated by the right residual $\alpha\backslash\gamma$ of of γ by α.*

(b) *The set of all solutions to the inequality $\chi \circ \beta \leq \gamma$, where χ is an unknown fuzzy relation between A and B, is the principal ideal of $\mathcal{R}(A,B)$ generated by the left residual γ/β of of γ by β.*

Proof. These are also results by E. Sanchez (cf. [29–31]). □

In the sequel we recall some well known results concerning fuzzy equivalence relations, which will be needed in the further work:

Lemma 2.3. *Let $E, F \in \mathcal{E}(A)$ be fuzzy equivalences on A. Then, relation $E \wedge F$ is also a fuzzy equivalence.*

Lemma 2.4. *Let $E, F \in \mathcal{E}(A)$ be fuzzy equivalences on A such that $E \leq F$, then $E \circ F = F$.*

Lemma 2.5. *Let $f \in \mathcal{R}(A)$ be fuzzy relation on A. Then, relation $f \mid f$ is a fuzzy equivalence.*

Lemma 2.6. *Let A and B be non-empty sets and let $R \in \mathcal{R}(A)$ and $G \in \mathcal{R}(A, B)$*

The set of all solutions to the inequality $R \circ G \leq G$, where R is an unknown fuzzy relation on A, is the principal ideal of $\mathcal{R}(A)$ generated by the right residual G/G. The set of all fuzzy equivalences which are solutions to the inequality , where is the principal ideal of $\mathcal{E}(A)$ generated by the right residual $G \mid G$.

The set of all solutions to the inequality $G \circ R \leq G$, where R is an unknown fuzzy relation on A, is the principal ideal of $\mathcal{R}(A)$ generated by the right residual $G \setminus G$. The set of all fuzzy equivalences which are solutions to the inequality , where is the principal ideal of $\mathcal{E}(A)$ generated by the residual $G \mid G$.

Lemma 2.7. *Let $E, F \in \mathcal{E}(A)$ such that $E \leq F$. Then, $E \circ F \leq F$.*

As the consequence of Lemma 2.6 and 2.7, we have the following result:

Lemma 2.8. *Let $E, F \in \mathcal{E}(A)$ such that $E \leq F$. Then, $E \leq F \mid F$.*

3. Regular fuzzy equivalence

Fuzzy social network is an ordered pair (A, R), where A is a non-empty set of the actors or the nodes of the network and $R \in \mathcal{R}(A)$ is a fuzzy relation on A, which present relationship among the actors.Since the social network, in general, have large number of actors, for understanding the structure of the network it is convenient to observe the equivalence classes on the set of actors.

Let (A, R) be a fuzzy social network. A fuzzy equivalence relation $E \in \mathcal{R}(A)$, is a regular equivalence on the network if the following holds:

$$E \circ R = R \circ E.$$

Two mode fuzzy social network is an ordered triple (A, B, R), where A and B are non-empty sets of the actors or the nodes of the network and $R \in \mathcal{R}(A, B)$ is a fuzzy relation between A and B, which present relationship among these two groups of actors.

Let (A, B, R) be a two mode fuzzy network. A pair of fuzzy equivalences (E, F), where $E \in \mathcal{R}(A)$ and $F \in \mathcal{R}(B)$, is a pair of regular equivalence on (A, B, R) if it satisfies:

$$E \circ R = R \circ F. \tag{8}$$

This kind of equivalence perform even better than classical regular fuzzy equivalence in presenting the structure of the network.

Theorem 3.1. *A pair of fuzzy equivalence relations (E, F) is a pair of regular equivalences if and only if the following holds:*

$$E \circ R \circ F = E \circ R \wedge R \circ F. \tag{9}$$

Proof. Let (E, F) be a pair of structural equivalences, then (E, F) satisfies (8). Therefore,

$$E \circ R \circ F = R \circ F \circ F = R \circ F,$$

and similarly $E \circ R \circ F = E \circ R$. Thus, (9) holds.

On the other hand, let (9) holds. Hence, $E \circ R \circ F \leq R \circ F$ and $E \circ R \circ F \leq E \circ R$ holds. Directly from the fact E and F are reflexive we obtain $R \circ F \leq E \circ R \circ F$ and $E \circ R \leq E \circ R \circ F$

So, $R \circ F = E \circ R \circ F$ and $E \circ R = E \circ R \circ F$ holds, which means (8) holds. □

The following theorem provides a method for computing the pair of greatest regular equivalences on the given two mode network.

Theorem 3.2. *Let (A, B, R) be a network and let $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$ be equivalences on A and B respectively. Define the sequences $\{(E_k, F_k)\}_{k\in N}$ and $\{(X_k, Y_k)\}_{k\in N}$ as follows: Initially for $k = 1$*

$$(X_1, Y_1) = (U_A, U_B), \qquad (E_1, F_1) = (E, F) \wedge \Big((R \circ U_B^b) \,|\, (R \circ U_B^b), (U_A^a \circ R) \,|\, (U_A^a \circ R)\Big)$$

where $a \in A$ and $b \in B$ are arbitrary elements.

Further, for each $k \in N$ repeat the following step: Find $a \in A$ and $b \in B$ such that $(X_k^a, Y_k^b) \neq (E_k^a, F_k^b)$ and set

$$(X_{k+1}, Y_{k+1}) = (X_k, Y_k) \wedge (E_k^a \,|\, E_k^a, F_k^b \,|\, F_k^b),$$

$$(E_{k+1}, F_{k+1}) = (E_k, F_k) \bigwedge \Big((R \circ Y_{k+1} | R \circ Y_{k+1}\,), \,(X_{k+1} \circ R \,|\, X_{k+1} \circ R)\Big),$$

until $(X_k, Y_k) = (E_k, F_k)$. Then:

(a) Sequences $\{(E_k, F_k)\}_{k\in N}$ and $\{(X_k, Y_k)\}_{k\in N}$ are descending;

(b) For every $k \in N$, $E_k \leq X_k$ and $F_k \leq Y_k$;

(c) For every $k \in N$, the following holds :

$$E_k \leq (R \circ Y_k) \,|\, (R \circ Y_k), \quad F_k \leq (X_k \circ R) \,|\, (X_k \circ R); \tag{10}$$

(d) If there exists $n \in N$ such that $(X_n, Y_n) = (E_n, F_n)$ then (E_n, F_n) is the greatest pair of regular equivalences contained in (E, F);

(e) If $\mathcal{A}$ is finite and $\mathcal{L}(\mathcal{A}, R)$ satisfies DCC, then there exists $n \in N$ such that $(X_n, Y_n) = (E_n, F_n)$.

Proof. (a) Follows directly from the definition of these sequences;

(b) We will show only $E_k \leq X_k$, $k \in N$, the other inequality can be showed in analogue way.

We prove it by induction on $k \in N$.

For $k = 1$, evidently $E_1 \leq X_1$.

Suppose for $k = m$, $E_m \leq X_m$, and prove $E_{m+1} \leq X_{m+1}$.

According to Lemma 2.7 we have $E_m \leq (E_m^a \,|\, E_m^a)$, and by induction assumption $E_m \leq R_m$, therefore $E_m \leq X_m \wedge (E_m^a \,|\, E_m^a) = X_{m+1}$, and since $\{E_k\}_{k\in N}$ is descending, we have that $E_{m+1} \leq X_{m+1}$, which was to be proved.

(c)We will prove only the first inequality, the secund one can be proved in analogue way. We will consider only the case $k = 1$. For $k > 1$ it is evident from the definition of (E_{k+1}, F_{k+1}). Let us first note that all Y_1^c of Y_1 are equal to each other, that is, for any $c \in B$, P_1^c is defined by $Y_1^c(b) = 1$, for every $b \in A$. According to this fact and the definition of E_1 we have :

$$E_1 \leq (R \circ Y_1^c \,|\, R \circ Y_1^c),$$

for every $c \in A$, and hence for $k = 1$ (c) holds.

(d) If $(E_k, F_k) = (X_k, Y_k)$, for some $k \in N$, then according to (c) the following holds:

$$(E_k, F_k) \leq ((R \circ F_k) \,|\, (R \circ F_k), (E_k \circ R) \,|\, (E_k \circ R)).$$

which means that (E_k, F_k) is a pair of regular equivalences. In order to show that (E_k, F_k) is the greatest pair, let us consider an arbitrary pair of regular fuzzy equivalences (E', F') contained in (E, F) on the network.

We will prove that $(E', F') \leq (E_n, F_n)$ for every $n \in N$, by induction on n.

For $n = 1$, (E', F') and (U_A, U_b) are fuzzy equivalences such that $(E', F') \leq (U_A, U_B) = (R_1, P_1)$. According to Theorem 3.1 fuzzy equivalence E' satisfies inequality $E' \circ R \circ F' \leq R \circ F'$. Since $F' \leq Y_1$, multiplying this inequality with Y_1 on the right, we obtain:

$$E' \circ R \circ F' \circ Y_1 \leq R \circ F' \circ Y_1$$

Next according to Lemma 2.4, we have

$$E' \circ R \circ Y_1 \leq R \circ Y_1,$$

and by Lemma 2.6 we have that $E' \leq (R \circ Y_1) \mid (R \circ Y_1)$, and since $E' \leq E$ we conclude $E' \leq E_1$. In the similar way we show that $F' \leq F_1$.

Suppose that assumption $(E', F') \leq (E_m, F_m)$ holds for $n = m$, and prove $(E', F') \leq (E_{m+1}, F_{m+1})$.

Since $E' \leq E_m$, using (b), we obtain $E' \leq R_m$ and by Lemma 2.7 it follows $E' \leq X_{m+1}$, and similarly $F' \leq Y_{m+1}$. Now, if we again use inequality $E' \circ R \circ F' \leq R \circ F'$ and fact $F' \leq Y_{m+1}$ we obtain $E' \leq X_{m+1}$, and similarly $F' \leq F_{m+1}$ which was to be proved.

(e) Let $\mathcal{A}$ be a finite fuzzy transition system and let $\mathcal{L}(\delta, \tau, R)$ satisfy DCC. Then fuzzy relations $\{R_k\}_{k\in\mathbb{N}}$ can be considered as fuzzy matrices with entries in $\mathcal{L}(\delta, \tau, R)$, and for any pair $(a, b) \in A \times A$, the (a, b)-entries of these matrices form a decreasing sequence $\{R_k(a, b)\}_{k\in\mathbb{N}}$ of elements of $\mathcal{L}(\delta, \tau, R)$. By the hypothesis, all these sequences stabilize, and since there is a finite number of these sequences, there exists $s \in \mathbb{N}$ such that after s steps all these sequences are stabilized. This means that the sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences also stabilizes after s steps, i.e., $E_k = E_{k+1}$.

Next, we will prove that if $X_k = X_{k+1}$ then $E_k = X_k$. If $X_k = X_{k+1}$ then

$$X_k = X_{k+1} = X_k \wedge (E_k^a \mid E_k^a),$$

and thus, $X_k \leq E_k^a \mid E_k^a$. Consequently, $X_k^a \leq E_k^a$, and since $X_k^a \leq E_k^a$ we obtain $X_k^a = E_k^a$. This means that there is no class X_k^a of X_k such that $X_k^a \neq E_k^a$, or equivalently $X_k = E_k$. □

4. Regular fuzzy quasi-orders

Note that if we consider inequality (8) and require relations E and F be fuzzy quasi-orders, we will obtain even the greater solution.

Let (A, B, R) be a fuzzy network. A pair of fuzzy quasi orders (P, Q), where $P \in \mathcal{R}(A)$ and $Q \in \mathcal{R}(B)$, is a pair of regular fuzzy quasi-orders if it satisfies:

$$P \circ R = R \circ Q.$$

The following theorem can be proved in the similar way as Theorem 3.2, so we will omit the proof.

Theorem 4.1. *Let (A, B, R) be a network and let $P \in \mathcal{Q}(A)$ and $Q \in \mathcal{Q}(B)$ be fuzzy quasi-oreds on A and B respectively. Define the sequences $\{(P_k, Q_k)\}_{k\in N}$ and $\{(X_k, Y_k)\}_{k\in N}$ as follows: Initially for $k = 1$*

$$(X_1, Y_1) = (U_A, U_B), \qquad (P_1, Q_1) = (P, Q) \wedge \Big((R \circ U_B^b) \,/\, (R \circ U_B^b), (U_A^a \circ R) \setminus (U_A^a \circ R)\Big)$$

where $a \in A$ and $b \in B$ are arbitrary elements.

Further, for each $k \in N$ repeat the following step: Find $a \in A$ and $b \in B$ such that $(X_k^a, Y_k^b) \neq (P_k^a, Q_k^b)$ and set

$$(X_{k+1}, Y_{k+1}) = (X_k, Y_k) \wedge (P_k^a \,/\, P_k^a, Q_k^b \setminus Q_k^b),$$

$$(P_{k+1}, Q_{k+1}) = (P_k, Q_k) \bigwedge \Big((R \circ Y_{k+1} / R \circ Y_{k+1}), (X_{k+1} \circ R \setminus X_{k+1} \circ R)\Big),$$

until $(X_k, Y_k) = (P_k, Q_k)$. Then:

(a) Sequences $\{(P_k, Q_k)\}_{k\in N}$ and $\{(X_k, Y_k)\}_{k\in N}$ are descending;

(b) For every $k \in N$, $P_k \leq X_k$ and $Q_k \leq Y_k$;

(c) For every $k \in N$, the following holds :

$$P_k \leq (R \circ Y_k) \,/\, (R \circ Y_k), \quad Q_k \leq (X_k \circ R) \setminus (X_k \circ R); \tag{11}$$

(d) If there exists $n \in N$ such that $(X_n, Y_n) = (P_n, Q_n)$ then (P_n, Q_n) is the greatest pair of regular equivalences contained in (P, Q);

(e) If $\mathcal{A}$ is finite and $\mathcal{L}(\mathcal{A}, R)$ satisfies DCC, then there exists $n \in N$ such that $(X_n, Y_n) = (P_n, Q_n)$.

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