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Minimum Property of Condition Numbers for the Drazin Inverse and Singular Linear Equations

Haifeng Ma[a]

[a]*School of Mathematical Science, Harbin Normal University, Harbin 150025, P. R. China.*

Abstract. For a singular linear equation $Ax = b, x \in \mathcal{R}(A^D)$, a small perturbation matrix E and a vector δb are given to A and b, respectively. We then have the perturbed singular linear equation $(A+E)\widetilde{x} = b+\delta b$, $\widetilde{x} \in \mathcal{R}[(A+E)^D]$. This note is devoted to show the minimum property of the condition numbers on the Drazin inverse A^D and the Drazin-inverse solution $A^D b$.

1. Introduction

The theory and applications of the Drazin inverse has been a substantial growth over the past few decades [1–7], which is useful in various applications, for example, applications in singular linear systems, Markov chains and iterative methods were found in the literature [8–12].

In this note, let $\mathbb{C}^{n\times n}$ denote all n by n complex matrices, $\mathbb{C}_r^{n\times n}$ denote all n by n complex matrices with the rank r. The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^D \in \mathbb{C}^{n\times n}$ satisfying the relations

$$A^D A = AA^D, \quad A^D AA^D = A^D, \quad A^{l+1}A^D = A^l, \quad \text{for all } l \geq r,$$

where r is the smallest nonnegative integer satisfying $rank(A^{r+1}) = rank(A^r)$, which is called the *Drazin index* of A and is denoted by $\mathrm{ind}(A)$. If $r = 1$, then the Drazin inverse reduces to the group inverse. Clearly, $\mathrm{ind}(A) = 0$ if and only if A is nonsingular. The symbols $rank(A), A^*, \mathcal{R}(A)$ and $\mathcal{N}(A)$ will stand for the rank, conjugate transpose, range space and null space of A, respectively.

Various normwise relative condition numbers measure the sensitivity of Drazin inverse and the solution of singular linear systems are characterized. The sensitivity of condition number itself is investigated in [14–17].

In this note, for the singular linear system $Ax = b$, we consider the perturbation E of A and δb of b, respectively, and consider the perturbed system $(A+E)\widetilde{x} = b+\delta b$. It should be stressed that we assume that $\mathcal{R}(A^D) = \mathcal{R}[(A+E)^D]$, so that $b, b+\delta b$ and $\mathcal{R}[(A+E)^D]$ are all kept inside original $\mathcal{R}(A^D)$.

In [14], the sensitivity of condition number of the Drazin inverse and the Drazin inverse solution of the singular linear systems are investigated. In this note, we focus on the minimum property of condition number for the Drazin inverse A^D and the Drazin inverse solution $A^D b$ of the doubly perturbed singular linear equations.

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Email address: haifengma@aliyun.com (Haifeng Ma)

2. Preliminaries

For $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, the following condition

$$B = A + E, \quad E = AA^D EAA^D, \quad \text{and} \quad \| A^D E \| < 1$$

is called $(\mathcal{W})$ condition, which is induced by Wei and Wang in [18].

It is well known that for any complex matrix $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and $rank(A^k) = r$, there is nonsingular matrix P such that

$$A = P\begin{pmatrix} C & O \\ O & N \end{pmatrix}P^{-1},$$

where C is an invertible matrix with order r, N is nilpotent, that is, $N^k = O$.

Then the Drazin inverse of A could be given by

$$A^D = P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix}P^{-1}.$$

In [13, 14], the P-norm $\|\cdot\|_P$ is defined by Wei et al. with respect to 2-norm,

$$\| A \|_P = \| P^{-1}AP \|_2, \quad \| x \|_P = \| P^{-1}x \|_2,$$

and

$$\| A^D \|_P = \| P^{-1}A^D P \|_2 = \| C^{-1} \|_2 .$$

Lemma 2.1. **([18])** *If, in addition to the hypotheses of $(\mathcal{W})$, $\| A^D \| \| E \| < 1$, then*

$$\frac{\| (A+E)^D - A^D \|}{\| A^D \|} \le \frac{k_D(A)\frac{\|E\|}{\|A\|}}{1 - k_D(A)\frac{\|E\|}{\|A\|}},$$

where $k_D(A) = \| A \| \| A^D \|$ is defined as the condition number with respect to the Drazin inverse for any consistent norms.

In Lemma 2.1, $k_D(A)$ reflects the sensitivity to the perturbations of A. If $k_D(A)$ is larger, then the relative error will be larger.

Lemma 2.2. **([19])** *Let $A \in \mathbb{C}^{n\times n}$. Consider $Ax = b$, where $b \in \mathcal{R}(A^D)$. Then there exists a solution in $\mathcal{R}(A^D)$. Moreover the Drazin inverse solution is unique, which is given by $x = A^D b$.*

Lemma 2.3. **([19])** *Suppose condition $(\mathcal{W})$ holds. Then $\mathcal{R}(B^k) = \mathcal{R}(A^k)$, $\mathcal{N}(B^k) = \mathcal{N}(A^k)$, where $\mathrm{ind}(B) = k$.*

For a singular linear system $Ax = b$, Wei and Wang consider the perturbed system $(A + E)\widetilde{x} = b + \delta b$, and give the upper bound.

Lemma 2.4. **([18])** *Suppose condition $(\mathcal{W})$ holds and $\| A^D \| \| E \| < 1$, then*

$$\frac{\| \widetilde{x} - x \|}{\| x \|} \le \frac{k_D(A)}{1 - k_D(A)\frac{\|E\|}{\|A\|}}\left(\frac{\| E \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|}\right),$$

where $k_D(A) = \| A \| \| A^D \|$ is defined as the condition number with respect to the Drazin inverse.

From the above lemmas, we know that $k_D(A)$ reflects the sensitivity of the Drazin inverse solution $A^D b$ of $Ax = b$ to the perturbations of A and b. If $k_D(A)$ is large, then solving $Ax = b$ will be an ill-conditioned problem.

3. Main results

In this section, we explore the minimum property of condition number for the Drazin inverse and the Drazin inverse solution of doubly perturbed singular linear equations with respect to the P-norm.

Theorem 3.1. *Let $A \in \mathbb{C}_r^{n\times n}$, E be any perturbation of A, $B = A + E$. Suppose A and E satisfy condition ($\mathcal{W}$). If $\| E \|_P < 1/\|A^D\|_P$ such that*

$$\frac{\| (A+E)^D - A^D \|_P}{\| A^D \|_P} \leq \frac{\mu(A)\frac{\|E\|_P}{\|A\|_P}}{1-\mu(A)\frac{\|E\|_P}{\|A\|_P}},$$

then $k_P(A) = \| A \|_P \| A^D \|_P \leq \mu(A)$, where $\mu(A)$ is only dependant on A.

Proof. It follows from $AA^DE = EAA^D = E$ that

$$(A+E)^D = (I + A^DE)^{-1}A^D = [I - I + (I + A^DE)^{-1}]A^D = A^D - [I - (I + A^DE)^{-1}]A^D.$$

By $\| A^D \|_P \| E \|_P < 1$, we have $(I + A^DE)^{-1} = \sum\limits_{k=0}^{\infty}(-A^DE)^k$.

Hence

$$\begin{aligned}
\| (A+E)^D - A^D \|_P &= \| [I - (I + A^DE)^{-1}]A^D \|_P \\
&= \| [I - \sum_{k=0}^{\infty}(-A^DE)^k]A^D \|_P \\
&= \| [\sum_{k=0}^{\infty}(-1)^k(A^DE)^k]A^D \|_P \\
&= \| A^DEA^D - (A^DE)^2\sum_{k=0}^{\infty}(-A^DE)^kA^D \|_P \\
&\geq \| A^DEA^D \|_P - \| A^D \|_P^3 \| E \|_P^2 \sum_{k=0}^{\infty} \| A^DE \|_P^k,
\end{aligned}$$

that is

$$\| A^DEA^D \|_P \leq \| (A+E)^D - A^D \|_P + \| A^D \|_P^3 \| E \|_P^2 \sum_{k=0}^{\infty} \| A^DE \|_P^k .$$

As we know, the Jordan decomposition of A is $A = P\begin{pmatrix} C & O \\ O & N \end{pmatrix}P^{-1}$, where C is an invertible matrix of order r and N is nilpotent.

Hence the Drazin inverse of A is $A^D = P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix}P^{-1}$.

Take $E = \varepsilon \| A \|_P \widehat{\delta A}$, and

$$\widehat{\delta A} = P\begin{pmatrix} y \\ 0 \end{pmatrix}(x^* \ \ 0)\,P^{-1},$$

where ε is a small positive number and there exist vectors x and y such that

$$\| C^{-1}y \|_2 = \| x^*C^{-1} \|_2 = \| C^{-1} \|_2,$$

and $\| x \|_2 = \| y \|_2 = 1$.

Hence we obtain

$$AA^DE = EAA^D = E$$

and

$$\begin{aligned}\| A^D\widehat{\delta A}A^D \|_P &= \left\| P^{-1}P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix}P^{-1}P\begin{pmatrix} y \\ 0 \end{pmatrix}(x^* \;\; 0)\, P^{-1}P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix}P^{-1}P\right\|_2 \\ &= \|C^{-1}yx^*C^{-1}\|_2 \\ &= \|C^{-1}y\|_2\|x^*C^{-1}\|_2 \\ &= \|C^{-1}\|_2^2 \\ &= \| A^D \|_P^2 .\end{aligned}$$

We also have

$$\begin{aligned}\| E \|_P &= \varepsilon \| A \|_P \| \widehat{\delta A} \|_P \\ &= \varepsilon \| A \|_P \left\| P^{-1}P\begin{pmatrix} y \\ 0 \end{pmatrix}(x^* \;\; 0)\, P^{-1}P\right\|_2 \\ &= \varepsilon \| A \|_P \|yx^*\|_2 \\ &= \varepsilon \| A \|_P \|y\|_2\|x\|_2 \\ &= \varepsilon \| A \|_P,\end{aligned}$$

and $\| A^DEA^D \|_P = (\varepsilon \| A \|_P) \| A^D \|_P^2$.

Moreover, we have

$$\begin{aligned}k_P(A) &= \| A \|_P \| A^D \|_P \\ &= \frac{\| A^D \|_P^2 \| A \|_P \| E \|_P}{\| A^D \|_P \| E \|_P} \\ &= \frac{\| A^DEA^D \|_P \| A \|_P \| E \|_P}{\| A^D \|_P \| E \|_P \; \varepsilon \| A \|_P} \\ &\le \frac{\| A \|_P}{\| E \|_P}\left(\frac{\| (A+E)^D - A^D \|_P + \| A^D \|_P^3 \| E \|_P^2 \sum_{k=0}^{\infty} \| A^DE \|_P^k}{\| A^D \|_P}\right) \\ &= \frac{\mu(A)}{1-\mu(A)\frac{\|E\|_P}{\|A\|_P}} + \| A^D \|_P^2 \| E \|_P \sum_{k=0}^{\infty} \| A^DE \|_P^k \\ &= \frac{\mu(A)}{1-\mu(A)\frac{\|E\|_P}{\|A\|_P}} + \varepsilon \| A^D \|_P^2 \sum_{k=0}^{\infty} \| A^DE \|_P^k .\end{aligned}$$

In the above inequality, take $\varepsilon \to 0$, hence we obtain $k_P(A) \le \mu(A)$. It finishes the proof. □

Theorem 3.2. *Let $A \in \mathbb{C}_r^{n\times n}$. For any small pertubation E of A and δb of b such that $Ax = b$ and $(A+E)\widetilde{x} = b + \delta b$, respectively. If $\| E \|_P < 1/\|A^D\|_P$, such that*

$$\frac{\| x - \widetilde{x} \|_P}{\| x \|_P} \le \frac{\lambda(A)}{1-\lambda(A)\frac{\|E\|_P}{\|A\|_P}}\left(\frac{\| E \|_P}{\| A \|_P} + \frac{\| \delta b \|_P}{\| b \|_P}\right),$$

where $\lambda(A)$ is only related with A, then $k_P(A) = \| A \|_P \| A^D \|_P \le \lambda(A)$.

Proof. By the Jordan decomposition of A, $A = P\begin{pmatrix} C & O \\ O & N \end{pmatrix}P^{-1}$, where C is a nonsingular matrix of order r and N is nilpotent.

Hence the Drazin inverse of A is $A^D = P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix}P^{-1}$.

Taking E as

$$E = \varepsilon \frac{\| A \|_P}{\|x\|_P} yx^* P^{-*} P^{-1},$$

where $\varepsilon > 0$,

$$\| C^{-1} z \|_2 = \| C^{-1} \|_2 = \| A^D \|_P, \quad y = P \begin{pmatrix} z \\ 0 \end{pmatrix},$$

and $\| z \|_2 = \| y \|_P = 1$, then

$$\begin{aligned}
\| E \|_P &= \varepsilon \frac{\| A \|_P}{\|x\|_P} \| yx^* P^{-*} P^{-1} \|_P \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \| P^{-1} yx^* P^{-*} P^{-1} P \|_2 \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \| \begin{pmatrix} z \\ 0 \end{pmatrix} (P^{-1}x)^* \|_2 \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \| \begin{pmatrix} z \\ 0 \end{pmatrix} \|_2 \| (P^{-1}x) \|_2 \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \| z \|_2 \| x \|_P \\
&= \varepsilon \| A \|_P
\end{aligned}$$

and

$$\begin{aligned}
\| A^D E x \|_P &= \left\| P^{-1} P \begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} P^{-1} \varepsilon \frac{\| A \|_p}{\|x\|_P} y\, x^* P^{-*} P^{-1} x \right\|_2 \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \|C^{-1} z (P^{-1}x)^* (P^{-1}x)\|_2 \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \|C^{-1} z\|_2 \|P^{-1}x\|_2^2 \\
&= \varepsilon \frac{\| A \|_P}{\|x\|_P} \|C^{-1}\|_2 \|x\|_P^2 \\
&= \varepsilon \| A \|_P \| A^D \|_P \| x \|_P \\
&= \| A^D \|_P \|E\|_P \| x \|_P .
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
x - \widetilde{x} &= A^D b - (A + E)^D (b + \delta b) \\
&= [I - (I + A^D E)^{-1}] A^D b - (A + E)^D \delta b \\
&= -\sum_{k=1}^{\infty} (-A^D E)^k x - (A + E)^D \delta b \\
&= A^D E x - (A^D E)^2 \sum_{k=0}^{\infty} (-A^D E)^k x - (A + E)^D \delta b.
\end{aligned}$$

Taking P-norm in the above equation, we have

$$\begin{aligned}
\| x - \widetilde{x} \|_P &\geq \| A^D E x \|_P - (\| A^D \|_P)^2 (\| E \|_P)^2 \sum_{k=0}^{\infty} (\| A^D E \|_P)^k \| x \|_P \\
&- \| (A + E)^D \|_P \| \delta b \|_P,
\end{aligned}$$

that is

$$\begin{aligned}\| A^D Ex \|_P &\leq \| x - \widetilde{x} \|_P + (\| A^D \|_P)^2 (\| E \|_P)^2 \sum_{k=0}^{\infty} (\| A^D E \|_P)^k \| x \|_P \\ &+ \| (A+E)^D \|_P \| \delta b \|_P .\end{aligned}$$

Therefore, we get

$$\begin{aligned} k_P(A) &= \| A \|_P \| A^D \|_P = \frac{\| A \|_P \| A^D Ex \|_P}{\| E \|_P \| x \|_P} \\ &\leq \frac{\| A \|_P}{\| E \|_P \| x \|_P} (\| x - \widetilde{x} \|_P + \| A^D \|_P^2 \| E \|_P^2 \sum_{k=0}^{\infty} (\| A^D E \|_P)^k \| x \|_P \\ &+ \| (A+E)^D \|_P \| \delta b \|_P) \\ &\leq \frac{\| A \|_P}{\| E \|_P \| x \|_P} (\frac{\lambda(A) \frac{\|E\|_P}{\|A\|_P} \| x \|_P}{1 - \lambda(A) \frac{\|E\|_P}{\|A\|_P}} \\ &+ \| A^D \|_P^2 \| E \|_P^2 \sum_{k=0}^{\infty} \| A^D E \|_P^k \| x \|_P + \| (A+E)^D \|_P \| \delta b \|_P) \\ &\leq \frac{\lambda(A)}{1 - \lambda(A) \frac{\|E\|_P}{\|A\|_P}} + \varepsilon (\| A^D \|_P)^3 \sum_{k=0}^{\infty} (\| A^D E \|_P)^k \\ &+ \frac{\| A \|_P \| (A+E)^D \|_P \| \delta b \|_P}{\| E \|_P \| x \|_P} \end{aligned}$$

Take $\| \delta b \|_P = o(\varepsilon)$, let $\varepsilon \to 0$ in the above equation. Hence, we obtain $k_P(A) \leq \lambda(A)$. □

Corollary 3.1. **([20])** *Let $A \in \mathbb{C}_n^{n\times n}$ be a nonsingular matrix. For any small pertubation E and δb, let $x, \widetilde{x}$ satisfy $Ax = b$ and $(A+E)\widetilde{x} = b + \delta b$, respectively. If $\| E \| < 1/\|A^{-1}\|$, such that*

$$\frac{\| x - \widetilde{x} \|}{\| x \|} \leq \frac{\lambda(A)}{1 - \lambda(A) \frac{\|E\|}{\|A\|}} \left(\frac{\| E \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|} \right),$$

where $\lambda(A)$ is only relating to A, then $k(A) = \| A \| \| A^{-1} \| \leq \lambda(A)$.

Remark 3.1. Our results are more general than that of [21], which assumed that $\mathcal{R}(A^k) = \mathcal{R}(A^{k^*})$.

4. Concluding remarks

In this note, we characterize the condition number of Drazin inverse and the Drazin inverse solution of singular linear systems. It is of interest to extend our results to the W-weighted Drazin inverse of a rectangular matrix [22–29] and the bounded linear operator [30–32].

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