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Homoclinic Solutions for Fractional Hamiltonian Systems with Indefinite Conditions

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Abstract. In this paper we are concerned with the existence of infinitely many homoclinic solutions for the following fractional Hamiltonian systems

$$\begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) - L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n), \end{cases} \tag{FHS}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. The novelty of this paper is that, when $L(t)$ is allowed to be indefinite and $W(t,u)$ satisfies some new superquadratic conditions, we show that (FHS) possesses infinitely many homoclinic solutions via a variant fountain theorem. Recent results are generalized and significantly improved.

1. Introduction

The purpose of this paper is to investigate the existence of homoclinic solutions for the following fractional Hamiltonian systems

$$\begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) - L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n), \end{cases} \tag{FHS}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. As usual, we say that a solution $u(t)$ of (FHS) is homoclinic (to 0) if $u(t) \not\equiv 0$ and $u(t) \to 0$ as $|t| \to \infty$.

During the last two decades, the study of fractional calculus (differentiation and integration of arbitrary order) has emerged as an important and popular field of research. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics,

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chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc., [2, 15, 21, 25, 30, 44]. An important characteristic of a fractional-order differential operator that distinguishes it from an integer-order differential operator is its nonlocal behavior, that is, the future state of a dynamical system or process involving fractional derivatives depends on its current state as well its past states. In other words, differential equations of arbitrary order describe memory and hereditary properties of various materials and processes. This is one of the features that has contributed to the popularity of the subject and has motivated researchers to focus on fractional order models, which are more realistic and practical than the classical integer-order models.

Recently, equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives are interesting and new fields in fractional differential equations theory. In this topic, many results are obtained to deal with the existence and multiplicity of solutions by using techniques of nonlinear analysis, such as fixed point theory (including Leray-Schauder nonlinear alternative) [4], topological degree theory (including co-incidence degree theory) [19] and comparison method (including upper and lower solutions and monotone iterative method) [45] and so on.

It should be noted that critical point theory and variational methods have been tested to be effective in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions of differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [24], Rabinowitz [32], Schechter [36] and the references listed therein.

In addition, it should be mentioned that the fractional variational principles have been investigated considerably. The fractional calculus of variations was introduced by Riewe in [35] where he presented a new approach to mechanics that allows one to obtain the equations for a nonconservative systems using certain functionals. Kilmek [22] gave another approach by considering fractional derivatives, and corresponding Euler-Lagrange equations were obtained, using both Lagrangian and Hamiltonian formalisms. Agrawal [1] presented Euler-Lagrange equations for unconstrained and constrained fractional variational problems, and as a continuation of Agrawal's work, the generalized mechanics were considered to obtain the Hamiltonian formulation for the Lagrangian depending on fractional derivative of coordinates. It is worth mentioning that the fractional Hamiltonian is not uniquely defined and many researchers have explored this area giving new insight into this problem [5, 38]. The recent book [23] provides a broad description of the important subject of fractional calculus of variations.

In what follows, we give briefly description on the study related to the existence of homoclinic solutions of (FHS). Motivated by the above works, in paper [20], for the first time, Jiao and Zhou showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla W(t,u(t)), \quad a.e.\ t \in [0,T], \\ u(0) = u(T), \end{cases}$$

where $\alpha \in (1/2, 1)$, $u \in \mathbb{R}^n$, $W \in C^1([0,T]\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u and obtained the existence of at least one nontrivial solution. Inspired by this work, Torres [39] showed that (FHS) possesses at least one nontrivial homoclinic solution via Mountain Pass Theorem, assuming that $L(t)$ satisfies

(L) $L(t)$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R},(0,\infty))$ such that $l(t) \to \infty$ as $|t| \to \infty$ and

$$(L(t)u,u) \geq l(t)|u|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n, \tag{1}$$

and $W(t,u)$ is supposed to fulfil

(AR) $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and there is a constant $\theta > 2$ such that

$$0 < \theta W(t,u) \leq (\nabla W(t,u),u) \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n\backslash\{0\},$$

and some other reasonable hypotheses. As far as the fractional differential equation (FHS) is concerned, a strong motivation investigating it comes from fractional advection-dispersion equation (ADE for short). The fractional ADE is a generalization of the classical ADE in which the second-order derivative is replaced by a fractional-order derivative. In contrast to the classical ADE, the fractional ADE has solutions that resemble the highly skewed and heavy-tailed breakthrough curves observed in field and laboratory studies (see, [6–8]), in particular in contaminant transport of ground-water flow (see, [8]). Benson et al. stated that solutes moving through a highly heterogeneous aquifer violations violates the basic assumptions of local second-order theories because of large deviations from the stochastic process of Brownian motion.

In (FHS), if $\alpha = 1$, it reduces to the following second order Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t, u) = 0. \tag{HS}$$

It is well known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been recognized from Poincaré [29]. They may be "organizing centers" for the dynamics in their neighborhood. From their existence one may, under certain conditions, infer the existence of chaos nearby or the bifurcation behavior of periodic orbits. During the past two decades, with the works of [28] and [33] variational methods and critical point theory have been successfully applied for the search of the existence and multiplicity of homoclinic solutions of (HS). Assuming that $L(t)$ and $W(t, u)$ are independent of t or periodic in t, many authors have studied the existence of homoclinic solutions for (HS), see for instance [11, 13, 33] and the references therein and some more general Hamiltonian systems are considered in recent papers [16, 17]. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t, u)$ are neither autonomous nor periodic in t, the existence of homoclinic solutions of (HS) is quite different from the periodic systems, because of the lack of compactness of the Sobolev embedding, such as [13, 18, 28, 34, 37] and the references mentioned there.

(AR) is the so-called global Ambrosetti-Rabinowitz condition, which implies that $W(t, u)$ is of superquadratic growth as $|u| \to \infty$. Motivated by [39], in [10] Chen et al. gave some more general superquadratic conditions on $W(t, u)$ and obtained that (FHS) possesses infinitely many nontrivial homoclinic solutions. Furthermore, using the genus properties of critical point theory, in [46] Zhang and Yuan established some new criterion to guarantee the existence of infinitely many homoclinic solutions of (FHS) for the case that $W(t, u)$ is subquadratic as $|u| \to \infty$. In [10, 46], the condition (L) is needed to guarantee that the functional corresponding to (FHS) satisfies the (PS) condition. More recently, for one dimensional case Naymoradi and Zhou [26] considered the bifurcation results for the following class of fractional Hamiltonian systems

$$\begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + b(t)u(t) - \lambda u(t) = \mu f(t, u(t)), \\ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}), \end{cases} \tag{2}$$

where $b(t)$ satisfies the coercive condition with $\inf_{t\in\mathbb{R}} b(t) > 0$, λ and μ are real parameters, and $f(t, u)$ is subquadratic at infinity.

As is well-known, the condition (L) is the so-called coercive condition and is very restrictive. In fact, for a simple choice like $L(t) = \tau_0 I_n$, the condition (1) is not satisfied, where $\tau_0 > 0$ is a constant and I_n is the $n \times n$ identity matrix. Therefore, in [47] Zhang and Yuan focused their attention on the case that $L(t)$ is bounded in the sense that

(L)′ $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there are constants $0 < \tau_1 < \tau_2 < \infty$ such that

$$\tau_1|u|^2 \leq (L(t)u, u) \leq \tau_2|u|^2 \quad \text{for all } (t, u) \in \mathbb{R} \times \mathbb{R}^n.$$

If the potential $W(t, u)$ is supposed to be subquadratic as $|u| \to \infty$, they showed that (FHS) possessed infinitely many homoclinic solutions. Furthermore, Naymoradi and Zhou [27] considered the existence and multiplicity of homoclinic solutions of (FHS) when $L(t)$ satisfies the following uniformly positive definite hypothesis

(L)″ $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there exists some constant $\tau > 0$ such that

$$(L(t)u, u) \geq \tau |u|^2 \quad \text{for all } (t, u) \in \mathbb{R} \times \mathbb{R}^n,$$

and $W(t, u)$ is supposed to fulfil some superquadratic assumptions. We note that the results in [27] have been improved in recent paper [48], where some classes of uniformly positive definite hypothesis for $L(t)$ are also needed. In addition, Torres [41] dealt with the one dimensional case, studied the existence of positive solution and analysed the radial symmetric property of these solutions. More recently, the authors in [40] and [43] investigated the following perturbed fractional Hamiltonian systems

$$\begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) - L(t)u(t) + \nabla W(t, u(t)) = f(t), \\ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^n), \end{cases} \tag{PFHS}$$

for the cases that (L) and (L)′ are satisfied respectively. Some reasonable assumptions on $W(t, u)$ and f are established to guarantee the existence of at least two homoclinic solutions of (PFHS).

Motivated mainly by [10, 26, 27, 39–41, 43, 46–48], in this paper we are interested in the existence of infinitely many homoclinic solutions for (FHS) when $L(t)$ is allowed to be indefinite, that is, $L(t)$ is unnecessarily required to be either uniformly positive definite or coercive as before, and $W(t, u)$ satisfies some new superquadratic conditions at infinity weaker than (AR) condition. As far as we know, maybe this is the first time to investigate (FHS) for the case that $L(t)$ is indefinite. To investigate the existence of infinitely homoclinic solutions of (FHS) when $L(t)$ is indefinite, we will translate (FHS) into an equivalent system, see Remark 2.3 below, and then check the hypothesis of a variant fountain theorem due to Zou [49]. Therefore, we present the following hypothesis on $L(t)$ and $W(t, u)$.

Before presenting our assumptions, we introduce some notations. In what follows, for two $n \times n$ symmetric matrices M_1 and M_2, we say that $M_1 \geq M_2$ if

$$((M_1 - M_2)u, u) \geq 0, \quad \forall u \in \mathbb{R}^n, \text{ with } |u| = 1,$$

and that $M_1 \ngeq M_2$ if $M_1 \geq M_2$ does not hold. Now we make the following assumptions on $L(t)$ and $W(t, u)$:

$(\mathcal{L})_1$ $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric matrix for all $t \in \mathbb{R}$ and the smallest eigenvalue of $L(t)$ is bounded from below;

$(\mathcal{L})_2$ there exists a constant $r_0 > 0$ such that

$$\lim_{|s| \to \infty} meas(\{t \in (s - r_0, s + r_0) : L(t) \ngeq MI_n\}) = 0, \quad \forall M > 0,$$

where *meas* denotes the Lebesgue measure in $\mathbb{R}$;

$(\text{FHS})_1$ $\lim_{|u| \to \infty} \frac{W(t,u)}{|u|^2} = \infty$ uniformly with respect to $t \in \mathbb{R}$;

$(\text{FHS})_2$ $W(t, 0) = 0$ for all $t \in \mathbb{R}$, and there exist constants $c > 0$ and $\nu > 2$ such that

$$|W_u(t, u)| \leq c(|u| + |u|^{\nu - 1}), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^n;$$

$(\text{FHS})_3$ there exists a constant $\vartheta \geq 1$ such that

$$\vartheta \widetilde{W}(t, u) \geq \widetilde{W}(t, su), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^n \text{ and } s \in [0, 1],$$

where $\widetilde{W}(t, u) = (W_u(t, u), u) - 2W(t, u)$;

$(\text{FHS})_4$ $W(t, u) = W(t, -u)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$.

Now, we are in the position to state our main result.

Theorem 1.1. *Suppose that* $(\mathcal{L})_1$, $(\mathcal{L})_2$ *and* (FHS)$_1$-(FHS)$_4$ *are satisfied, then* (FHS) *possesses a sequence of homoclinic solutions* $\{u_k\}_{k\in\mathbb{N}}$ *satisfying*

$$\int_{\mathbb{R}}\Big[\frac{1}{2}|{}_{-\infty}D_t^{\alpha}u_k(t)|^2+\frac{1}{2}(L(t)u_k(t),u_k(t))-W(t,u_k(t))\Big]dt\to\infty \quad as \quad k\to\infty.$$

Remark 1.2. It is obvious that the conditions $(\mathcal{L})_1$ and $(\mathcal{L})_2$ are weaker than the coercive condition (L), the bounded conditions (L)′ and (L)″. In addition, the well-known (AR) condition is not required in our Theorem 1.1. Therefore, the previous results mentioned above are generalized and improved significantly. There are $L(t)$ and $W(t,u)$ satisfying all the conditions in our Theorem 1.1. For an example, let

$$L(t)=(|t|\sin^2 t-1)I_n,\quad \forall t\in\mathbb{R}$$

and

$$W(t,u)=\pi(t)\ln(1+|u|^2)(1+|u|^2)-|u|^2,\quad \forall(t,u)\in\mathbb{R}\times\mathbb{R}^n,$$

where $\pi:\mathbb{R}\to\mathbb{R}$ is a continuously bounded function with positive lower bound, then simple computation shows that $(\mathcal{L})_1$, $(\mathcal{L})_2$ and (FHS)$_1$-(FHS)$_4$ are satisfied.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1.

2. Preliminary Results

In this section, for the reader's convenience, firstly we introduce some basic definitions of fractional calculus. The Liouville-Weyl fractional integrals of order $0<\alpha<1$ are defined as

$${}_{-\infty}I_x^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}u(\xi)d\xi$$

and

$${}_{x}I_{\infty}^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}u(\xi)d\xi.$$

The Liouville-Weyl fractional derivative of order $0<\alpha<1$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$${}_{-\infty}D_x^{\alpha}u(x)=\frac{d}{dx}{}_{-\infty}I_x^{1-\alpha}u(x) \tag{3}$$

and

$${}_{x}D_{\infty}^{\alpha}u(x)=-\frac{d}{dx}{}_{x}I_{\infty}^{1-\alpha}u(x). \tag{4}$$

The definitions of (3) and (4) may be written in alternative forms as follows:

$${}_{-\infty}D_x^{\alpha}u(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}}d\xi$$

and

$${}_{x}D_{\infty}^{\alpha}u(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}}d\xi.$$

Moreover, recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w)=\int_{-\infty}^{\infty}e^{-iwx}u(x)dx.$$

In order to establish the variational structure which enables us to reduce the existence of homoclinic solutions of (FHS) to find critical points of the corresponding functional, it is necessary to construct an appropriate function space. In what follows, we introduce some fractional spaces, for more details see [14, 39]. To this end, denote by $L^p(\mathbb{R}, \mathbb{R}^n)$ $(1 \le p < \infty)$ the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|u\|_p = \Big(\int_{\mathbb{R}} |u(t)|^p dt\Big)^{1/p},$$

and $L^\infty(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_\infty = ess\sup\{|u(t)| : t \in \mathbb{R}\}.$$

For $\alpha > 0$, define the semi-norm

$$|u|_{I^\alpha_{-\infty}} = \|{}_{-\infty}D^\alpha_x u\|_2$$

and the norm

$$\|u\|_{I^\alpha_{-\infty}} = \Big(\|u\|_2^2 + |u|^2_{I^\alpha_{-\infty}}\Big)^{1/2} \tag{5}$$

and let

$$I^\alpha_{-\infty} = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_{-\infty}}},$$

where $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^n$ with vanishing property at infinity.

Now we can define the fractional Sobolev space $H^\alpha(\mathbb{R}, \mathbb{R}^n)$ in terms of the Fourier transform. Choose $0 < \alpha < 1$, define the semi-norm

$$|u|_\alpha = \||w|^\alpha \widehat{u}\|_2$$

and the norm

$$\|u\|_\alpha = \Big(\|u\|_2^2 + |u|_\alpha^2\Big)^{1/2}$$

and let

$$H^\alpha = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_\alpha}.$$

Notice that a function $u \in L^2(\mathbb{R}, \mathbb{R}^n)$ belongs to $I^\alpha_{-\infty}$ if and only if

$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R}, \mathbb{R}^n).$$

Especially, we have

$$|u|_{I^\alpha_{-\infty}} = \||w|\widehat{u}\|_2.$$

Subsequently, $I^\alpha_{-\infty}$ and H^α are equivalent with equivalent semi-norm and norm, see [14, Theorem 2.1]. Analogous to $I^\alpha_{-\infty}$, we introduce I^α_∞. Define the semi-norm

$$|u|_{I^\alpha_\infty} = \|{}_x D^\alpha_\infty u\|_2$$

and the norm

$$\|u\|_{I^\alpha_\infty} = \Big(\|u\|_2^2 + |u|^2_{I^\alpha_\infty}\Big)^{1/2} \tag{6}$$

and let

$$I^\alpha_\infty = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_\infty}}.$$

Then $I^\alpha_{-\infty}$ and I^α_∞ are equivalent with equivalent semi-norm and norm, see [14, Theorem 2.3].

Let $C(\mathbb{R}, \mathbb{R}^n)$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$. Then we obtain the following lemma.

Lemma 2.1. *[39, Theorem 2.1] If $1/2 < \alpha < 1$, $H^\alpha \subset C(\mathbb{R}, \mathbb{R}^n)$ and there is a constant $C = C_\alpha$ such that*

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(x)| \le C\|u\|_\alpha.$$

In addition, from the definition of H^α, it is easy to see that

$$\lim_{|t|\to\infty} u(t) = 0, \tag{7}$$

see Corollary 8.9 in [9].

Remark 2.2. From Lemma 2.1, we know that if $u \in H^\alpha$ with $1/2 < \alpha < 1$, $u \in L^p(\mathbb{R}, \mathbb{R}^n)$ for all $p \in [2, \infty)$, since

$$\int_{\mathbb{R}} |u(t)|^p dt \le \|u\|_\infty^{p-2} \|u\|_2^2.$$

Remark 2.3. According to $(\mathrm{FHS})_2$, it is easy to verify that

$$|W(t,u)| = \Big|\int_0^1 (W_u(t,su), u)ds\Big| \le \frac{c}{2}|u|^2 + \frac{c}{\nu}|u|^\nu, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^n. \tag{8}$$

From $(\mathcal{L})_1$, $(\mathrm{FHS})_1$ and (8), we know that there exists a positive constant l_0 such that $L(t) + l_0 I_n \ge I_n$ for all $t \in \mathbb{R}$ and $W(t,u) + l_0|u|^2 \ge 0$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$. Define $\overline{L}(t) = L(t) + l_0 I_n$ and $\overline{W}(t,u) = W(t,u) + l_0|u|^2$, consider the following fractional Hamiltonian systems

$$\begin{cases} -{}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) - \overline{L}(t)u(t) + \nabla\overline{W}(t,u(t)) = 0, \\ u \in H^\alpha(\mathbb{R}, \mathbb{R}^n), \end{cases} \tag{9}$$

then (9) is equivalent to (FHS). Moreover, it is obvious that the hypotheses $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathrm{FHS})_1$-$(\mathrm{FHS})_4$ still hold for $\overline{L}(t)$ and $\overline{W}(t,u)$.

In view of Remark 2.3, in what follows, we investigate the equivalent problem (9) and make the following assumption instead of $(\mathcal{L})_1$:

$(\mathcal{L})'_1$ $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix such that $L(t) \ge I_n$ for all $t \in \mathbb{R}$.

Meanwhile, due to Remark 2.3, we can also assume that $W(t,u) \ge 0$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$.

Remark 2.4. Before we introduce the fractional space in which we will construct the variational framework of (FHS), we must mention the recent paper [3]. In [3], with the aid of the genus properties in the critical point theory, the authors showed that (FHS) possesses infinitely many homoclinic solutions when $(\mathcal{L})'_1$ and $(\mathcal{L})_2$ are satisfied, $W(t,u)$ is of subquadratic growth as $|u| \to \infty$, see its Theorem 1.1.

In present paper, under the assumptions of $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathrm{FHS})_1$-$(\mathrm{FHS})_4$, we deal with the case that $L(t)$ is allowed to be indefinite, and $W(t,u)$ satisfies some new superquadratic conditions at infinity weaker than (AR) condition. Therefore, the results in [3] are complemented.

Moreover, based on Remark 2.3, $(\mathcal{L})_1$, $(\mathcal{L})_2$ for $L(t)$ and $(\mathrm{FHS})_1$-$(\mathrm{FHS})_4$ for $W(t,u)$ can be replaced by $(\mathcal{L})'_1$, $(\mathcal{L})_2$ and $(\mathrm{FHS})_1$-$(\mathrm{FHS})_4$ for $\overline{L}(t)$ and $\overline{W}(t,u)$ respectively. However, if $W(t,u)$ is of subquadratic growth as $|u| \to \infty$, we could not do the same thing. Thus, one natural question is that whether one can obtain the existence of (infinitely many) homoclinic solutions of (FHS) for the case that $L(t)$ satisfies $(\mathcal{L})_1$ and $(\mathcal{L})_2$ (that is, $L(t)$ is allowed to be indefinite), and $W(t,u)$ is subquadratic at infinity or not.

Now, going back to our Theorem 1.1, to finish its proof, we need the following function space as our working space. Let

$$X^\alpha = \Big\{u \in H^\alpha : \int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt < \infty\Big\},$$

then X^α is a reflexive and separable Hilbert space with the inner product

$$\langle u, v\rangle_{X^\alpha} = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))]dt$$

and the corresponding norm is

$$\|u\|^2 = \langle u, u\rangle_{X^\alpha}.$$

To verify the conditions of the variant fountain theorem, see Lemma 2.9 below, the following two lemmas play an essential role. Although one can find their proofs in [3], for the reader's convenience, we give the details of their proofs.

Lemma 2.5. *Suppose $L(t)$ satisfies $(\mathcal{L})_1'$, then X^α is continuously embedded in H^α.*

Proof. In view of $(\mathcal{L})_1'$, we have

$$(L(t)u, u) \geq (u, u) = |u|^2, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^n.$$

Thus, it deduces that

$$\begin{aligned}\|u\|_\alpha^2 &= \int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (u(t), u(t))]dt \\ &\leq \int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt \\ &= \|u\|^2, \quad \forall u \in X^\alpha,\end{aligned}$$

which implies that the conclusion holds true. □

Remark 2.6. From Lemma 2.1 and Lemma 2.5, the embedding of X^α into $L^\infty(\mathbb{R}, \mathbb{R}^n)$ is continuous. On the other hand, it is obvious that the embedding $X^\alpha \hookrightarrow L^2(\mathbb{R}, \mathbb{R}^n)$ is also continuous. Therefore, combining this with Remark 2.2, for any $p \in [2, \infty]$, there exists $C_p > 0$ such that

$$\|u\|_p \leq C_p\|u\|. \tag{10}$$

In addition, we can further have the following lemma.

Lemma 2.7. *If $(\mathcal{L})_1'$ and $(\mathcal{L})_2$ are satisfied, X^α is compactly embedded into $L^2(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X^\alpha$ be a bounded sequence such that $u_n \rightharpoonup u$ in X^α. We will show that $u_n \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. Suppose, without loss of generality, that $u_n \rightharpoonup 0$ in X^α. The Sobolev embedding implies that $u_n \to 0$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^n)$. Thus it suffices to show that, for any $\epsilon > 0$, there is $r > 0$ such that

$$\int_{\mathbb{R}\setminus[-r,r]} |u_n(t)|^2 dt < \epsilon, \quad \forall n \in \mathbb{N}. \tag{11}$$

To this end, for any $s \in \mathbb{R}$, we denote by $\mathcal{B}_{r_0}(s)$ the interval in $\mathbb{R}$ centered at s with radius r_0, i.e., $\mathcal{B}_{r_0}(s) = (s - r_0, s + r_0)$, where r_0 is defined in $(\mathcal{L})_2$. Let $\{s_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of points such that $\mathbb{R} = \cup_{i=1}^\infty \mathcal{B}_{r_0}(s_i)$ and each $t \in \mathbb{R}$ is contained in at most two such intervals. For any $r > 0$ and $M > 0$, let

$$C(r, M) = \{t \in \mathbb{R} \setminus [-r, r] : L(t) \geq MI_n\}$$

and

$$\mathcal{D}(r, M) = \{t \in \mathbb{R} \setminus [-r, r] : L(t) \ngeq MI_n\}.$$

Then, one deduces that

$$\int_{C(r,M)} |u_n(t)|^2 dt \leq \frac{1}{M}\int_{C(r,M)} (L(t)u_n(t), u_n(t))dt \leq \frac{1}{M}\int_{\mathbb{R}} (L(t)u_n(t), u_n(t))dt,$$

and moreover this can be made arbitrarily small by choosing M large enough. In addition, making use of Hölder inequality and (10), for a fixed $M > 0$, we have

$$\begin{aligned}
\int_{\mathcal{D}(r,M)} |u_n(t)|^2 dt &\le \sum_{i=1}^{\infty} \int_{\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)} |u_n(t)|^2 dt \\
&\le \sum_{i=1}^{\infty} \Big(\int_{\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)} |u_n(t)|^4 dt \Big)^{1/2} \Big(meas(\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)) \Big)^{1/2} \\
&\le \epsilon_r \sum_{i=1}^{\infty} \Big(\int_{\mathcal{B}_{r_0}(s_i)} |u_n(t)|^4 dt \Big)^{1/2} \\
&\le \epsilon_r \sum_{i=1}^{\infty} \int_{\mathcal{B}_{r_0}(s_i)} |u_n(t)|^4 dt \\
&\le 2\epsilon_r \int_{\mathbb{R}} |u_n(t)|^4 dt \\
&\le 2C_4^4 \epsilon_r \int_{\mathbb{R}} [|{}_{-\infty}D_t^{\alpha} u_n(t)|^2 + (L(t)u_n(t), u_n(t))] dt,
\end{aligned}$$

where $\epsilon_r = \sup_{i\in\mathbb{N}} (meas(\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)))^{1/2}$. On account of $(\mathcal{L})_2$, $\epsilon_r \to 0$ as $r \to \infty$ and noting that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X^{α}, we can make this term small small enough by choosing r large. This completes the proof. □

Remark 2.8. From Remark 2.2 and Lemma 2.7, it is easy to check that X^{α} is compactly embedded into $L^p(\mathbb{R}, \mathbb{R}^n)$ for any $p \in [2, \infty)$.

As mentioned above, to obtain the existence of infinitely many homoclinic solutions of (FHS), we need the following variant fountain theorem established in [49].

Let $\mathcal{B}$ be a Banach space with the norm $\|\cdot\|$ and $\mathcal{B} = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^{k} X_j$ and $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. Consider the following C^1-functional $\Phi_\lambda : \mathcal{B} \to \mathbb{R}$ defined by

$$\Phi_\lambda = A(u) - \lambda B(u), \quad \lambda \in [1, 2].$$

Lemma 2.9. *([49, Theorem 2.1]) Assume that the above functional Φ_λ satisfies*

$(\mathcal{A})_1$ *Φ_λ maps bounded sets to bounded sets for $\lambda \in [1, 2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times \mathcal{B}$;*

$(\mathcal{A})_2$ *$B(u) \ge 0$ for all $u \in \mathcal{B}$, and $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$;*

$(\mathcal{A})_3$ *there exist $\rho_k > \sigma_k > 0$ such that*

$$\alpha_k(\lambda) = \inf_{u\in Z_k, \|u\|=\sigma_k} \Phi_\lambda(u) > \beta_k(\lambda) = \max_{u\in Y_k, \|u\|=\rho_k} \Phi_\lambda(u), \quad \lambda \in [1, 2].$$

Then

$$\alpha_k(\lambda) \le \zeta_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2],$$

where $B_k = \{u \in Y_k : \|u\| \le \rho_k\}$ and $\Gamma_k = \{\gamma \in C(B_k, \mathcal{B}) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$. Moreover, for almost every $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^{\infty}$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \to 0, \quad \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \quad as \quad m \to \infty.$$

3. Proof of Theorem 1.1

The aim of this section is to establish the proof of Theorem 1.1. For this purpose, we are going to establish the corresponding variational framework to obtain homoclinic solutions of (FHS). To this end, we choose $\mathcal{B} = X^\alpha$ as our working function space. Due to the fact that X^α is a reflexive and separable Hilbert space, we can select an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of X^α and let $X_j = span\{e_j\}$ for all $j \in \mathbb{N}$. Define the functionals A, B and Φ_λ on X^α by

$$A(u) = \frac{1}{2}\|u\|^2, \quad B(u) = \int_{\mathbb{R}} W(t, u(t))dt, \tag{12}$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u\| - \lambda \int_{\mathbb{R}} W(t, u(t))dt \tag{13}$$

for all $u \in X^\alpha$ and $\lambda \in [1, 2]$. Especially, we denote by $\Phi_1 = \Phi$, that is,

$$\begin{aligned}\Phi(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t))\Big]dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, u(t))dt.\end{aligned} \tag{14}$$

(8) and (10) imply that Φ_λ is well defined on X^α. Furthermore, under the conditions of Theorem 1.1, as usual, we see that $\Phi_\lambda \in C^1(X^\alpha, \mathbb{R})$, that is, Φ_λ is a continuously Fréchet-differentiable functional defined on X^α. Moreover, we have

$$\Phi'_\lambda(u)v = \int_{\mathbb{R}} \Big[({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - \lambda(\nabla W(t, u(t)), v(t))\Big]dt \tag{15}$$

for all $u, v \in X^\alpha$, which yields that

$$\Phi'_\lambda(u)u = \|u\|^2 - \lambda \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{16}$$

In addition, any nontrivial critical points of Φ are homoclinic solutions of (FHS), see (7).

To obtain the existence of infinitely many homoclinic solutions by using the fountain theorem, in the sequel we establish some technical lemmas.

Lemma 3.1. *For any finite dimensional subspace $E \subset X^\alpha$, there exists a constant $\varrho > 0$ such that*

$$meas(\{t \in \mathbb{R} : |u(t)| \geq \varrho\|u\|\}) \geq \varrho, \quad \forall u \in E \setminus \{0\}.$$

Proof. On the contrary, assume that, for any $n \in \mathbb{N}$, there exists $u_n \in E \setminus \{0\}$ such that

$$meas(\{t \in \mathbb{R} : |u_n(t)| \geq \frac{\|u_n\|}{n}\}) < \frac{1}{n}.$$

For each $n \in \mathbb{N}$, let $v_n = \frac{u_n}{\|u_n\|} \in E$, then we have $\|v_n\| = 1$ and

$$meas(\{t \in \mathbb{R} : |v_n(t)| \geq \frac{1}{n}\}) < \frac{1}{n}. \tag{17}$$

Passing to a subsequence if necessary, we may assume that $v_n \to v_0$ in X^α for some $v_0 \in E$, since E is of finite dimension. Combining this with (10), we have

$$\int_{\mathbb{R}} |v_n(t) - v_0(t)|^2 dt \to 0 \quad \text{as} \quad n \to \infty. \tag{18}$$

Noting that $\|v_0\| = 1$, there must exist a constant $\delta_0 > 0$ such that

$$meas(\{t \in \mathbb{R} : |v_0(t)| \geq \delta_0\}) \geq \delta_0. \tag{19}$$

Otherwise, for each fixed $n \in \mathbb{N}$, we have

$$meas(\{t \in \mathbb{R} : |v_0(t)| \geq \frac{1}{n}\}) \leq meas(\{t \in \mathbb{R} : |v_0(t)| \geq \frac{1}{m}\}) \leq \frac{1}{m}, \quad \forall m \geq n.$$

Letting $m \to \infty$, we obtain that

$$meas(\{t \in \mathbb{R} : |v_0(t)| \geq \frac{1}{n}\}) = 0.$$

Consequently, one deduces that

$$\begin{aligned} 0 &\leq meas(\{t \in \mathbb{R} : |v_0(t)| \neq 0\}) \\ &= meas(\cup_{n=1}^{\infty}\{t \in \mathbb{R} : |v_0(t)| \geq \frac{1}{n}\}) \\ &\leq \sum_{n=1}^{\infty} meas(\{t \in \mathbb{R} : |v_0(t)| \geq \frac{1}{n}\}) = 0, \end{aligned}$$

which yields that $v_0 = 0$, a contradiction to $\|v_0\| = 1$. Thus (19) holds. In what follows, set $\Omega_0 = \{t \in \mathbb{R} : |v_0(t)| \geq \delta_0\}$, where δ_0 is the constant given in (19). For any $n \in \mathbb{N}$, let

$$\Omega_n = \{t \in \mathbb{R} : |v_n(t)| < \frac{1}{n}\} \quad \text{and} \quad \Omega_n^c = \mathbb{R} \setminus \Omega_n = \{t \in \mathbb{R} : |v_n(t)| \geq \frac{1}{n}\}.$$

Then, for n large enough, by (17) and (19), we have

$$meas(\Omega_n \cap \Omega_0) \geq meas(\Omega_0) - meas(\Omega_n) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

Consequently, for n large enough, it holds that

$$\begin{aligned} \int_{\mathbb{R}} |v_n(t) - v_0(t)|^2 dt &\geq \int_{\Omega_n \cap \Omega_0} |v_n(t) - v_0(t)|^2 dt \\ &\geq \int_{\Omega_n \cap \Omega_0} (|v_0(t)| - |v_n(t)|)^2 dt \\ &\geq \left(\delta_0 - \frac{1}{n}\right)^2 meas(\Omega_n \cap \Omega_0) \\ &\geq \frac{\delta_0^3}{8}, \end{aligned}$$

which contradicts to (18). The proof is completed. □

Lemma 3.2. *Assume that* $(\mathcal{L})_1'$, $(\mathcal{L})_2$ *and* $(\mathrm{FHS})_2$ *hold, then there exist a positive integer* k_1 *and a sequence* $\{\sigma_k\}_{k\in\mathbb{N}}$ *satisfying* $\sigma_k \to \infty$ *as* $k \to \infty$ *such that*

$$\alpha_k(\lambda) = \inf_{u \in Z_k, \|u\| = \sigma_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1,$$

where $Z_k = \overline{\oplus_{j=k}^{\infty} X_j} = \overline{span\{e_k, \dots\}}$ *for all* $k \geq k_1$.

Proof. Note that (8) and (13) imply that

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} W(t,u(t))dt \\ &\geq \frac{1}{2}\|u\|^2 - c\|u\|_2^2 - \frac{2c}{\nu}\|u\|_\nu^\nu, \quad \forall(\lambda,u) \in [1,2]\times X^\alpha.\end{aligned} \tag{20}$$

For each $k \in \mathbb{N}$, define

$$\ell_2(k) = \sup_{u\in Z_k,\|u\|=1} \|u\|_2 \quad \text{and} \quad \ell_\nu(k) = \sup_{u\in Z_k,\|u\|=1} \|u\|_\nu. \tag{21}$$

Since X^α is compactly embedded into both $L^2(\mathbb{R},\mathbb{R}^n)$ and $L^\nu(\mathbb{R},\mathbb{R}^n)$, then it deduces that (see [42, Lemma 3.8])

$$\ell_2(k) \to 0 \quad \text{and} \quad \ell_\nu(k) \to 0 \quad \text{as} \quad k\to\infty. \tag{22}$$

On account of (20) and (21), we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - c\ell_2^2(k)\|u\|^2 - \frac{2c}{\nu}\ell_\nu^\nu(k)\|u\|^\nu, \quad \forall(\lambda,u)\in[1,2]\times Z_k. \tag{23}$$

In view of (22), there exists a positive integer k_1 such that

$$c\ell_2^2(k) \leq \frac{1}{4}, \quad \forall k \geq k_1. \tag{24}$$

For each $k \geq k_1$, choose

$$\sigma_k = \Big(\frac{16c\ell_\nu^\nu(k)}{\nu}\Big)^{1/(2-\nu)}. \tag{25}$$

Then, it follows from (22) that

$$\sigma_k \to \infty \quad \text{as} \quad k\to\infty, \tag{26}$$

since $\nu > 2$. Based on (23)-(25), a direct computation shows that

$$\alpha_k(\lambda) = \inf_{u\in Z_k,\|u\|=\sigma_k} \Phi_\lambda(u) \geq \frac{\sigma_k^2}{8} > 0, \quad \forall k\geq k_1,$$

which implies that the proof is completed. □

Lemma 3.3. *Suppose that* $(\mathcal{L})_1'$, $(\mathcal{L})_2$, $(\mathrm{FHS})_1$ *and* $(\mathrm{FHS})_2$ *are satisfied. Then, for the positive integer* k_1 *and the sequence* $\{\sigma_k\}_{k\in\mathbb{N}}$ *determined in Lemma 3.2, there exists* $\rho_k > \sigma_k$ *for each* $k\geq k_1$ *such that*

$$\beta_k = \max_{u\in Y_k,\|u\|=\rho_k} \Phi_\lambda(u) < 0,$$

where $Y_k = \oplus_{j=1}^k X_j = span\{e_1,\ldots,e_k\}$ *for all* $k \geq k_1$.

Proof. Note that Y_k is a finite dimensional for all $k\geq k_1$. Then, by Lemma 3.1, for all $k\geq k_1$, there exists a constant $\varrho_k > 0$ such that

$$meas(\Omega_u^k) \geq \varrho_k, \quad \forall u \in Y_k\setminus\{0\}, \tag{27}$$

where $\Omega_u^k = \{t \in \mathbb{R} : |u(t)| \geq \varrho_k \|u\|\}$ for all $k \geq k_1$ and $u \in Y_k \setminus \{0\}$. By $(\mathrm{FHS})_1$, for each $k \geq k_1$, there exists a constant $b_k > 0$ such that

$$W(t,u) \geq \frac{|u|^2}{\varrho_k^3}, \quad \forall t \in \mathbb{R} \quad \text{and} \quad |u| \geq b_k. \tag{28}$$

Combining (13), (27) and (28), for all $k \geq k_1$ and $\lambda \in [1,2]$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u(t))dt \\
&\leq \frac{1}{2}\|u\|^2 - \int_{\Omega_u^k} \frac{|u(t)|^2}{\varrho_k^3} dt \\
&\leq \frac{1}{2}\|u\|^2 - \varrho_k^2 \|u\|^2 \frac{meas(\Omega_u^k)}{\varrho_k^3} \\
&\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2
\end{aligned} \tag{29}$$

for all $u \in Y_k$ with $\|u\| \geq \frac{b_k}{\varrho_k}$. Here we use the fact that $W(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$. As a result, for each $k \geq k_1$, if we choose $\rho_k > \max\{\sigma_k, \frac{b_k}{\varrho_k}\}$, (29) implies that

$$\beta_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} \Phi_\lambda(u) \leq -\frac{\rho_k^2}{2},$$

which deduces that the conclusion holds true. □

Now we are in the position to establish the proof of Theorem 1.1.

Proof. Firstly, from (8), (10) and (13), it follows that Φ_λ maps bounded sets to bounded sets uniformly with respect to $\lambda \in [1,2]$. In addition, $(\mathrm{FHS})_4$ implies that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X^\alpha$. Thus, $(\mathcal{A})_1$ holds. Next, using again the fact that $W(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$, we know that $(\mathcal{A})_2$ holds by the definitions of functionals A and B in (12). Finally, Lemmas 3.2 and 3.3 show that $(\mathcal{A})_3$ holds for $k \geq k_1$, where k_1 is given in Lemma 3.2. Therefore, for each $k \geq k_1$, applying Theorem 2.9, for almost every $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \to 0, \quad \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \quad \text{as} \quad m \to \infty, \tag{30}$$

where

$$\zeta_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1,2]$$

with $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$ and $\Gamma_k = \{\gamma \in C(B_k, X^\alpha) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$. From the proof of Lemma 3.2, we infer that

$$\zeta_k(\lambda) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall k \geq k_1 \text{ and } \lambda \in [1,2], \tag{31}$$

where $\overline{\zeta}_k = \max_{u \in B_k} \Phi_1(u)$ and $\overline{\alpha}_k = \frac{\sigma_k^2}{8} \to \infty$ as $k \to \infty$ by (26). In view of (30), for each $k \geq k_1$, we can choose a sequence $\lambda_n \to 1$ (dependent on k) and get the corresponding sequences satisfying

$$\sup_m \|u_m^k(\lambda_n)\| < \infty \quad \text{and} \quad \Phi'_{\lambda_n}(u_m^k(\lambda_n)) \to 0 \quad \text{as} \quad m \to \infty. \tag{32}$$

Claim 1. For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m=1}^\infty$ has a strong convergent subsequence.

For notational simplicity, we set $u_m = u_m^k(\lambda_n)$ for $m \in \mathbb{N}$ throughout the proof of Claim 1. By (32), without loss of generality, we assume that

$$u_m \rightharpoonup u \quad \text{as} \quad m \to \infty \tag{33}$$

for some $u \in X^\alpha$. According to (15), we have

$$\begin{aligned} \|u_m - u\|^2 =& \Phi'_{\lambda_n}(u_m)(u_m - u) - \Phi'_{\lambda_n}(u)(u_m - u) \\ &+ \lambda_n \int_{\mathbb{R}} (\nabla W(t, u_m(t)) - \nabla W(t, u(t)), u_m(t) - u(t))dt. \end{aligned} \tag{34}$$

By (32) and (33), we have

$$\Phi'_{\lambda_n}(u_m)(u_m - u) \to 0 \quad \text{and} \quad \Phi'_{\lambda_n}(u)(u_m - u) \to 0 \tag{35}$$

as $m \to \infty$. In addition, according to (10), Lemma 2.7 and the Hölder inequality, we infer that

$$\begin{aligned} &\left| \int_{\mathbb{R}} (\nabla W(t, u_m) - \nabla W(t, u)), u_m - u)dt \right| \\ &\le \Big(\int_{\mathbb{R}} |\nabla W(t, u_m) - \nabla W(t, u)|^2 dt \Big)^{1/2} \|u_m - u\|_2 \\ &\le c \Big(\int_{\mathbb{R}} (|u_m(t)| + |u(t)| + |u_m(t)|^{\nu-1} + |u(t)|^{\nu-1})^2 dt \Big)^{1/2} \|u_m - u\|_2 \\ &\le 4c \Big(\int_{\mathbb{R}} (|u_m(t)|^2 + |u(t)|^2 + |u_m(t)|^{2\nu-2} + |u(t)|^{2\nu-2})dt \Big)^{1/2} \|u_m - u\|_2 \\ &\le 4c \Big(\|u_m\|_2^2 + \|u\|_2^2 + \|u_m\|_{2\nu-2}^{2\nu-2} + \|u\|_{2\nu-2}^{2\nu-2} \Big)^{1/2} \|u_m - u\|_2 \to 0 \end{aligned} \tag{36}$$

as $m \to \infty$. Here we use the fact that $\{u_m\}_{m\in\mathbb{N}}$ is bounded in X^α. Combining (34), (35) and (36), we obtain that $u_m \to u$ in X^α. Thus, Claim 1 holds.

By Claim 1, without loss of generality, we may assume that

$$\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathbb{N} \quad \text{and} \quad k \ge k_1. \tag{37}$$

On account of (37), (30), and (31), we deduce that

$$\Phi'_{\lambda_n}(u_n^k) = 0 \quad \text{and} \quad \Phi_{\lambda_n}(u_n^k) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall n \in \mathbb{N} \quad \text{and} \quad k \ge k_1. \tag{38}$$

Claim 2. For each $k \ge k_1$, the sequence $\{u_n^k\}_{n=1}^\infty$ in (37) is bounded in X^α.

As in the proof of Claim 1, for notational simplicity, we set $u_n = u_n^k$ for all $n \in \mathbb{N}$. On the contrary, if Claim 2 is not true, without loss of generality, we may assume that

$$\|u_n\| \to \infty \quad \text{and} \quad \omega_n = \frac{u_n}{\|u_n\|} \rightharpoonup \omega \in X^\alpha \quad \text{as} \quad n \to \infty. \tag{39}$$

By (39) and Remark 2.8, passing to a subsequence if necessary, we have

$$\omega_n \to \omega \quad \text{in } L^p(\mathbb{R}, \mathbb{R}^n) \quad \text{for} \quad 2 \le p < \infty, \tag{40}$$

and

$$\omega_n(t) \to \omega(t) \quad \text{a.e. } t \in \mathbb{R}. \tag{41}$$

When $\omega \ne 0$ occurs, $\Theta = \{t \in \mathbb{R} : \omega(t) \ne 0\}$ has a positive Lebesgue measure. By (39), it holds that

$$u_n(t) \to \infty, \quad \forall t \in \Theta. \tag{42}$$

Combining (13), (41), (42) and $(FHS)_1$, by Fatou's Lemma, we deduce that

$$\begin{aligned}\frac{1}{2}-\frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} &= \lambda\int_{\mathbb{R}}\frac{W(t,u_n(t))}{\|u_n\|^2}dt\\ &\geq \int_{\Theta}|\omega_n(t)|^2\frac{W(t,u_n(t))}{|u_n(t)|^2}dt\to\infty \quad \text{as} \quad n\to\infty,\end{aligned}$$

which is a contradiction to (38) and (39). When $\omega = 0$ occurs, we choose a sequence $\{s_n\}_{n\in\mathbb{N}}\subset[0,1]$ such that

$$\Phi_{\lambda_n}(s_nu_n)=\max_{s\in[0,1]}\Phi_{\lambda_n}(su_n). \tag{43}$$

For $M>0$, let $\widetilde{\omega}_n=\sqrt{4M}\omega_n=\frac{\sqrt{4M}}{\|u_n\|}u_n$, then (40) yields that

$$\widetilde{\omega}_n\to\sqrt{4M}\omega=0 \quad \text{in } L^p(\mathbb{R},\mathbb{R}^n) \quad \text{for} \quad 2\leq p<\infty, \tag{44}$$

which, combining with (8) and (40), imply that

$$\Big|\int_{\mathbb{R}}W(t,\widetilde{\omega}_n(t))dt\Big|\leq c\int_{\mathbb{R}}\Big(\frac{1}{2}|\widetilde{\omega}_n(t)|^2+\frac{1}{\nu}|\widetilde{\omega}_n(t)|^\nu\Big)dt\to 0 \quad \text{as} \quad n\to\infty.$$

Note that $0<\frac{\sqrt{4M}}{\|u_n\|}<1$ holds by (39) for n large enough. On account of (13) and (43), we obtain that

$$\begin{aligned}\Phi_{\lambda_n}(s_nu_n) &\geq \Phi_{\lambda_n}(\widetilde{\omega}_n)\\ &= \frac{1}{2}\|\widetilde{\omega}_n\|^2-\lambda_n\int_{\mathbb{R}}W(t,\widetilde{\omega}_n(t))dt\\ &= 2M-\lambda_n\int_{\mathbb{R}}W(t,\widetilde{\omega}_n(t))dt\geq M\end{aligned}$$

for n large enough. It follows that $\lim_{n\to\infty}\Phi_{\lambda_n}(s_nu_n)=\infty$. Observing that $\Phi_{\lambda_n}(0)=0$ and $\Phi_{\lambda_n}(u_n)\in[\overline{\alpha}_k,\overline{\zeta}_k]$ in (38), we know that $s_n\in(0,1)$ in (43) for n large enough. Hence, one deduces that

$$0=s_n\frac{d}{ds}\Big|_{s=s_n}\Phi_{\lambda_n}(su_n)=\Phi'_{\lambda_n}(s_nu_n)s_nu_n. \tag{45}$$

In view of (13), (15), (29), (45) and $(FHS)_3$, we have

$$\begin{aligned}\Phi_{\lambda_n}(u_n) &= \Phi_{\lambda_n}(u_n)-\frac{1}{2}\Phi'_{\lambda_n}(u_n)u_n\\ &= \frac{\lambda_n}{2}\int_{\mathbb{R}}\widetilde{W}(t,u_n(t))dt\\ &\geq \frac{\lambda_n}{2\vartheta}\int_{\mathbb{R}}\widetilde{W}(t,s_nu_n(t))dt\\ &= \frac{1}{\vartheta}\Phi_{\lambda_n}(s_nu_n)-\frac{1}{2\vartheta}\Phi'_{\lambda_n}(s_nu_n)s_nu_n\\ &= \frac{1}{\vartheta}\Phi_{\lambda_n}(s_nu_n)\to\infty \quad \text{as} \quad n\to\infty,\end{aligned}$$

where ϑ is the constant in $(FHS)_3$. It provides a contradiction to (38). Thus, Claim 2 is true.

In view of Claim 2 and (38), for each $k\geq k_1$, using the similar arguments in the proof of Claim 1, we can also show that the sequence $\{u_n^k\}_{n=1}^{\infty}$ has a strong convergent subsequence with the limit u^k being just a critical point $\Phi=\Phi_1$. Evidently, $\Phi(u^k)\in[\overline{\alpha}_k,\overline{\zeta}_k]$ for all $k\geq k_1$. Since $\overline{\alpha}_k\to\infty$ as $k\to\infty$ in (31), we obtain infinitely many nontrivial critical points of Φ. Therefore, (FHS) possesses infinitely many nontrivial homoclinic solutions. □

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