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Spectral Properties of a Fourth Order Eigenvalue Problem with Spectral Parameter in the Boundary Conditions

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Abstract. In this paper we consider the eigenvalue problem for fourth order ordinary differential equation that describes the bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end of which is fixed rigidly and on the right end an inertial mass is concentrated. We characterize the location of the eigenvalues on the real axis, we investigate the structure of root spaces and oscillation properties of eigenfunctions and their derivatives, we study the basis properties in the space L_p, $1 < p < \infty$, of the system of eigenfunctions of considered problem.

1. Introduction

We consider the following boundary value problem problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \ \ 0 < x < 1, \tag{1}$$

$$y(0) = y'(0) = 0, \tag{2}$$

$$y''(1) - a_1 \lambda y'(1) = 0 \tag{3}$$

$$Ty(1) - a_2 \lambda y(1) = 0, \tag{4}$$

where $\lambda \in \mathbb{C}$ is spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive and absolutely continuous function on $[0, 1]$, a_1 and a_2, are real constants.

The problem (1)-(4) arises when variables are separated in the dynamical boundary value problem describing bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end of which is fixed rigidly and on the right end an inertial mass is concentrated (see [19, Ch. 8, § 5]).

Boundary value problems for ordinary differential operators with spectral parameter in the boundary conditions have been considered in various formulations by many authors (see, e.g., [2-9, 12, 17, 18, 20-23, 25-30, 32-36]). In [2, 5, 7, 20-23, 28, 30, 32, 33, 36] studied the basis property in various function spaces

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of the system of root functions of the Sturm-Liouville problem with spectral parameter in the boundary conditions. Problem (1)-(4) was considered in [3, 9, 25-27, 29] for $a_1 = 0$, in [2] for $a_2 = 0$, and in [6] for $a_1 > 0$, $a_2 < 0$. In these papers the oscillation properties of eigenfunctions (and their derivatives) were investigated. Moreover, in [3, 4, 6, 26, 27] the basis properties of the system of root functions in $L_p(0,1)$, $1 < p < \infty$, also studied, necessary and sufficient conditions for the basicity of subsystems of root functions is obtained.

Note that the signs of the parameters a_1 and a_2 play an important role. If $a_1 > 0$ and $a_2 < 0$, then problem (1)-(4), can be treated as a spectral problem for a self-adjoint operator in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$. If $a_1 < 0$ and $a_2 < 0$, then this problem is equivalent to a spectral problem for the self-adjoint operator in the Pontryagin space $\Pi_1 = L_2(0,1) \oplus \mathbb{C}^2$ with the corresponding inner product (e.g., see [4, 7, 13, 17, 32]). In the case $a_1 > 0$ and $a_2 < 0$ all eigenvalues of problem (1)-(4) are positive and simple. But in the case $a_1 < 0$ and $a_2 < 0$ we show that problem (1)-(4) have one negative and simple eigenvalue and a sequence of positive and simple eigenvalues tending to infinity.

Throughout what follows we shall assume that the following conditions are fulfilled:

$$a_1 < 0, \; a_2 < 0. \tag{5}$$

In the present paper we study the location of the eigenvalues on the real axis, the structure of the root subspaces, the oscillation properties of the eigenfunctions and their derivatives, and the basis property in the space $L_p(0,1)$, $1 < p < \infty$, of the system of root functions of the boundary value problem (1)-(4) under condition (5).

2. Preliminaries

Consider the boundary condition

$$y'(1)\cos\gamma - y''(1)\sin\gamma = 0, \tag{6}$$

where $\gamma \in [0, \frac{\pi}{2}]$.

Alongside the boundary value problem (1)-(4) we shall consider the spectral problem (1), (2), (6), (4). A more general form of the problem (1), (2), (6), (4) has been considered in [26, 27], where the authors study the oscillation properties of the eigenfunctions and the basis properties of subsystems of root functions in the space L_p, $1 < p < \infty$.

The next theorem is a special case of the central result of [26].

Theorem 2.1. [26, Theorem 5.1] *The eigenvalues of the boundary value problem* (1), (2), (6), (4) *are real, simple and form an infinitely increasing sequence* $\{\lambda_k(\gamma)\}_{k=1}^{\infty}$ *such that* $\lambda_k(\gamma) > 0$ *for all* $k \in \mathbb{N}$. *Moreover, the eigenfunction* $u_k^{(\gamma)}(x)$ *corresponding to the eigenvalue* $\lambda_k(\gamma)$ *has* $k-1$ *simple zeros in the interval* $(0,1)$.

In view of [6, Theorem 3.1] for each fixed $\lambda \in \mathbb{C}$ there exists a unique (up to a constant factor) nontrivial solution $y(x,\lambda)$ of problem (1), (2), (4). The solution $y(x,\lambda)$ for each fixed $x \in [0,1]$ is an entire function of λ.

Let $\mathcal{A}_k = (\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$, where $\lambda_0(0) = -\infty$.

It is obvious that the eigenvalues $\lambda_k(0)$ and $\lambda_k(\pi/2)$, $k \in \mathbb{N}$, of the boundary value problem (1), (2), (6), (4) for $\gamma = 0$ and $\gamma = \pi/2$ are zeros of the entire functions $y'(1,\lambda)$ and $y''(1,\lambda)$, respectively. We observe that the function

$$F(\lambda) = y''(1,\lambda)/y'(1,\lambda)$$

is will defined for

$$\lambda \in \mathcal{A} \equiv \left(\bigcup_{k=1}^{\infty} \mathcal{A}_k\right) \bigcup (\mathbb{C} \backslash \mathbb{R}),$$

and is meromorphic function of finite order, $\lambda_k(\pi/2)$ and $\lambda_k(0)$, $k \in \mathbb{N}$, are the zeros and poles of this function, respectively.

In Eq. (1) we set $\lambda = \rho^4$. As is known (see [31, Ch. 2, §4.5, Theorem 1]), in each subdomain T of the complex ρ-plane Eq. (1) has four linearly independent solutions $z_k(x,\rho)$, $k = 1, 2, 3, 4$, which are regular with respect to ρ (for sufficiently large ρ) and satisfy the relations

$$z_k^{(s)}(x,\rho) = (\rho\omega_k)^s\, e^{\rho\omega_k x}\left[1 + O\left(1/\rho\right)\right],\ k = 1, 2, 3, 4,\ s = 0, 1, 2, 3, \tag{7}$$

where ω_k, $k = 1, 2, 3, 4$, are the distinct fourth roots of unity.

We shall seek the solution $y(x,\lambda)$ in the following form:

$$y(x,\lambda) = \sum_{k=1}^{4} C_k z_k(x,\rho),$$

where C_k, $k = 1, 2, 3, 4$, are constants depending only on λ. Taking into account (7) and boundary conditions (2) and (4), we obtain for large $|\lambda|$ the asymptotic estimate

$$\begin{aligned} y\,(x,\lambda) = (\sin\rho x - \cos\rho x\ +\ e^{-\rho x} - \\ - \sqrt{2}\sin\left(\rho - \pi/4\right) e^{\rho(x-1)}\Big)\left(1 + O\left(\tfrac{1}{\rho}\right)\right). \end{aligned} \tag{8}$$

It follows by (8) that

$$F(\lambda) = \rho\,\frac{\cos\rho\ - \sin\rho}{\cos\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right). \tag{9}$$

By virtue of (9) we obtain the following asymptotic formulae

$$F(\lambda) = \sqrt{2}\,\sqrt[4]{|\lambda|}\,\left(1 + O\left(\frac{1}{\sqrt[4]{|\lambda|}}\right)\right)\ \text{ as }\ \lambda \to -\infty. \tag{10}$$

Hence, in view of (10) we have

$$\lim_{\lambda\to-\infty} F(\lambda) = +\infty\,. \tag{11}$$

Remark 2.2. It follows by (8) that the number of zeros in $(0,1)$ of solution $y(x,\lambda)$ of problem (1), (2), (4) tends to $+\infty$ as $\lambda \to \pm\infty$.

Lemma 2.3. [6, Lemma 3.3, formula (3.5)] *The following formula holds:*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{1}{y'^2(1,\lambda)}\left\{\int_0^1 y^2(x,\lambda)dx - a_2 y^2(1,\lambda)\right\},\ \lambda \in \mathcal{A}. \tag{12}$$

Remark 2.4. It follows by (11) and (12) that $y'(1,\lambda)y''(1,\lambda) > 0$ for $\lambda \in (-\infty, \lambda_1(\pi/2))$.

Lemma 2.5. *The following representation holds:*

$$F(\lambda) = F(0) + \sum_{k=1}^{\infty}\frac{\lambda c_k}{\lambda_k(0)\,(\lambda - \lambda_k(0))}\,, \tag{13}$$

where $c_k = \underset{\lambda=\lambda_k(0)}{res} F(\lambda) > 0$, $k \in \mathbb{N}$.

Proof. The proof of this lemma is similar to that of [8, Propostion 4].

Corollary 2.6. *The function $F(\lambda)$ is concave in the interval $\mathcal{A}_1$.*

Proof. By differentiating the right-hand side of relation (13) with respect to λ, we obtain

$$F'(\lambda) = -\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^2},\; F''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^3}\,.$$

which implies that $F''(\lambda) < 0$ for $\lambda \in (-\infty,\, \lambda_1(0)) = \mathcal{A}_1$. The proof of Corollary 2.1 is complete.

By $\tau(\lambda)$ and $s(\lambda)$ we denote the number of zeros in the interval $(0,1)$ of functions $y(x,\lambda)$ and $y'(x,\lambda)$, respectively.

Lemma 2.7. [6, Theorem 3.2] (see also [3, Lemma 2.11]) *If $\lambda \in (0, \lambda_1(0)]$, then $\tau(\lambda) = s(\lambda) = 0$, if $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ and $k \geq 2$, then $\tau(\lambda) = k-2$ or $\tau(\lambda) = k-1$, if $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ and $k \geq 2$, then $\tau(\lambda) = k-1$, if $\lambda \in (\lambda_{k-1}(0), (\lambda_k(0)]$ and $k \geq 2$, then $s(\lambda) = k-1$.*

3. Main properties of eigenvalues of problem (1)-(4)

Lemma 3.1. *All eigenvalues of the boundary value problem* (1)-(4) *are real.*

Proof. It is easy to see that the eigenvalues of problem (1)-(4) are the roots of the equation

$$y''(1,\lambda) - a_1\lambda\, y'(1,\lambda) = 0. \tag{14}$$

If λ is a nonreal eigenvalue of problem (1)-(4), then $\bar{\lambda}$ is also eigenvalue of this problem, because the coefficients $q(x)$, a_1, a_2 are real. In this case $y(x,\bar{\lambda}) = \overline{y(x,\lambda)}$, so that if equality (14) holds for λ, then it also holds for $\bar{\lambda}$.

By virtue of (1) we have

$$(Ty(x,\mu))'\, y(x,\lambda) - (Ty(x,\lambda))'\, y(x,\mu) = (\mu - \lambda) y(x,\mu) y(x,\lambda).$$

Integrating this relation from 0 to 1 (using the formula for the integration by parts) and taking into account boundary conditions (2) and (4) we obtain

$$-y''(1,\mu)y'(1,\lambda) + y''(1,\lambda)y'(1,\mu) = (\mu - \lambda)\left\{\int_0^1 y(x,\mu)\, y(x,\lambda)dx - a_2 y(1,\mu)\, y(1,\lambda)\right\}. \tag{15}$$

Setting $\mu = \bar{\lambda}$ in (15), we have

$$-\,\overline{y''(1,\lambda)}\, y'(1,\lambda) + y''(1,\lambda)\, \overline{y'(1,\lambda)} \;=\; (\bar{\lambda} - \lambda)\left\{\int_0^1 |y(x,\lambda)|^2 dx - a_2 |y(1,\lambda)|^2\right\}. \tag{16}$$

By virtue of (3) from (16) we get

$$-\,a_1\,(\bar{\lambda} - \lambda)\,|y'(1,\lambda)|^2 = (\bar{\lambda} - \lambda)\left\{\int_0^1 |y(x,\lambda)|^2 dx - a_2 |y(1,\lambda)|^2\right\}.$$

Since $\bar{\lambda} \neq \lambda$, it follows that

$$\int_0^1 |y(x,\lambda)|^2 dx + a_1\, |y'(1,\lambda)|^2 - a_2|y(1,\lambda)|^2 = 0. \tag{17}$$

In the other hand multiplying both sides of equation (1) to $\overline{y(x,\lambda)}$, and integrating resulting equality from 0 to 1, using the formula of integration by parts and taking into account conditions (2)-(4) we find

$$\int_0^1 |y''(x,\lambda)|^2 dx + \int_0^1 q(x)|y'(x,\lambda)|^2 dx = \lambda \left\{ \int_0^1 |y(x,\lambda)|^2 dx + a_1|y'(1,\lambda)|^2 - a_2|y(1,\lambda)|^2 \right\} \tag{18}$$

By (17) from (18) we obtain

$$\int_0^1 |y''(x,\lambda)|^2 dx + \int_0^1 q(x)|y'(x,\lambda)|^2 dx = 0.$$

which implies (by (2)) that $y(x,\lambda) \equiv 0$. The resulting contradiction shows that the eigenvalues of problem (1)-(4) are real. The proof of this lemma is complete.

Lemma 3.2. *The eigenvalues of the boundary value problem* (1)-(4) *are simple and form an at most countable set without finite limit point.*

Proof. The entire function occurring on the left-hand side in equation (14) does not vanish for non-real λ. Consequently, it does not vanish identically. Therefore, its zeros form an at most countable set without finite limit point.

Let us show that Eq. (14) has only simple roots. Indeed, if $\lambda = \tilde{\lambda}$ is a multiple root of (14), then

$$y''(1,\tilde{\lambda}) - a_1\tilde{\lambda}\, y'(1,\tilde{\lambda}) = 0, \tag{19}$$

$$\frac{\partial y''(1,\tilde{\lambda})}{\partial \lambda} - a_1 y'(1,\tilde{\lambda}) - a_1\tilde{\lambda}\, \frac{\partial y'(1,\tilde{\lambda})}{\partial \lambda} = 0. \tag{20}$$

Dividing both sides of relation (15) by $\mu - \lambda$ ($\mu \neq \lambda$) and passing to the limit as $\mu \to \lambda$ we obtain

$$-\frac{\partial y''(1,\lambda)}{\partial \lambda}\, y'(1,\lambda) + y''(1,\lambda)\, \frac{\partial y'(1,\lambda)}{\partial \lambda} = \int_0^1 y^2(x,\lambda)dx - a_2 y^2(1,\lambda). \tag{21}$$

Setting $\lambda = \tilde{\lambda}$ in equality (21), we have

$$-\frac{\partial y''(1,\tilde{\lambda})}{\partial \tilde{\lambda}}\, y'(1,\tilde{\lambda}) + y''(1,\tilde{\lambda})\, \frac{\partial y'(1,\tilde{\lambda})}{\partial \tilde{\lambda}} = \int_0^1 y^2(x,\tilde{\lambda})dx - a_2 y^2(1,\tilde{\lambda}). \tag{22}$$

Taking (19) and (20) into account from (22) we obtain

$$\int_0^1 y^2(x,\tilde{\lambda})dx - a_2 y^2(1,\tilde{\lambda}) + a_1 y'^2(1,\tilde{\lambda}) = 0. \tag{23}$$

In the other hand, since $\tilde{\lambda}$ is a real eigenvalue it follows from (18) that

$$\int_0^1 y''^2(x,\tilde{\lambda})dx + \int_0^1 q(x)y'^2(x,\tilde{\lambda})dx = \tilde{\lambda} \left\{ \int_0^1 y^2(x,\tilde{\lambda})dx - a_2 y^2(1,\tilde{\lambda}) + a_1 y'^2(1,\tilde{\lambda}) \right\}. \tag{24}$$

Hence, by virtue of (23) and (24), we have

$$\int_0^1 y''^2(x,\tilde{\lambda})dx + \int_0^1 q(x)y'^2(x,\tilde{\lambda})dx = 0$$

which implies (by (2)) that $y(x,\tilde{\lambda}) \equiv 0$. The resulting contradiction completes the proof of Lemma 3.2.

By (2) from (18) follows directly the following assertion.

Lemma 3.3. *$\lambda = 0$ is not an eigenvalue of the boundary value problem* (1)-(4)*.*

By virtue of Property 1 in [16] and formula (12), we have

$$\lambda_1\left(\frac{\pi}{2}\right) < \lambda_1(0) < \lambda_2\left(\frac{\pi}{2}\right) < \lambda_2(0) < \ldots . \tag{25}$$

Remark 3.4. If λ is an eigenvalue of problem (1)-(4), then by relation (25), we have $y'(1,\lambda) \neq 0$.

By virtue of Remark 3.4, each root (with regard of multiplicities) of equation (14) is a root of the equation

$$F(\lambda) = a_1\lambda \tag{26}$$

as well.

Lemma 3.5. *The spectral problem* (1)-(4) *can has only one eigenvalue in each interval $\mathcal{A}_k$, $k = 2, 3, 4, \ldots$.*

Proof. Let $\tilde{\lambda} \in \mathcal{A}_{k_0}$ is an eigenvalue of problem (1)-(4) for some $k_0 \in \mathbb{N}\backslash\{1\}$. Then it follows from (24) that

$$\int_0^1 y^2(x,\tilde{\lambda})\,dx - a_2y^2(1,\tilde{\lambda}) + a_1y'^2(1,\tilde{\lambda}) > 0.$$

Using formula (12) from this relation we obtain

$$\frac{d}{d\lambda}\left(F(\lambda) - a_1\lambda\right)\Big|_{\lambda=\tilde{\lambda}} < 0.$$

Since $F(\tilde{\lambda}) - a_1\tilde{\lambda} = 0$ it follows from this inequality that the function $F(\lambda) - a_1\lambda$ takes zero value only strictly decreasing in the interval $\mathcal{A}_{k_0}$. Consequently, equation (26) has a unique solution $\tilde{\lambda}$ in the interval $\mathcal{A}_{k_0}$. The proof of Lemma 3.5 is complete.

4. Oscillatory properties of eigenfunctions of the boundary problem (1)-(4)

Lemma 4.1. *Let $y(x,\lambda)$ is a solution of problem* (1), (2), (4) *and $\lambda < 0$. Then the function $y(x,\lambda)$ has no multiple roots in the interval* $(0,1)$*.*

Proof. We suppose that $x_0 \in (0,1)$ and $\lambda_0 < 0$ such that $y(x_0,\lambda_0) = y'(x_0,\lambda_0) = 0$. Then the function $y(x,\lambda_0)$ solves the Eq. (1) in $(0,x_0)$ with the boundary conditions (2) and $y(x_0) = y'(x_0) = 0$ which contradicts the condition $\lambda_0 < 0$ in view of [16, Theorem 5.4]. The proof of this lemma is complete.

In view of Remarks 2.2 and 2.4, as $\lambda < 0$ varies, the functions $y(x,\lambda)$ and $y'(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ only through the endpoint $x = 0$. If these zeros pass through the point $x = 0$, then $x = 0$ would be a triple zero of function $y(x, \lambda)$, i.e. $y(0, \lambda) = y'(0, \lambda) = y''(0, \lambda) = 0$.

Let $\lambda < 0$ and μ is a real eigenvalue of the following boundary value problem

$$\begin{gathered} \ell(y)(x) = \lambda y(x), \; x \in (0,1), \\ y(0) = y'(0) = y''(0) = Ty(1) - a_2 \lambda y(1) = 0. \end{gathered} \tag{27}$$

The oscillation index of this eigenvalue is the difference between the number of zeros of the solution $y(x, \lambda)$ of the problem (1), (2), (4) for $\lambda = \mu - 0$ belonging to the interval $(0, 1)$ and the number of the same zeros for $\lambda = \mu + 0$ [9]. From this definition, it directly follows that the number of zeros of the function $y(x, \lambda)$ belonging to the interval (0, 1) is equal to the sum of the oscillation indices of all eigenvalues of problem (27) belonging to the interval $(\lambda, 0)$.

Lemma 4.2. *Then there exists $\zeta < 0$ such that the eigenvalues μ_k, $k = 1, 2, \ldots$, of problem* (27) *lying on the ray $(-\infty, \zeta)$ and enumerated in the decreasing order are simple, admit the asymptote*

$$\mu_k = -4\left(k\pi + \frac{\pi}{2}\right)^4 + o(k^4).$$

and have oscillation index 1.

Proof. The proof of this lemma is similar to that of [9, Theorem 4.1] by using asymptotic formula (8).

Let $\lambda < 0$ and $i(\mu_k)$ be the oscillation index of the eigenvalue μ_k, $k \in \mathbb{N}$ of problem (27). Then, by condition (2), it follows from the above consideration that

$$s(\lambda) = \tau(\lambda) = \sum_{\mu_k \in (\lambda, 0)} i\,(\mu_k). \tag{28}$$

Theorem 4.3. *There exists an unboundedly increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues of the boundary value problem* (1)-(4)*; moreover, $\lambda_1 < 0$ and $\lambda_k > 0$ for $k \geq 2$. The corresponding eigenfunctions $y_k(x)$, $k = 1, 2, \ldots$ and their derivatives have the following oscillation properties: the functions $y_k(x)$ and $y_k'(x)$ for $k \geq 3$ have exactly $k - 2$ simple zeros, for $k = 2$ have no zeros and for $k = 1$ have $\sum\limits_{\mu_k \in (\lambda_1, 0)} i\,(\mu_k)$ simple zeros in the interval* $(0, 1)$.

Proof. Recall [see (26)] that the eigenvalues of problem (1)-(4) are the roots of the equation $F(\lambda) = a_1 \lambda$. Since $a_2 < 0$ it follows from Lemma 2.3 that $F(\lambda) = \frac{y''(1,\lambda)}{y'(1,\lambda)}$ is a continuous decreasing function in the interval $\mathcal{A}_k = (\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$. Taking into account of the relations (11), (12) and the representation (13), we have

$$\lim_{\lambda \to \lambda_{k-1}(0) + 0} F(\lambda) = +\infty,$$

$$\lim_{\lambda \to \lambda_k(0) - 0} F(\lambda) = -\infty\,.$$

Hence the function $F(\lambda)$ assumes each value in $(-\infty, +\infty)$ at a unique point in the interval $\mathcal{A}_k$, $k \in \mathbb{N}$. Moreover, $F(\lambda) > 0$ if $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ and $F(\lambda) < 0$ if $\lambda \in (\lambda_k(\pi/2), \lambda_k(0))$, $k = 1, 2, \ldots$. Since $a_1 < 0$ (see (5)), it follows that the function $G(\lambda) = a_1 \lambda$ is strictly decreasing in the interval $(-\infty, +\infty)$. Consequently, $G(\lambda) > 0$ if $\lambda < 0$ and $G(\lambda) < 0$ if $\lambda > 0$.

By the relations $\lambda_1(\pi/2) > 0$, $F(\lambda_1(\pi/2)) = 0$ and by (12) we have $F(0) > 0$. In view of Corollary 2.6 the function $F(\lambda)$ is concave in $\mathcal{A}_1$.

It follows from the preceding considerations that in the interval $\mathcal{A}_1$ Eq. (26) has two roots $\lambda_1 \in (-\infty, 0)$ and $\lambda_2 \in (\lambda_1(\pi/2), \lambda_1(0))$. Hence, by Lemma 2.7 and formula (28) we have $\tau(\lambda_2) = s(\lambda_2) = 0$ and $\tau(\lambda_1) = s(\lambda_1) = \sum\limits_{\mu_k \in (\lambda_1, 0)} i\,(\mu_k)$. Moreover, by virtue of Lemma 3.5, in the interval A_k, $k = 2, 3, \ldots$, Eq. (26) has unique root $\lambda_{k+1} \in (\lambda_k(\pi/2), \lambda_k(0))$. Hence, it follows from Lemma 2.7 that $m(\lambda_{k+1}) = s(\lambda_{k+1}) = k-1,\ k = 2,3,\ldots$. The proof of Theorem 4.3 is complete.

From [27; § 3, formulas (3.3) and (3.4)] follows the asymptotic formulas

$$\sqrt[4]{\lambda_k(0)} = (k-1/2)\,\pi + O\,(1/k)\,, \tag{29}$$

$$\sqrt[4]{\lambda_k(\pi/2)} = (k-3/4)\,\pi + O\,(1/k)\,, \tag{30}$$

$$y_k^{(0)}(x) = \sin(k-1/2)\pi x - \cos(k-1/2)\pi x + e^{-(k-1/2)\pi x} + (-1)^k e^{-(k-1/2)\pi(1-x)} + O(1/k), \tag{31}$$

$$y_k^{(\pi/2)}(x) = \sin(k-3/4)\pi x - \cos(k-3/4)\pi x + e^{-(k-3/4)\pi x} + O(1/k), \tag{32}$$

where relations (31)-(32) hold uniformly for $x \in [0,1]$.

Theorem 4.4. *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k-3/2)\,\pi + O\,(1/k)\,, \tag{33}$$

$$y_k(x) = \sin(k-3/2)\pi x - \cos(k-3/2)\pi x + e^{-(k-3/2)\pi x} + (-1)^k e^{-(k-3/2)\pi(1-x)} + O(1/k), \tag{34}$$

where relation (34) *holds uniformly for* $x \in [0,1]$.

Proof. The proof of this theorem is similar to that of [27, Theorem 3.1].

5. Basis property of the system of eigenfunctions of the boundary value problem (1)-(4) in the space $L_p(0,1)$, $1 < p < \infty$

In the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with the inner product

$$(\hat{u},\hat{v}) = (\{y,m,n\},\{v,s,t\}) = (y,v)_{L_2} + |a_1|^{-1}m\bar{s} + |a_2|^{-1}n\bar{t}, \tag{35}$$

we define the operator

$$L \hat{=} L\{y,m,n\} = \{(Ty(x))',\ y''(1),\ Ty(1)\},$$

on the domain

$$D(L) = \{\{y\,(x),\ m,\ n\}\ :\ y \in W_2^4(0,1),\ (Ty(x))' \in L_2(0,1),\\ y(0) = y'(0) = 0,\ m = a_1 y'(1),\ n = a_2 y(1)\}$$

dense everywhere in H (see [33, 35]), where $(u,v)_{L_2}$ is an inner product in $L_2(0,1)$ and $W_l^p(0,1)$ is the Sobolev function space having a generalized (in the sense of distributions) lth derivative in $L_p(0,1)$. Obviously, the operator L is well defined in H. Problem (1)-(4) takes the form

$$L\hat{y} = \lambda\hat{y},\ \ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of the operator L and problem (1)-(4) coincide, and between the eigenfunctions, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow\ \{y_k(x),\ m_k,\ n_k\},\ \ m_k = a_1 y_k'(1),\ \ n_k = a_2 y_k\,(1).$$

Since $a_1 < 0$ and $a_2 < 0$, L is a closed (nonself-adjoint) operator in H with compact resolvent. In this case we define an operator $J : H \to H$ as the following:

$$J\{y, m; n\} = \{y, -m, n\}.$$

J is a unitary, symmetric operator on H. Its spectrum consists of two eigenvalues: -1 with multiplicity 1, and $+1$ with infinite multiplicity. This operator generates the Pontryagin space $\Pi_1 = L_2(0, 1) \oplus \mathbb{C}^2$ with inner product [13]

$$[\hat{u}, \hat{v}] = (\hat{u}, \hat{v})_{\Pi_1} = (\{y, m, n\}, \{u, s, t\})_{\Pi_1} = (u, v)_{L_2} + a_1^{-1} m\bar{s} - a_2^{-1} n\bar{t}, \tag{36}$$

Theorem 5.1. *L is J–self-adjoint operator in Π_1.*

Proof. JL is self-adjoint in H by virtue [17, Theorem 2.2]. Then, J-self-adjointness of L on Π_1 follows from [14, Section 3, Proposition 3^0].

Theorem 5.2. *If L^* be the adjoint operator of L in H, then $L^* = JLJ$. The system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k\}$, of the operator L (after normalizing) forms a Riesz basis in H.*

Proof. The proof of the first part of this theorem follows from [14, Section 3, Propostion 5^0] and the second part - from [15].

Each element $\hat{y}_k = \{y_k, m_k, n_k\}$, $k \in \mathbb{N}$, where $m_k = a_1 y_k'(1)$ and $n_k = a_2 y_k(1)$, of the system of root vectors $\{\hat{y}_k\}_{k=1}^{\infty}$ of the operator L satisfies the relation

$$L\hat{y}_k = \lambda_k \hat{y}_k. \tag{37}$$

An element $\hat{v}_k^* = \{v_k^*, s_k^*, t_k^*\}$ of the system of eigenvectors $\{\hat{v}_k^*\}_{k=1}^{\infty}$ of the operator L^* satisfies the relation

$$L^* \hat{v}_k^* = \lambda_k \hat{v}_k^*. \tag{38}$$

By Theorem 5.2 and relations (37) and (38), we have

$$\hat{v}_k^* = J\hat{y}_k,\ k \in \mathbb{N}. \tag{39}$$

Since operator L is J-self-adjoint in Π_1, it follows that the eigenvectors $\hat{y}_k$ and $\hat{y}_l$, $k \neq l$, of this operator corresponding to eigenvalues λ_k and λ_l are J-orthogonal in Π_1; consequently, by (36), we obtain

$$[\hat{y}_k, \hat{y}_l] = 0. \tag{40}$$

Since for each $k \in \mathbb{N}$ the eigenvalue λ_k of operator L is simple it follows by (26) that

$$F'(\lambda_k) - a_1 \neq 0. \tag{41}$$

Using Remark 3.4 from (12) we obtain

$$||y_k||_{L_2}^2 + a_1 y_k'^2(1) - a_2 y_k^2(1) \neq 0.$$

By (36) it follows from this relation that

$$[\hat{y}_k, \hat{y}_k] = ||y_k||_{L_2}^2 + a_1 y_k'^2(1) - a_2 y_k^2(1) \neq 0. \tag{42}$$

As an immediate consequence of (36), (39), (40) and (42) we obtain the following result.

Lemma 5.3. *An element $\hat{v}_k = \{v_k, s_k, t_k\}$ of the system $\{\hat{v}_k\}_{k=1}^{\infty}$ adjoint to the system $\{\hat{y}_k\}_{k=1}^{\infty}$ is given by the formula*

$$\hat{v}_k = \delta_k^{-1} \hat{y}_k,\ k \in \mathbb{N}, \tag{43}$$

where $\delta_k = [\hat{y}_k, \hat{y}_k]$, $k \in \mathbb{N}$.

Let r and l be arbitrary fixed positive integers, and let

$$\Delta_{r,l} = \begin{vmatrix} s_r & s_l \\ t_r & t_l \end{vmatrix}. \tag{44}$$

Theorem 5.4. *If $\Delta_{r,l} \neq 0$, then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,l}^{\infty}$ of problem* (1)-(4) *forms a Riesz basis in the space $L_2(0,1)$; if $\Delta_{r,l} = 0$, then this system is incomplete and nonminimal in the space $L_2(0,1)$.*

Proof. The proof of this theorem follows from [1, Theorems 3.1, 3.2 and Corollary 3.1] on the base of Theorem 5.2.

For brevity, we introduce the notation $\sigma_{r,l} = a_1 a_2 \delta_r^{-1} \delta_l^{-1} y_r'(1) y_l'(1)$. Then, by Remark 3.4 and (43) it follows from (44) that

$$\Delta_{r,l} = \delta_r^{-1}\delta_l^{-1} \begin{vmatrix} m_r & m_l \\ n_r & n_l \end{vmatrix} = \sigma_{r,l} \left\{ \frac{y_r(1)}{y_r'(1)} - \frac{y_l(1)}{y_l'(1)} \right\}. \tag{45}$$

Let

$$\tilde{\Delta}_{r,l} = \left\{ \frac{y_r(1)}{y_r'(1)} - \frac{y_l(1)}{y_l'(1)} \right\}.$$

Then it follows from (45) that

$$\Delta_{r,l} = \sigma_{r,l}\, \tilde{\Delta}_{r,l}. \tag{46}$$

Theorem 5.5. *If $\tilde{\Delta}_{r,l} \neq 0$, then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,l}^{\infty}$ of problem* (1)-(4) *forms a basis in the space $L_p(0,1)$, $1 < p < \infty$ (a Riesz basis for $p = 2$); if $\tilde{\Delta}_{r,l} = 0$, then this system is incomplete and nonminimal in the space $L_p(0,1)$, $1 < p < \infty$.*

Proof. The proof of this theorem is similar to that of [27, Theorem 4.1] by using (46), Theorem 5.4 and asymptotic formulas (29)-(34).

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