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On a Generalized Variational Inequality Problem

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Abstract. In this paper, a sufficient condition in order to have $C-$ pseudomonotone property for multifunctions is presented. By applying a special minimax theorem and KKM theory some existence results of solutions of a generalized variational inequality problem are established. Some examples in order to illustrate the main results are given. The results of this paper can be considered as extension and improvement of some articles in this area.

1. Introduction and Preliminaries

The variational inequality problem with its extensions and its numerous applications has been intensively studied in the last years. Historically, the variational inequality was introduced in [10]. Later it was extended for multifunctions in finite dimensional setting (see [9]) as follows: given a nonempty set $K \subseteq \mathbb{R}^n$ and a multifunction $\Phi : K \to 2^{\mathbb{R}^n}$, with 2^Y denoting the family of all subsets of the set Y, find a vector $\overline{x} \in K$ and $x^* \in \Phi(\overline{x})$ such that

$$\langle x^*, y - \overline{x} \rangle \geq 0, \qquad \forall y \in K,$$

where $\langle ., . \rangle$ indicates the usual product of the space $\mathbb{R}^n$. Such problem is usually referred to as the generalized variational inequality problem associated with K and Φ [in short, $GVI(\Phi, K)$]. Observe that, in most all previous results on the solution existence of $GVI(\Phi, K)$ the continuity of Φ is often required. In fact in order to reduce the continuity on Φ some authors have proposed monotonicity or many kinds of pseudomonotonicity together with upper semicontinuity. In [18] the authors considered the $GVI(\Phi, K)$ in the setting of infinite dimensional case and answered to the question whether the solution existence of $GVI(\Phi, K)$ for $C-$ pseudomonotone operators is still valid if operators are not continuous on finite dimensional subspaces. In this note, by an approach that is completely different from the one used in [18] we extend the main existence result of [18].

The rest of this section will deal with some definitions and basic facts which are needed in the sequel.

Let X be a Hausdorff topological vector space and $X^* = L(X, \mathbb{R})$ the space of all continuous linear mappings from X into the real line and $\langle l, x \rangle$, the evaluation of $l \in X^*$ at $x \in X$. If X^* is equipped with $\sigma-$ topology (see, for instance, [5]) then from the corollary of Schaefer [20], X^* becomes a locally convex

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space. Moreover, by [5] the duality mapping $\langle .,. \rangle : X^* \times X \to X$ is continuous. In this paper we consider $\sigma-$ topology on X^*. Suppose that K is a nonempty convex subset of X and $\Phi : K \to 2^{X^*}$ (the set of all nonempty subsets of X^*) is a multifunction from K to X^*. The graph of Φ is denoted by $Gr(\Phi)$ and is defined by

$$Gr(\Phi) = \{(x, x^*) \in K \times X^* : x^* \in \Phi(x)\}.$$

The generalized variational inequality defined by Φ and K, denoted by $GVI(\Phi, K)$ (see [21, 22]), is the problem of finding a point $x \in K$ and $x^* \in \Phi(x)$ such that

$$\langle x^*, y - x \rangle \geq 0, \qquad \forall y \in K.$$

Definition 1.1. *([1]) A multifunction $\Phi : X \to 2^Y$ between topological spaces is called:*

(a) ***upper semi-continuous*** *(u.s.c.) at $x \in X$ if for each open set V containing $\Phi(x)$, there is an open set U containing x such that for each $t \in U$, $\Phi(t) \subseteq V$; Φ is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.*

(b) ***lower semi-continuous*** *(l.s.c.) at $x \in X$ if for each open set V with $\Phi(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U$, $\Phi(t) \cap V \neq \emptyset$; Φ is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.*

(c) ***continuous*** *on X if it is at the same time u.s.c. and l.s.c. on X.*

(d) ***closed*** *if the graph $Gr(\Phi)$ of Φ, i.e., $\{(x, y) : x \in X, y \in \Phi(x)\}$, is a closed set in $X \times Y$.*

(e) ***compact*** *if the closure of the range, i.e., $\overline{\Phi(X)}$, is compact, where $\Phi(X) = \cup_{x \in X} \Phi(x)$.*

(f) ***transfer closed*** *if $\cap_{x \in X} \overline{\Phi(x)} = \cap_{x \in X} \Phi(x)$.*

(g) ***intersectionally closed*** *on $A \subseteq X$ if $\cap_{x \in A} \overline{\Phi(x)} = \overline{\cap_{x \in A} \Phi(x)}$.*

We need the following facts in the next section.

Theorem 1.2. *([1, Theorem 16.11]) Let X and Y be two topological spaces. Then an upper semicontinuous multifunction $\Phi : X \to 2^Y$ is closed if either:*
(a) Φ is closed-valued and Y is regular, or
(b) Φ is compact-valued and Y is Hausdorff.

Theorem 1.3. *([1]). For a multifunction $\Phi : X \to 2^Y$ between topological spaces and a point $x \in X$ the following statements are equivalent:*
(a) The multifunction Φ is upper semicontinuous at x and $\Phi(x)$ is compact.
(b) For any net $\{x_i\} \subseteq X$ such that $x_i \to x$ and for every $y_i \in \Phi(x_i)$, there exist $y \in \Phi(x)$ and a subnet $\{y_j\}$ of $\{y_i\}$ such that $y_j \to y$.

Theorem 1.4. *([1, Theorem 2.40]) A real-valued lower (resp. upper) semicontinuous function on a compact space attains a minimum (resp. maximum) value, and the nonempty set of all minimizers (resp. maximizers) is compact.*

Considering several kinds of generalized monotonicity for multifunctions in order to study generalized variational inequality problems is needed. Hence we introduce the following definition which is used in proving our main results.

Definition 1.5. *Let $\Phi : K \to 2^{X^*}$ be a multifunction. We recall that Φ is said to be:*
(a) ***monotone*** *if, for all $(x, x^*), (y, y^*) \in Gr(\Phi)$, one has*

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

(b) ***K-pseudomonotone*** *(in the sense of Karamardian) ([17]) if, for any $(x, x^*), (y, y^*) \in Gr(\Phi)$, the following implication holds:*

$$\langle y^*, x - y \rangle \geq 0 \Rightarrow \langle x^*, x - y \rangle \geq 0.$$

(c) ***B-pseudomonotone*** *(in the sense of Brézis) if for any $u \in K$ and any net $\{u_i\}$ with $u_i \to u$, $u_i^* \in \Phi(u_i)$, and $\limsup\langle u_i^*, u_i - u\rangle \le 0$ then for any $v \in K$ there exists $u_v^* \in \Phi(u)$ such that $\langle u_v^*, u - v\rangle \le \liminf\langle u_i^*, u_i - v\rangle$.*

(d) ***C-pseudomonotone*** *if for any $x, y \in K$ and net $\{x_i\}$ in K with $x_i \to x$,*

$$\sup_{x^*\in\Phi(x_i)} \langle x^*, (1-t)x + ty - x_i\rangle \ge 0, \quad \forall t \in [0,1],\ \forall i \in I$$

implies

$$\sup_{x^*\in\Phi(x)} \langle x^*, y - x\rangle \ge 0.$$

Note that part (c) of the aforementioned definition was first introduced in [4], when X is a Banach space and X is equipped by the weak topology. The name C-pseudomonotone was first given in [15], the notion appears with the name 0-segmentary closed in [7].

It is clear that monotonicity implies K-pseudomonotonicity while it does not imply B-pseudomonotonicity. It is a well known result (see [11]) that if Φ is monotone and hemicontinuous (that is the mapping $t \to \langle\Phi((1-t)x + ty), y - x\rangle$ is continuous at 0^+) then Φ is B-pseudomonotone. There is no relationship between K-pseudomonotonicity and B-pseudomonotonicity without adding some additional conditions. For example in the finite dimensional case it is easy to see that any continuous single-valued mapping $\Phi : K \to X^*$ is B-pseudomonotone, nevertheless it is not necessary to be K-pseudomonotone. It is worth noting that the definition of K-pseudomonotonicity is based upon the algebraic structure while the definition of B-pseudomonotonicity is based on the topological structure. Then if one would like to establish a link between them the space has to have both a linear and a topological structure. Maybe the motivation of introducing the C-pseudomonotonicity was the combination of two definitions. We note that the notion of B-pseudomonotonicity was introduced by H. Brézis [4] and this type of pseudomonotonicity can be used to prove the solvability of nonlinear integral equations and partial differential equations (see, for example, [10]) while K-pseudomonotonicity was introduced by S. Karamardian [17] and has been frequently used in optimization problems.

The following result establishes a link between upper semicontinuity and C-pseudomonotonicity. In fact, the next result provides a sufficient condition under which a multifunction is C-pseudomonotone.

Proposition 1.6. *If $\Phi : X \to 2^{X^*}$ is upper semicontinuous with compact values (with respect to $\sigma-$ topology), then it is C-pseudomonotone.*

Proof. Let $\{x_i\}_{i\in I}$ be a net in X such that $x_i \to x$. Then for all $t \in [0,1]$ and $y \in K$ we have $(1-t)x + ty - x_i \to (1-t)x + ty - x$. If for every $i \in I$, $\sup_{x^*\in\Phi(x_i)} \langle x^*, (1-t)x + ty - x_i\rangle \ge 0$, then by Theorem 1.4 for each $i \in I$ there exists $x_i^* \in \Phi(x_i)$ such that

$$\langle x_i^*, (1-t)x + ty - x_i\rangle = \sup_{x^*\in\Phi(x_i)} \langle x^*, (1-t)x + ty - x_i\rangle \ge 0.$$

It follows from Theorem 1.3 that there exist a subnet $\{x_{i_j}^*\}_j$ of $\{x_i^*\}_i$ and $x^* \in \Phi(x)$ such that $x_{i_j}^* \to x^*$. Hence

$$\langle x_{i_j}^*, (1-t)x + ty - x_{i_j}\rangle \to \langle x^*, (1-t)x + ty - x\rangle$$

$$\Rightarrow \langle x^*, (1-t)x + ty - x\rangle \ge 0 \Rightarrow \sup_{x^*\in\Phi(x)} \langle x^*, (1-t)x + ty - x\rangle \ge 0.$$

Since $t \ge 0$ and $\sup_{x^*\in\Phi(x)} \langle x^*, y - x\rangle \ge 0$ we get that Φ is C-pseudomonotone. This completes the proof. □

The following example shows that the upper semicontinuity in Proposition 1.6 is essential.

Example 1.7. *Take $X = \mathbb{R}$, $K = [0,1]$ and define $\Phi : K \to 2^{\mathbb{R}}$ by*

$$\begin{cases} \{1\}, & x \in \mathbb{Q} \cap K, \\ \{0\}, & x \in \mathbb{Q}^c \cap K, \end{cases}$$

where $\mathbb{Q}$ denotes the set of rational numbers. We claim that Φ is not $C-$ pseudomonotone. Because if we take $x = \frac{1}{10}$, $y = 0$ and $x_n = e - (1 + \frac{1}{n})^n + \frac{1}{10}$ then $\phi(x_n) = 0$, for all n and

$$\sup_{x^* \in \phi(x_n)} \langle x^*, tx + (1-t)y - x_n \rangle = 0, \quad \forall t \in [0,1],$$

while

$$\sup_{x^* \in \Phi(x)} \langle x^*, y - x \rangle < 0.$$

This completes the proof of the claim. It is obvious that Φ is not upper semicontinuous.

The following definition and lemmas play crucial role in the next section.

Definition 1.8. *([8]) Let E be a topological vector space. A mapping $F : M \subseteq E \to 2^E$ is said to be a KKM mapping, if, for any finite set $A \subseteq M$,*

$$coA \subseteq F(A) = \bigcup_{a \in A} F(a),$$

where coA denotes the convex hull of A.

One can verify that the KKM property for a multifunction implies the finite intersection property for the values of the multifunction.

Lemma 1.9. *([8]) Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ be a KKM mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then $\bigcap_{x \in K} F(x) \neq \emptyset$.*

The following lemma is an application of a minimax theorem.

Lemma 1.10. *([3]) Let D be a compact convex and K be a nonempty convex subset of topological vector space X and let $P : D \times K \to \mathbb{R}$ be concave and upper semicontinuous in the first variable and convex in the second variable such that*

$$\max_{\xi \in D} P(\xi, y) \geq 0, \quad \forall y \in K.$$

Then there exists $\bar{\xi} \in D$ such that

$$P(\bar{\xi}, y) \geq 0, \quad \forall y \in K.$$

2. Main Results

In this section we are going to present some existence results of solutions of GVI by applying the definitions and basic results given in the previous section. Moreover some examples in order to illustrate the main results are given. The results of this section can be viewed as a refinement version of the main results given in [9, 15, 18, 21, 22] relaxing $C-$ pseudomonotonicity and using mild assumptions.

Theorem 2.1. *Suppose that $K \subset X$ is a nonempty convex set and $\Phi : K \to 2^{X^*}$ is a multifunction which satisfies the following conditions:*

(a) the set-valued mapping $y \mapsto \{x \in K : \sup_{x^* \in \Phi(x)} \langle x^*, y - x \rangle \geq 0\}$ is transfer closed and intersectionally closed on compact and convex subsets of K ,

(b) there exist a nonempty compact subset $D \subseteq K$ and a nonempty convex compact subset $B \subseteq K$ such that, for each $x \in K \setminus D$, there exists $z \in B$ such that $\sup_{x^* \in \Phi(x)} \langle x^*, z - x \rangle < 0$,
(c) Φ has convex and compact values.
Then there exists $x_0 \in D$ and $x_0^* \in \Phi(x_0)$ such that

$$\langle x_0^*, x - x_0 \rangle \geq 0, \quad \forall x \in K.$$

Proof. Define $\Gamma : K \to 2^K$, as follows:

$$\Gamma(y) = \{x \in K : \sup_{x^* \in \Phi(x)} \langle x^*, y - x \rangle \geq 0\}.$$

We claim that Γ is a *KKM* mapping. Otherwise, there exist a finite set $\{y_1, y_2, ..., y_n\} \subseteq K$ and $z = \sum_{i=1}^n \lambda_i y_i \in co(\{y_1, y_2, ..., y_n\})$ (where $\lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1$) such that $z \notin \bigcup_{i=1}^{n} \Gamma(y_i)$. So

$$\sup_{x^* \in \Phi(z)} \langle x^*, y_i - z \rangle < 0, \ \forall i = 1, 2, ..., n.$$

Hence, for any fixed element $w \in \Phi(z)$, we have $\langle w, y_i - z \rangle < 0$, for all $i = 1, 2, ..., n$. Multiplying by λ_i and adding them we deduce that

$$0 = \langle w, \Sigma_{i=1}^n \lambda_i y_i - z \rangle < 0,$$

which is a contradiction. Then Γ is a *KKM* mapping. Hence it is obvious that the the mapping $\overline{\Gamma} : K \to 2^K$ defined by $y \mapsto \overline{\Gamma(y)}$ is a KKM mapping.

It is clear that $\bigcap_{y \in B} \Gamma(y) \subseteq D$. If we assume on the contrary that $\bigcap_{y \in B} \Gamma(y) \nsubseteq D$, there exists $\xi \in K \setminus D$ such that $\xi \in \Gamma(y)$ for each $y \in B$. By condition (b) there exists $z \in B$ so that $\sup_{x^* \in \Phi(\xi)} \langle x^*, z - \xi \rangle < 0$. Hence $\xi \notin \Gamma(z)$ which is a contradiction. By (b) we have $\bigcap_{y \in B} \overline{\Gamma(y)} \subset D$. Now $\overline{\Gamma}$ satisfies all the conditions of Lemma 1.9 and so the intersection of the family $\{\overline{\Gamma(y)} : y \in K\}$ is nonempty. Hence it follows from (a) that the intersection of the family $\{\Gamma(y) : y \in K\}$ is nonempty. This means that there exists $x_0 \in \bigcap_{y \in K} \Gamma(y) \subseteq \bigcap_{y \in B} \Gamma(y) \subseteq D$. Then

$$\sup_{x^* \in \Phi(x_0)} \langle x^*, y - x_0 \rangle \geq 0, \quad \forall y \in K.$$

Define $P : \Phi(x_0) \times K \to \mathbb{R}$ by

$$P(x^*, y) = \langle x^*, y - x_0 \rangle, \ \ \forall (x^*, y) \in \Phi(x_0) \times K.$$

Since $\Phi(x_0)$ is compact, we have

$$\sup_{x^* \in \Phi(x_0)} P(x^*, y) = \max_{x^* \in \Phi(x_0)} P(x^*, y) \geq 0, \ \ \forall y \in K.$$

Hence by Lemma 1.10 there exists $x_0^* \in \Phi(x_0)$ such that

$$P(x_0^*, y) = \langle x_0^*, y - x_0 \rangle \geq 0, \quad \forall y \in K.$$

This completes the proof. □

The following example shows that Theorem 2.1 is different from Theorem 1.1 in [18] (further from the main theorem given in [9, 15, 21, 22]).

Example 2.2. *Let $X = \mathbb{R}$, $K = [0,1]$ and $\Phi : K \to 2^{\mathbb{R}}$ defined as*

$$\begin{cases} \{1\}, & x = 0; \\ \{0\}, & 0 < x \leq 1. \end{cases}$$

Clearly, for any $y \in K$, the set $\{x \in K : \langle \Phi(x), y - x\rangle = \Phi(x)(y - x) \geq 0\} = K$ which is closed, while if we let $y = -1 \in K - K$, then

$$\{x \in K : \langle \Phi(x), y\rangle = \Phi(x)(y) \geq 0\} =]0,1],$$

which is not closed. This means that the example does not satisfy the condition (iii) of Theorem 1.1 in [18]. The example fulfills all the assumptions of Theorem 2.1 and the solution set of GVI equals to K.

The next example shows that condition (a) of Theorem 2.1 is essential.

Example 2.3. *Let*

$$X = l^2 = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\},\ \|(x_n)\| = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}},$$

and let K be the unit ball of X. Define $\Phi : K \to 2^X$ by

$$\Phi(x = (x_1, x_2, ...)) = \{(\sqrt{1 - \|x\|}, x_1, x_2, ...)\}.$$

If we take $x = \theta = (0,0,...)$, $y = (-1,0,0,...)$ and the unit vectors, for any positive integer number n,

$$e_n = (0,0,...,1\underbrace{{}_{n^{th} - place}},0,0,...),$$

then (e_n) weakly converges to $x = \theta$. Indeed,

$$\langle z, e_n\rangle = z_n \to 0 = \langle x = \theta, z\rangle, \quad \forall z = (z_1, z_2, ...) \in X,$$

(note $z \in X$ and so $z_n \to 0$). It is easy to check that

$$e_n \in \{z \in K : \sup_{z^* \in \Phi(z)} \langle z^*, (-1,0,0,\dots) - z\rangle \geq 0\} := \Gamma(-1,0,0,\dots)$$

which means that θ belongs to the weak closure of the set $\Gamma(-1,0,0,\dots)$. But $\theta = (0,0,\dots) \notin \Gamma(-1,0,0,\dots)$, so the set $\Gamma(-1,0,0,\dots)$ is not weakly closed.

It can be proved that condition (a) of Theorem 2.1 is not valid. Indeed, we have that $\theta \notin \Gamma(-1,0,0,\dots)$ and let $y' = (y_1, y_2, \dots)$ be arbitrary fixed in K. Denoting $\sigma(t) = 1$ if $t \geq 0$ and $\sigma(t) = -1$ if $t < 0$, consider the vectors

$$e'_n = (0,0,\dots,\sigma(y_{n+1}),0,\dots)$$

where $\sigma(y_{n+1})$ is on the n-th position. Like above, (e'_n) converges weakly to $x = \theta$. Also it is easy to check that $e'_n \in \Gamma(y')$. This means θ belongs to the weak closure of the set $\Gamma(y')$, so Γ is not transfer closed (in the weak topology) and then condition (a) of Theorem 2.1 is not valid, while it is straightforward to check that the rest of the conditions of Theorem 2.1 are true and the solution set of $GVI(\Phi, K)$ is empty.

One can check that conditions (i) and (iii) in Theorem 1.1 of [18] have been reduced by condition (a) in Theorem 2.1. Moreover, verifying condition (a) in Theorem 2.1. is easier than checking conditions (i) and (iii) Theorem 1.1 of [18]. For instance, if we take

$$X = l^2 = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\},\ \|(x_n)\| = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}},$$

K the unit ball of X, and define $\Phi : K \to 2^X$ by

$$\begin{cases} \{\theta\}, & x = \theta; \\ \{(1,0,0,...)\}, & x \neq \theta. \end{cases}$$

then, for all $y = (y_1, y_2, ...) \in K$, we have

$$\begin{aligned} \{x = (x_1, x_2, ...) \in K : \sup_{x^* \in \Phi(x)} \langle x^*, y - x \rangle \geq 0\} &= \{\theta\} \cup \{x = (x_1, x_2, ...) : y_1 - x_1 \geq 0\} \\ &= \{\theta\} \cup \{x = (x_1, x_2, ...) : y_1 \geq x_1\}, \end{aligned}$$

which is closed (note the mapping $(x_1, x_2, ...) \to x_1$ is weakly continuous). This means condition (a) of Theorem 2.1 holds.

Remark 2.4. *(a) One can omit condition (a) in Theorem 2.1 when the mapping Φ is upper semicontinuous. Furthermore in this case the solution set of $GVI(\Phi, K)$ is compact and Γ is closed valued. For this, let $(x_i)_{i \in I}$ be a convergent net to $z \in K$ such that*

$$x_i \in \Gamma(y) = \{x \in K : \sup_{x^* \in \Phi(x)} \langle x^*, y - x \rangle \geq 0\}$$

We have to show that $z \in \Gamma(y)$. To see this, since $\Phi(x)$ is a compact set and $x_i \in \Gamma(y)$ for all $i \in I$, there is $x_i^* \in \Phi(x_i)$ such that

$$\sup_{x^* \in \Phi(x_i)} \langle x^*, y - x_i \rangle = \langle x_i^*, y - x_i \rangle \geq 0, \tag{1}$$

and so by Theorem 1.3 there exist a subnet $(x_{i_j}^*)_j$ of $(x_i^*)_i$ and $w \in \Phi(z)$ such that $x_{i_j}^* \to w$. From this and $x_{i_j} \to z$, through continuity of the duality pairing between X and X^* (it is easy, by using Theorem 2.5 in [19], to see that the duality pairing is a continuous mapping) and (1) we get $\langle w, y - x \rangle \geq 0$ and hence the proof of the claim is completed. The compactness of the solution set of $GVI(\Phi, K)$ follows from Theorems 1.3, 1.4 and condition (b) of Theorem 2.1.

(b) Notice that for fixed $y \in K$,

$$\Gamma(y) = \{x \in K : \sup_{x^* \in \Phi(x)} < x^*, y - x > \geq 0\} = \{x \in K : \inf_{x^* \in \Phi(x)} < x^*, x - y > \leq 0\}$$

Now if we define $u_y : X \times X^* \longrightarrow \mathbb{R}$ as $u_y(x, x^*) = < x^*, x - y >$ and

$$\Gamma(y) = \{x \in K : \; V(x) \geq 0\}$$

where $V : X \to \mathbb{R}$ is defined by $V(x) = \inf_{x^* \in \Phi(x)} u_y(x, x^*)$, with the assumption of continuity of Φ and Berge's Maximum Theorem [6] we have that $\Gamma(y)$ is closed.
(c) It follows from the proof of Theorem 2.1 that if we omit convexity of the values of Φ then we get an existence theorem for the following problem:

$$\text{Find} \quad \overline{x} \in K : \sup_{x^* \in \Phi(\overline{x})} \langle x^*, z - x \rangle \geq 0, \quad \forall z \in K,$$

which has been studied by Yao and Guo in [22] for finite dimension spaces. Hence one can consider Theorem 2.1 is a generalized version of it with mild assumptions.

(d) The following coercivity condition given in Theorem 1.1 of [18] (in fact, condition (ii) of Theorem 1.1 in [18]): "There exist weakly compact subsets B_0, B_1 of K, where $B_0 \subseteq B_1$ and B_0 lies in a finite dimensional subspace, such that for every $x \in K \backslash B_1$ there exists $z \in B_0$ satisfying

$$\sup_{x^* \in \Phi(x)} \langle x^*, z - x \rangle < 0.\text{''} \quad (**)$$

is a special case of condition (b) of Theorem 2.1. Because it is a well known result that the closure of the convex hull of a compact subset of a finite dimensional space is compact (see [19]) and so the condition (b) of Theorem 2.1 will be satisfied by taking $B = \overline{coB_0}$ and $D = B_1$ automatically satisfies in condition (b) of Theorem 2.1 and it can be removed.

We can summarize the above notes in an existence theorem for GVI(Φ, K) as follows.

Theorem 2.5. *Suppose that $K \subset X$ is a nonempty convex set and $\Phi : K \to 2^{X^*}$ is upper semicontinuous with compact and convex values. If Φ fulfils in (b) of Theorem 2.1 then the solution set of $GVI(\Phi, K)$ is nonempty and compact.*

The next result is an application of Theorem 2.5 which improves Theorem 15 of [15].

Theorem 2.6. *Suppose that $K \subset \mathbb{R}^n$ is a nonempty convex set and $\Phi : K \to 2^{X^*}$ is upper semicontinuous with compact values. If Φ satisfies in (b) of Theorem 2.1 then the solution set of $GVI(\Phi, K)$ is nonempty and compact.*

The following theorem is an application of Theorem 2.6 which extends the corresponding result given in [18, 21, 22].

Theorem 2.7. *Suppose $K \subset X$ is a nonempty convex set and for each finite dimensional subspace N of X, $\Phi_N : K \cap N \to 2^{X^*}$ is upper semicontinuous with compact values. Suppose also that there exist a non-empty compact subset $D_N \subseteq K \cap N$ and a non-empty convex compact subset $B_N \subseteq K \cap N$ such that, for each $x \in K \cap N \setminus D_N$, there exists $z \in B_N$ such that $\sup_{x^* \in \phi(x)} \langle x^*, z - x\rangle < 0$. Then the solution set of $GVI(\Phi, K)$ is nonempty .*

Proof. Let $F(X)$ denote the set of all finite dimensional linear subspaces of X. By Theorem 2.6, the solution set of the $GVI(K \cap N, \Phi)$ (we denote by $S(N)$) is nonempty and compact. Hence, since

$$\bigcap_{i=1}^{m} S(N_i) \supseteq S(\bigcap_{i=1}^{m} N_i) \neq \emptyset,$$

where $\{N_i | i = 1, 2, ..., m\} \subset F(X)$, the family $\{S(N) : N \in F(X)\}$ has the finite intersection property and then there exists $\overline{x} \in K$ such that

$$\overline{x} \in \bigcap_{N \in F(X)} S(N).$$

Now if $y \in K$ then $\overline{x} \in S(\langle\{\overline{x}, y\}\rangle)$ where $\langle\{\overline{x}, y\}\rangle$ is the linear space generated by $\{\overline{x}, y\}$, and so there exists $x^*_y \in \Phi(\overline{x})$ such that $\langle x^*_y, y - \overline{x}\rangle \geq 0$. Now the rest of proof is similar to the proof given for Theorem 2.1. □

Similar conditions like in Theorem 2.1 could be used to obtain existence results for some other variational problems, for instance for problems of the sort treated in [2], [12–14], [16].

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