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Spectral Properties of the iterated Laplacian with a potential in a Punctured Domain

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Abstract. In the work we derive regularized trace formulas which were established in papers of Kanguzhin and Tokmagambetov for the Laplace and m-Laplace operators in a punctured domain with the fixed iterating order $m \in \mathbb{N}$. By using techniques of Sadovnichii and Lyubishkin, the authors in that papers described regularized trace formulae in the spatial dimension $d = 2$. In this note one claims that the formulas are also true for more general operators in the higher spatial dimensions, namely, $2 \leq d \leq 2m$. Also, we give the further discussions on a development of the analysis associated with the operators in punctured domains. This can be done by using so called 'nonharmonic' analysis.

1. Introduction

In the remark we investigate a class of elliptic differential equations in a punctured domain. For general motivation, we refer to the papers [1, 3, 4, 9, 11, 12, 18, 19] and references therein, where different differential operators with δ–like potentials are studied, and spectral properties, that is, formulas for the regularized traces and resolvents are given.

In this paper we observe that the results of the work [7] are valid, even when there is a potential and, the spatial dimension is greater than two.

Let $D \subset \mathbb{R}^d$ be a simply connected domain with the smooth boundary ∂D. Denote by $s = (s_1, \ldots, s_d)$ a fixed point of the domain D. Then we define a punctured domain $D_0 := D \setminus \{s\}$. During this manuscript, we study the differential expression

$$(-\Delta)^m u + qu \tag{1}$$

with real valued potential q in a punctured domain D_0. Here

$$(-\Delta)^m u := \left(- \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \right)^m .$$

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We assume that the operator corresponding to the equation (1) with the Dirichlet boundary condition on the "whole" domain D has only discrete spectrum.

Since D_0 is not simply connected, we need a special functional space for (1) to define an operator correctly. For this, we introduce the functional class $\mathcal{F}_m$ that can be represented in the following form

$$w(x) = w_0(x) + kG_m(x,s), \tag{2}$$

where k is some constant. The function w_0 is from the functional space $\mathbb{F}_m$ which is consisted of the functions $v \in \mathrm{H}^{2m}(D)$ such that

$$\left(\frac{\partial}{\partial n}\right)^j v\Big|_{\partial D} = 0, \tag{3}$$

for all $j = 0, \dots, m-1$, where $\frac{\partial}{\partial n}$ is the outer normal derivative. Here H^q stands for the usual Sobolev space with the parameters $(2, q)$, and $G_m(x,s)$ is the Green function of the Dirichlet problem for the equation (1) in the whole domain D with the boundary conditions (3).

Now, we define a functional for our further investigations. To this, we consider the parallelled

$$\Pi_{s,\delta} = \{x : -\delta \le |x-s| \le \delta\}.$$

Then for the function h from the space $\mathcal{F}_m$ defined as (2) we introduce the following functional

$$\alpha_m(h) = \lim_{\delta \to +0} \int\limits_{\partial \Pi_{s,\delta}} \left[\frac{\partial(-\Delta)^{m-1}h(\xi)}{\partial n_\xi}\right] ds_\xi. \tag{4}$$

Remark 1.1. *We note that the functional (4) is defined for all $d \in \mathbb{N}$. Moreover, the value of α_m from the function $G(x,s)$ exists.*

For our convenience, we denote

$$\gamma := \alpha_m(G(\cdot,s)), \quad \alpha(\cdot) := \frac{1}{\gamma}\alpha_m(\cdot),$$

and

$$\xi^-(w) := \alpha(w), \quad \xi^+(w) := w_0(s).$$

2. Main Results

In this section we repeat the results of the paper [7]. However, here we formulate them also for the case $d \le 2m$.

Now, we are in a way in the Hilbert space $\mathrm{H}^2(D)$ to introduce an operator associated with the differential equation (1), that is, $(-\Delta)^m u + qu$. We denote by $\mathcal{K}_M$ the operator defined as

$$\mathcal{K}_M u = (-\Delta)^m u + qu,$$

in the punctured domain D_0 for all functions $u \in \mathcal{F}_m$. Assign $\mathcal{K}_m$ as the restriction of the operator $\mathcal{K}_M$ to

$$D(\mathcal{K}_m) = \{u | u \in \mathcal{F}_m, \ \xi^-(u) = 0, \xi^+(u) = 0\}.$$

Discussing as in the works [5, 7, 8], we get the following statements:

Proposition 2.1. *Let $d \le 2m$. Assume that $u, v \in \mathcal{F}_m$. Then, we have*

$$< \mathcal{K}_M u, v > = < u, \mathcal{K}_M v > + \xi^-(u)\xi^+(v) - \xi^-(v)\xi^+(u).$$

Moreover, the operator $\mathcal{K}_\theta$ defined on $\mathcal{F}_m$ by the expression

$$(-\Delta)^m u + qu = f,$$

in the punctured domain D_0 with the condition

$$\theta_1 \xi^-(u) = \theta_2 \xi^+(u) \tag{5}$$

is a self-adjoint extension of $\mathcal{K}_m$ in the functional space $\mathcal{F}_m$. Here $\theta = (\theta_1, \theta_2)$, $\theta_1, \theta_2 \in \mathbb{R}$ with the property $\theta_1^2 + \theta_2^2 \neq 0$.

In the Hilbert space $\mathrm{H}^2(D)$ consider the operator

$$\mathcal{K}_Q u(x) := [(-\Delta)^m + q]\, u(x), \quad x \in D_0 \tag{6}$$

on $u \in \mathcal{F}_m$ with

$$\alpha(u) + \int\limits_D Q(x)([(-\Delta)^m + q]\, u_0)(x)dx = 0, \tag{7}$$

where $Q \in \mathrm{H}^2(D)$. Here we can write

$$\int\limits_D Q(x)([(-\Delta)^m + q]\, u_0)(x)dx =: \langle Q, [(-\Delta)^m + q]\, u_0 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product of $\mathrm{H}^2(D)$.

Now, we consider the operator $\mathcal{K}_Q$ as a perturbation of $\mathcal{K}_0$. Here $\mathcal{K}_0$ stands for the Dirichlet problem for m–Laplace operator in the whole domain D. Then, we assume that $\{\mu_n\}_{n=1}^\infty$ are the eigenvalues of $\mathcal{K}_Q$ ordered in the increasing order of their absolute values taking into account the multiplicities, and suppose that $\{\lambda_n\}_{n=1}^\infty$ are the eigenvalues of $\mathcal{K}_0$ ordered in the increasing order by taking into account their multiplicities.

Theorem 2.2. *Let the spatial dimension $d \leq 2m$. Suppose that $p, \epsilon > 0$ are fixed numbers. Assume that $Q \in D(\mathcal{K}_0^m)$, $\mathcal{K}_0^{m-1} Q \in \mathrm{H}^p(\Pi_{s,\epsilon})$, and $Q(s) \neq -1$. Then, we have the following regularized trace formula*

$$\sum_{n=1}^\infty (\mu_n - \lambda_n) = \frac{\widetilde{Q}(s)}{1 + Q(s)}. \tag{8}$$

Here $\widetilde{Q}(s) = -\lim\limits_{x \to s} \mathcal{K}_0^{m-1} Q(x)$.

The proof of Theorem 2.2 follows directly from the proofs of the main theorems of the papers [7, 17].

2.1. Further development

Finally, we note that Proposition 2.1 implies the following corollary, which gives a way to find out self–adjoint operators from the class of operators $\{\mathcal{K}_Q : Q \in \mathrm{H}^2(D)\}$, namely:

Corollary 2.3. *Suppose that $\theta_1 \neq 0$ and $Q(x) = -\mu G_m(x, s)$ with $\mu = \theta_2/\theta_1$. Then the operator $\mathcal{K}_Q$ is self–adjoint with the parameter (θ_1, θ_2) in the space $\mathcal{F}_m$:*

$$\mathcal{K}_{-\mu G_m} \sim \mathcal{K}_{(1,\mu)} = \mathcal{K}_{(\theta_1,\theta_2)}.$$

Thus, we observe that the class of operators given by the equation (6) and condition (7) has a huge number of self–adjoint operators in a punctured domain. One can be started a 'nonharmonic' analysis connected with the singular, in the above sense, operators. Note, that the nonharmonic analysis is developed in the works [2, 10, 13, 15] with applications given in [14, 16]. For more general setting of the nonharmonic analysis, see for instance [6].

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