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New Characterizations for the Integral-type Operator from BMOA to Bloch-type Spaces

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Abstract. Our aim in this paper is to consider some new characterizations for the boundedness of the integral-type operator T_gC_φ acting from $BMOA(VMOA)$ into Bloch-type spaces and give a brief expression for its essential norm.

1. Introduction

The set of positive integers without the element zero is denoted by $\mathbb{N}$. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$ and $S(\mathbb{D})$ be the set of holomorphic self-maps of $\mathbb{D}$. For $f \in H(\mathbb{D})$ with Taylor expansion $f(z) = \sum_{i=0}^{\infty} a_i z^i$, the Cesáro operator acting on f is

$$C[f](z) = \sum_{i=0}^{\infty}\left(\frac{1}{i+1}\sum_{k=0}^{i} a_k\right) z^k.$$

There are many papers studied the operator $C[.]$ acting on various spaces of analytic functions including the Hardy space [20] and Bloch space [12, 25]. Now the extended Cesáro operator T_g is defined by

$$T_g(f)(z) = \int_0^z f(t)g'(t)dt$$

acting on function $f \in H(\mathbb{D})$. When $g(z) = z$ or $g(z) = \log\left(\frac{1}{1-z}\right)$, T_g is the integral operator or the Cesáro operator, respectively.

For $\varphi \in S(\mathbb{D})$, the composition operator C_φ is defined as $C_\varphi(f) = f \circ \varphi,\ f \in H(\mathbb{D})$. The study of composition operators is a fairly active field. For general references on the theory of composition operators,

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see the two books [6] and [22]. In this paper, we consider the integral-type operator

$$T_gC_\varphi(f)(z) = \int_0^z (f\circ\varphi)(\zeta)g'(\zeta)d\zeta,\ \ f \in H(\mathbb{D}),\ \varphi \in S(\mathbb{D}).$$

We refer the interested readers to the paper [16] to know more about the operator.

The weighted Banach spaces of analytic functions is defined by

$$H^\infty_{\nu_\alpha} := \{f \in H(\mathbb{D}) :\ \|f\|_{\nu_\alpha} := \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty\}$$

endowed with the norm $\|.\|_{\nu_\alpha}$. For a weight ν the associated weight $\tilde{\nu}(z)$ is defined by

$$\tilde{\nu}(z) := \Big(\sup\{|f(z)| :\ f \in H^\infty_\nu, \|f\|_\nu \le 1\}\Big)^{-1},\ z \in \mathbb{D}.$$

For the standard weights ν_α, it is well known that its associated weight is $\tilde{\nu}_\alpha(z) = \nu_\alpha(z)$. We also need the weight $v_{\log} = \left(\log\frac{2}{1-|z|^2}\right)^{-1}$ satisfying $\tilde{v}_{\log} = v_{\log}$, too. We refer the interested readers to [14, P39]. Moreover, a weight ν is called radial if $\nu(z) = \nu(|z|), z \in \mathbb{D}$.

For $0 < \alpha < \infty$, an $f \in H(\mathbb{D})$ is said to be in the Bloch-type space $\mathcal{B}^\alpha$, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty,$$

endowed with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

Then the Bloch space $\mathcal{B}$ consists of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_1 := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

For $1 \le p < \infty$, let H^p be the classical Hardy space consisting of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{H^p} = \sup_{r\in(0,1)}\int_0^{2\pi}|f(re^{i\theta})|^p\frac{d\theta}{2\pi} < \infty.$$

The space $BMOA$ consists of all functions $f \in H^2$ such that

$$\|f\|_* = \sup_{a\in\mathbb{D}}\|f\circ L_a - f(a)\|_{H^2} < \infty,$$

where $L_a(z) = \frac{a-z}{1-\bar{a}z}$ for $z \in \mathbb{D}$. The corresponding $f \to \|f\|_*$ is a seminorm and $\|f\|_{BMOA} = |f(0)| + \|f\|_*$ yields a Banach space structure on $BMOA$. As we all know the set of all bounded analytic functions space H^∞ is properly contained in $BMOA$, which is in turn a proper subset of $\mathcal{B}$. That is,

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

In fact, $\|f\|_{\mathcal{B}} \le \|f\|_{BMOA}$. Thus the inclusion of $BMOA$ into $\mathcal{B}$ is continuous. Furthermore, $\|f\|_{\mathcal{B}} \le \|f\|_\infty$ and $\|f\|_* \le 2\|f\|_\infty$ for $f \in H^\infty$, where $\|f\|_\infty$ denotes the supremum norm of f. Moreover, if $f \in BMOA$, then

$$|f(z)| \le |f(0)| + \frac{1}{2}\log\frac{1+|z|}{1-|z|}\|f\|_1 \le |f(0)| + \frac{1}{2}\log\frac{1+|z|}{1-|z|}\|f\|_{BMOA} \tag{1}$$

The closed subspace $VMOA$ consists of those $f \in BMOA$ such that $\lim_{|a|\to 1}\|f\circ L_a - f(a)\|_{H^2} = 0$. For more information on the spaces $BMOA$, $VMOA$ and $\mathcal{B}$, we suggest [1, 2, 10].

Very recently, there are many papers about the operators on the space $BMOA$, such as [5], which gives the new characterization for the boundedness of weighted composition operator $W_{\psi,\varphi}: H^\infty \to BMOA$ as follows:

Theorem A Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. The following statements are equivalent:
(a) The operator $W_{\psi,\varphi}: H^\infty \to BMOA$ is bounded.
(b) $M := \sup_{n\in\mathbb{N}\cup\{0\}} \|\psi\varphi^n\|_{BMOA} < \infty$.
(c) $\psi \in BMOA$ and $\sup_{a\in\mathbb{D}} |\psi(a)|\|L_{\varphi(a)} \circ \varphi \circ L_a\|_{H^2} < \infty$.

In particular, it has been shown in [24] that C_φ is compact on $BMOA$ if and only if the single condition $\lim_{n\to\infty} \|\varphi^n\| = 0$ holds. And it has been proved in [15] that this condition is equivalent with $\limsup_{|\varphi(z)|\to 1} \|L_a\|_{H^2} = 0$. In a recent paper [9], it complements the above results by proving estimates for the essential norm $\|C_\varphi\|_{e,BMOA}$ as following:

Theorem B For $\varphi \in S(\mathbb{D})$, we have $\|C_\varphi\|_{e,BMOA} \asymp \limsup_{n\to\infty} \|\varphi^n\|$.

Recently, there have been an increasing interest in new characterizations for the boundedness and compactness of operators, one can refer to [3, 4, 7, 8, 14, 17, 18, 23]. Based on the above results, we continue to investigate the new characterizations for the integral-type operator T_gC_φ acting from $BMOA(VMOA)$ to $\mathcal{B}^\alpha$. The organization of the paper is as follows: section 2 devotes to some lemmas. The boundedness and the estimates for the essential norm of the operator T_gC_φ acting from $BMOA(VMOA)$ to $\mathcal{B}^\alpha$ are given in section 3.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notations $A \asymp B$, $A \preceq B$, $A \succeq B$ mean that there maybe different positive constants C such that $B/C \le A \le CB$, $A \le CB$, $A \ge CB$.

2. Some Lemmas

We will make extensive use of the following lemma when proving our main theorems. This lemma is due to Montes-Rodríguez [21, Theorem 2.1] and Hyvärinen, et al. [13, Theorem 2.4]. For $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the weighted composition operator is defined as $uC_\varphi(f)(z) = u(z)f(\varphi(z))$, $f \in H(\mathbb{D})$. Then we have

Lemma 1. *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then*
(i) *the weighted composition operator uC_φ maps H_v^∞ into H_w^∞ if and only if*

$$\sup_{n\ge 0} \frac{\|u\varphi^n\|_w}{\|z^n\|_v} \asymp \sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\tilde{v}(\varphi(z))} < \infty,$$

with norm comparable to the above supremum.
(ii) $\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \limsup_{n\to\infty} \frac{\|u\varphi^n\|_w}{\|z^n\|_v} = \limsup_{|\varphi(z)|\to 1} \frac{w(z)|u(z)|}{\tilde{v}(\varphi(z))}$.

Lemma 2. *[14, Lemma 2.1]* $\lim_{n\to\infty} (\log n)\|z^n\|_{v_{\log}} = 1$.

The following lemma is an easy result from (1).

Lemma 3. *For $f \in BMOA$,*

$$|f(z)| \le \log \frac{2}{1-|z|^2}\|f\|_{BMOA}.$$

The following lemma for compactness follows similarly from [6, Proposition 3.11].

Lemma 4. *The operator $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$ ($uC_\varphi : BMOA \to H_{v_\alpha}^\infty$) is compact if and only if $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$ ($uC_\varphi : BMOA \to H_{v_\alpha}^\infty$) is bounded and $\|T_gC_\varphi f_n\|_{\mathcal{B}^\alpha} \to 0$ ($\|uC_\varphi f_n\|_{v_\alpha} \to 0$), as $n \to \infty$, for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in BMOA converging to zero uniformly on compact subsets of $\mathbb{D}$.*

3. Main Results

3.1. Boundedness

In this part, we give a new characterization for the boundedness of $T_gC_\varphi : BMOA(VMOA) \to \mathcal{B}^\alpha$.

Theorem 1. *For $\alpha > 0$, $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then the following statements are equivalent:*
(a) $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$ is bounded.
(b) $T_gC_\varphi : VMOA \to \mathcal{B}^\alpha$ is bounded.
(c)

$$\sup_{n\geq 0} \log(n+1)\|g'\varphi^n\|_{v_\alpha} \asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g'(z)|\log\frac{2}{1-|\varphi(z)|^2} < \infty. \tag{2}$$

In each case the norm $\|T_gC_\varphi\|$ comparable to (2).

Proof. (a)⇒(b). This implication is obvious.
(b)⇒(c). For $w \in \mathbb{D}$, define the function

$$f_w(z) = \log\frac{2}{1-\overline{\varphi(w)}z}. \tag{3}$$

It is well known that $\|f_w\|_* \leq \left\|\log\frac{2}{1-z}\right\|_* \leq C < \infty$ and $f_w \in VMOA$. Since $T_gC_\varphi f_w(0) = 0$, thus

$$\begin{aligned}\|T_gC_\varphi f_w\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(T_gC_\varphi f_w)'(z)| \\ &\geq (1-|w|^2)^\alpha|f_w(\varphi(w))g'(w)| \\ &= (1-|w|^2)^\alpha|g'(w)|\log\frac{2}{1-|\varphi(w)|^2}.\end{aligned} \tag{4}$$

From (4) we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g'(z)|\log\frac{2}{1-|\varphi(z)|^2} < \infty.$$

By Lemma 1 (i) and above inequality it follows that $g'C_\varphi : H^\infty_{v_{\log}} \to H^\infty_{v_\alpha}$ is bounded. Then by Lemma 2 and Lemma 1 (i), it follows that

$$\begin{aligned}\sup_{n\geq 0}\log(n+1)\|g'\varphi^n\|_{v_\alpha} &\asymp \sup_{n\geq 0}\frac{\log(n+1)}{\log n}\log n\|g'\varphi^n\|_{v_\alpha} \\ &\leq \sup_{n\geq 0}\log n\|g'\varphi^n\|_{v_\alpha} \\ &\asymp \sup_{n\geq 0}\frac{\|g'\varphi^n\|_{v_\alpha}}{\|z^n\|_{v_{\log}}} \\ &\asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g'(z)|\log\frac{2}{1-|\varphi(z)|^2} < \infty.\end{aligned}$$

(c)⇒ (a). For every $f \in BMOA$, $T_gC_\varphi f(0) = 0$, by Lemma 3 we have that

$$\begin{aligned}\|T_gC_\varphi f\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(\varphi(z))g'(z)| \\ &\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g'(z)|\log\frac{2}{1-|\varphi(z)|^2}\|f\|_{BMOA} \\ &< \infty.\end{aligned}$$

From which it follows the boundedness of $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$. This completes the proof. □

3.2. Essential norm

The essential norm of a continuous linear operator T is the distance from T to the compact operators K, that is, $\|T\|_e = \inf\{\|T-K\| : K \text{ is compact }\}$. Notice that $\|T\|_e = 0$ if and only if T is compact, so estimates on $\|T\|_e$ lead to conditions for T to be compact. There are lots of papers concerning this topic, the interested readers can refer to [7, 11, 13, 14, 19, 26, 27].

In this part, we estimate the essential norms of the integral-type operator T_gC_φ acting from $BMOA(VMOA)$ to $\mathcal{B}^\alpha$. Since $(T_gC_\varphi f)' = g' f \circ \varphi$, then

$$\|T_gC_\varphi\|_{e,BMOA\to\mathcal{B}^\alpha} \le \|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}}. \tag{5}$$

The following lemma characterizes the essential norms of the weighted composition operator uC_φ from $BMOA$ to $H^\infty_{v_\alpha}$.

Lemma 5. *Let $0<\alpha<\infty$, the weighted composition operator $g'C_\varphi : BMOA \to H^\infty_{v_\alpha}$ is bounded. Then*

$$\|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}} \asymp \limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha}.$$

Proof. The upper estimate. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $BMOA$, then it has a subsequence denoting by $(f_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact subsets of $\mathbb{D}$. We can assume, without loss of generality, that $(f_n)_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Fix $0<\delta<1$ and let $(r_m)_{m\in\mathbb{N}}$ be an increasing sequence in $(0,1)$ converging to 1. We can easily obtain that $g'C_{r_m\varphi}$ is a compact operator by the boundedness of $g'C_\varphi$ and Lemma 4. Thus

$$\begin{aligned}
\|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}} &\le \|g'C_\varphi - g'C_{r_m\varphi}\|_{BMOA\to H^\infty_{v_\alpha}}\\
&= \sup_{z\in\mathbb{D}}\sup_{\|f\|_{BMOA}\le1}(1-|z|^2)^\alpha|g'(z)||f(\varphi(z))-f(r_m\varphi(z))|\\
&\le \sup_{|\varphi(z)|<\delta}\sup_{\|f\|_{BMOA}\le1}(1-|z|^2)^\alpha|g'(z)||f(\varphi(z))-f(r_m\varphi(z))| \qquad (6)\\
&+ \sup_{|\varphi(z)|\ge\delta}\sup_{\|f\|_{BMOA}\le1}(1-|z|^2)^\alpha|g'(z)||f(\varphi(z))-f(r_m\varphi(z))|. \qquad (7)
\end{aligned}$$

Case i $|\varphi(z)|<\delta$. since $f\in\mathcal{B}$, when $f\in BMOA$, thus $|f'(z)| \le \frac{1}{1-|z|^2}\|f\|_{BMOA}$.

$$\begin{aligned}
|f(\varphi(z))-f(r_m\varphi(z))| &\le \int_{r_m}^1 |\varphi(z)||f'(t\varphi(z))|dt\\
&\le \|f\|_{BMOA}\int_{r_m}^1|\varphi(z)|\frac{1}{1-|t\varphi(z)|^2}dt\\
&\le \|f\|_{BMOA}\frac{|\varphi(z)|}{1-|\varphi(z)|}(1-r_m).
\end{aligned}$$

Since $\frac{|\varphi(z)|}{1-|\varphi(z)|} < \frac{\delta}{1-\delta}$, then we have

$$\sup_{\|f\|_{BMOA}\le1}|f(\varphi(z))-f(r_m\varphi(z))| \le \frac{\delta}{1-\delta}(1-r_m).$$

Furthermore, $\|g'C_\varphi(1)\|_{v_\alpha}$ is finite by the boundness of $g'C_\varphi : BMOA\to H^\infty_{v_\alpha}$. Thus

$$\begin{aligned}
&\sup_{|\varphi(z)|<\delta}\sup_{\|f\|_{BMOA}\le1}(1-|z|^2)^\alpha|g'(z)||f(\varphi(z))-f(r_m\varphi(z))|\\
\le\ &\frac{\delta}{1-\delta}(1-r_m)\sup_{|\varphi(z)|<\delta}(1-|z|^2)^\alpha|g'(z)|\\
\le\ &\frac{\delta}{1-\delta}(1-r_m)\|g'C_\varphi(1)\|_{v_\alpha}.
\end{aligned}$$

From which it follows that (6) tends to zero as $m \to \infty$.

Case ii $|\varphi(z)| \geq \delta$. Given $f \in BMOA$ with $\|f\|_{BMOA} \leq 1$,

$$\begin{aligned}
|f(\varphi(z)) - f(r_m\varphi(z))| &\leq \int_{r_m}^{1} |\varphi(z)||f'(t\varphi(z))|dt \\
&\leq \int_{r_m}^{1} \frac{|\varphi(z)|}{1-|t\varphi(z)|^2} dt \|f\|_{BMOA} \\
&\leq \int_{r_m}^{1} \frac{1}{1-|t\varphi(z)|} d(t|\varphi(z)|) \\
&= \log \frac{1-|\varphi(z)|}{1-r_m|\varphi(z)|} \\
&\leq \log \frac{2}{1-r_m|\varphi(z)|}.
\end{aligned}$$

Therefore

$$\lim_{m\to\infty} \sup_{\|f\|_{BMOA}\leq 1} |f(\varphi(z)) - f(r_m\varphi(z))| \leq \log \frac{2}{1-|\varphi(z)|},$$

and letting $\delta \to 1$, from (7) it follows that

$$\|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}} \leq \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |g'(z)| \log \frac{2}{1-|\varphi(z)|^2}. \tag{8}$$

The lower estimate. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Define the sequence

$$f_n(z) = \left(\log \frac{2}{1-|\varphi(z_n)|^2}\right)^{-1} \left(\log \frac{2}{1-\overline{\varphi(z_n)}z}\right)^2, \ z \in \mathbb{D}. \tag{9}$$

Then $f_n \in VMOA$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{BMOA} < \infty$. Moreover, $(f_n)_{n\in\mathbb{N}}$ converges to zero unifromly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Then for every compact operator T by Lemma 4 it follows that

$$\begin{aligned}
\|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}} &\geq \limsup_{n\to\infty} \|(g'C_\varphi - T)f_n\|_{v_\alpha} \\
&\geq \limsup_{n\to\infty} \|g'C_\varphi f_n\|_{v_\alpha} \\
&= \limsup_{n\to\infty} \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |g'(z)||f_n(\varphi(z))| \\
&\geq \limsup_{n\to\infty} (1-|z_n|^2)^\alpha |g'(z_n)||f_n(\varphi(z_n))| \\
&= \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |g'(z)| \log \frac{2}{1-|\varphi(z)|^2}.
\end{aligned} \tag{10}$$

Combining (8) and (10) it follows that

$$\|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}} \asymp \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |g'(z)| \log \frac{2}{1-|\varphi(z)|^2}.$$

Further by Lemma 1 (ii) and Lemma 2,

$$\|g'C_\varphi\|_{e,BMOA\to H^\infty_{v_\alpha}} \asymp \limsup_{n\to\infty} (\log n) \|g'\varphi^n\|_{v_\alpha}.$$

This completes the proof. $\square$

Theorem 2. *For $\alpha > 0$, $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. If $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$ is bounded, then*

$$\|T_gC_\varphi\|_{e,BMOA\to\mathcal{B}^\alpha} \asymp \|T_gC_\varphi\|_{e,VMOA\to\mathcal{B}^\alpha} \asymp \limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha}.$$

Proof. It is obvious that $\|T_gC_\varphi\|_{e,BMOA\to\mathcal{B}^\alpha} \geq \|T_gC_\varphi\|_{e,VMOA\to\mathcal{B}^\alpha}$.

By (5) we obtain that $\|T_gC_\varphi\|_{e,BMOA\to\mathcal{B}^\alpha} \leq \|g'C_\varphi\|_{e,BMOA\to H_{v_\alpha}}$. Further by Lemma 5, we have that

$$\|T_gC_\varphi\|_{e,BMOA\to\mathcal{B}^\alpha} \leq \limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha}. \tag{11}$$

On the other hand, let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Take the function sequence defined in (9). Then we have

$$\begin{aligned}
\|T_gC_\varphi\|_{e,VMOA\to\mathcal{B}^\alpha} &\geq \limsup_{n\to\infty}\|T_gC_\varphi f_n\|_{\mathcal{B}^\alpha}\\
&\geq \limsup_{n\to\infty}(1-|z_n|^2)^\alpha|g'(z_n)|\log\frac{2}{1-|\varphi(z_n)|^2}\\
&= \limsup_{|\varphi(z)|\to1}(1-|z|^2)^\alpha|g'(z)|\log\frac{2}{1-|\varphi(z)|^2}\\
&\asymp \limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha}.
\end{aligned} \tag{12}$$

Combining (11) and (12) we obtain that

$$\begin{aligned}
\limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha} &\geq \|T_gC_\varphi\|_{e,BMOA\to\mathcal{B}^\alpha}\\
&\geq \|T_gC_\varphi\|_{e,VMOA\to\mathcal{B}^\alpha}\\
&\geq \limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha}.
\end{aligned}$$

From the above we obtain the desired results. This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.

Corollary 1. *For $\alpha > 0$, $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. If $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$ is bounded, then the following are equivalent:*

(*i*) $T_gC_\varphi : BMOA \to \mathcal{B}^\alpha$ *is compact.*

(*ii*) $T_gC_\varphi : VMOA \to \mathcal{B}^\alpha$ *is compact.*

(*iii*) $\limsup_{n\to\infty}(\log n)\|g'\varphi^n\|_{v_\alpha} = 0$.

(*iv*) $\limsup_{|\varphi(z)|\to1}(1-|z|^2)^\alpha|g'(z)|\log\frac{2}{1-|\varphi(z)|^2} = 0$.

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