Filomat 33:1 (2019), 121–134
https://doi.org/10.2298/FIL1901121A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Signed Double Roman Domination of Graphs

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Abstract. In this paper we continue the study of signed double Roman dominating functions in graphs. A signed double Roman dominating function (SDRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ having the property that for each $v \in V(G)$, $f[v] \geq 1$, and if $f(v) = -1$, then vertex v has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then vertex v must have at leat one neighbor w with $f(w) \geq 2$. The weight of a SDRDF is the sum of its function values over all vertices. The signed double Roman domination number $\gamma_{sdR}(G)$ is the minimum weight of a SDRDF on G. We present several lower bounds on the signed double Roman domination number of a graph in terms of various graph invariants. In particular, we show that if G is a graph of order n and size m with no isolated vertex, then $\gamma_{sdR}(G) \geq \frac{19n-24m}{9}$ and $\gamma_{sdR}(G) \geq 4\sqrt{\frac{n}{3}} - n$. Moreover, we characterize the graphs attaining equality in these two bounds.

1. Introduction

We consider finite, undirected and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V|$ of a graph G is called the *order* of G and is denoted by n. The *size* m of a graph G is the number of edges $|E|$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, we let $d_S(v)$ denote the number of neighbors of a vertex $v \in V$. In particular, $d_V(v) = \deg_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* of G is a vertex of degree one. We write P_n for the *path* of order n, C_n for the *cycle* of length n, $K_{p,q}$ for the complete bipartite graph and $\overline{G}$ for the complement graph of G.

A set $S \subseteq V$ in a graph G is called a *dominating set* if every vertex of G is either in S or adjacent to a vertex of S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G.

A *double Roman dominating function* (DRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = 0$ is adjacent to least two vertices assigned a 2 or to at least one vertex assigned a 3, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$.The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G. Double Roman domination was introduced in 2016 by Beeler et al. [5] and studied further in [1–3].

2010 *Mathematics Subject Classification*. 05C69, 05C05
Keywords. Double Roman domination; signed double Roman domination
Received: 05 December 2017; Accepted: 24 December 2018
Communicated by Francesco Belardo
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In this paper, we continue the study of the signed double Roman domination in graphs introduced in [4] as follows. A *signed double Roman dominating function* (SDRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f[v] = \sum_{u\in N[v]} f(u) \geq 1$ holds for any vertex v. The weight of a SDRDF f is the value $\omega(f) = \sum_{u\in V(G)} f(u)$. The *signed double Roman domination number* $\gamma_{sdR}(G)$ is the minimum weight of a SDRDF on G. For a graph G, let $f : V(G) \to\{-1, 1, 2, 3\}$ be a function, and let $V_i = \{v \in V | f(v) = i\}$ for $i \in \{-1, 1, 2, 3\}$. In the whole paper, the function f can be denoted $f = (V_{-1}, V_1, V_2, V_3)$.

In this paper, we present various lower bounds on the signed double Roman domination number of a graph. Moreover, we determine the signed double Roman domination of some classes of graphs including complete graphs, cycles, and complete bipartite graphs.

2. Preliminary results

In this section we investigate basic properties of signed double Roman domination number. The following observation is straightforward.

Observation 2.1. If $f = (V_{-1}, V_1, V_2, V_3)$ is a SDRDF of a graph G, then the following holds.
(i) Every vertex in $V_{-1} \cup V_1$ is dominated by a vertex in $V_2 \cup V_3$.
(ii) $w(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.
(iii) $V_2 \cup V_3$ is a dominating set in G.

Proposition 2.2. *Let $f = (V_{-1}, V_1, V_2, V_3)$ be a SDRDF in a graph G of order n. Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Then the following holds.*
(i) $(3\Delta + 2)|V_3| + (2\Delta + 1)|V_2| + \Delta|V_1| \geq (\delta + 2)|V_{-1}|$.
(ii) $(3\Delta + \delta + 4)|V_3| + (2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + 2)n$.
(iii) $(\Delta + \delta + 2)w(f) \geq (\delta - \Delta + 2)n + (\delta - \Delta)|V_2| + 2(\delta - \Delta)|V_3|$.
(iv) $w(f) \geq (\delta - 3\Delta)n/(3\Delta + \delta + 4) + |V_2| + 2|V_3|$.

Proof. (i) We have that

$$\begin{aligned}
|V_{-1}| + |V_1| + |V_2| + |V_3| &= n \\
&\leq \sum_{v\in V} f[v] \\
&= \sum_{v\in V} (\deg_G(v) + 1)f(v) \\
&= \sum_{v\in V_3} 3(\deg_G(v) + 1) + \sum_{v\in V_2} 2(\deg_G(v) + 1)+ \\
&\quad \sum_{v\in V_1} (\deg_G(v) + 1) - \sum_{v\in V_{-1}} (\deg_G(v) + 1) \\
&\leq 3(\Delta + 1)|V_3| + 2(\Delta + 1)|V_2| + (\Delta + 1)|V_1| - (\delta + 1)|V_{-1}|,
\end{aligned}$$

and the desired result follows.
(ii) By substituting $|V_{-1}| = n - |V_1| - |V_2| - |V_3|$ in Part (i), the result follows.
(iii) By Observation 2.1 and Part (ii), we have

$$\begin{aligned}
(\Delta + \delta + 2)w(f) &= (\Delta + \delta + 2)\left(2(|V_1| + |V_2| + |V_3|) - n + |V_2| + 2|V_3|\right) \\
&\geq 2(\delta + 2)n - 2(\Delta + 1)|V_2| - 4(\Delta + 1)|V_3| + (\Delta + \delta + 2)(|V_2| + 2|V_3| - n) \\
&= (\delta - \Delta + 2)n + (\delta - \Delta)|V_2| + 2(\delta - \Delta)|V_3|.
\end{aligned}$$

(iv) It follows from the proof of Part (i) that

$$\begin{aligned} n &\leq 3(\Delta+1)|V_3| + 2(\Delta+1)|V_2| + (\Delta+1)|V_1| - (\delta+1)|V_{-1}| \\ &\leq 3(\Delta+1)|V_1 \cup V_2 \cup V_3| - (\delta+1)|V_{-1}| \\ &= (3\Delta+\delta+4)|V_1 \cup V_2 \cup V_3| - (\delta+1)n. \end{aligned}$$

And so $|V_1 \cup V_2 \cup V_3| \geq n(\delta+2)/(3\Delta+\delta+4)$. Therefore,

$$\begin{aligned} w(f) &= 2|V_1 \cup V_2 \cup V_3| - n + |V_2| + 2|V_3| \\ &\geq (\delta - 3\Delta)n/(3\Delta+\delta+4) + |V_2| + 2|V_3|. \end{aligned}$$

□

Next result is an immediate consequence of Proposition 2.2(iii).

Corollary 2.3. If $r \geq 1$ is an integer and G is an r-regular graph of order n, then $\gamma_{sdR}(G) \geq n/(r+1)$.

If G is not a regular graph, then as a consequence of Proposition 2.2(iii) and (iv), we have the following result.

Corollary 2.4. If G is a graph of order n, minimum degree δ and maximum degree Δ where $\delta < \Delta$, then

$$\gamma_{sdR}(G) \geq \left(\frac{-3\Delta^2 + 3\Delta\delta + \Delta + 3\delta + 4}{(\Delta+1)(3\Delta+\delta+4)} \right) n.$$

Proof. Multiplying both sides of the inequality in Proposition 2.2(iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2.2(iii) we obtain the desired result. □

Proposition 2.5. For any graph G, $\gamma_{sdR}(G) \geq \Delta(G) + 2 - n$. This bound is sharp for complete graphs.

Proof. Let v be a vertex of maximum degree $\Delta(G)$. We have

$$\begin{aligned} \gamma_{sdR}(G) &= f[v] + \textstyle\sum_{x \in V(G)-N[v]} f(x) \\ &\geq 1 - (n - \Delta(G) - 1) \\ &= \Delta(G) + 2 - n. \end{aligned}$$

□

We note that the bound of Proposition 2.5 is sharp for non-trivial complete graphs except K_4 (see Proposition 4.1 for exact values of $\gamma_{sdR}(K_n)$).

Proposition 2.6. For any graph G, $\gamma_{sdR}(G) + \gamma_{sdR}(\overline{G}) \geq \Delta(G) - \delta(G) + 3 - n$.

Proof. By Proposition 2.5, we have

$$\begin{aligned} \gamma_{sdR}(G) + \gamma_{sdR}(\overline{G}) &\geq (\Delta(G) + 2 - n) + (\Delta(\overline{G}) + 2 - n) \\ &= (\Delta(G) + 2 - n) + (n - 1 - \delta(\overline{G}) + 2 - n) \\ &= \Delta(G) - \delta(\overline{G}) + 3 - n. \end{aligned}$$

□

3. Bounds

In this section we present some sharp bounds on the signed double Roman domination number in graphs. First we introduce some notation for convenience.

Let $V'_{-1} = \{v \in V_{-1} \mid N(v) \cap V_3 \neq \emptyset\}$ and $V''_{-1} = V_{-1} - V'_{-1}$. For disjoint subsets U and W of vertices, we let $[U, W]$ denote the set of edges between U and W. For notational convenience, we let $V_{12} = V_1 \cup V_2, V_{13} = V_1 \cup V_3, V_{123} = V_1 \cup V_2 \cup V_3$ and let $|V_{12}| = n_{12}, |V_{13}| = n_{13}, |V_{123}| = n_{123}$, and let $|V_1| = n_1, |V_2| = n_2$ and $|V_3| = n_3$. Then, $n_{123} = n_1 + n_2 + n_3$. Further, we let $|V_{-1}| = n_{-1}$, and so $n_{-1} = n - n_{123}$. Let $G_{123} = G[V_{123}]$ be the subgraph induced by the set V_{123} and let G_{123} have size m_{123}. For $i = 1, 2, 3$, if $V_i \neq \emptyset$, let $G_i = G[V_i]$ be the subgraph induced by the set V_i and let G_i have size m_i. Hence, $m_{123} = m_1 + m_2 + m_3 + |[V_1, V_2]| + |[V_1, V_3]| + |[V_2, V_3]|$.

For $k \geq 1$, let L_k be the graph obtained from a graph H of order k by adding $3d_H(v) + 2$ pendant edges to each vertex v of H. Note that $L_1 = K_{1,2}$. Let $\mathcal{H} = \{L_k \mid k \geq 1\}$.

Theorem 3.1. Let G be a graph of order n and size m with no isolated vertex. Then

$$\gamma_{sdR}(G) \geq \frac{19n - 24m}{9}$$

with equality if and only if $G \in \mathcal{H}$.

Proof. The proof is by induction on n. If $n = 2$, then $\gamma_{sdR}(K_2) = 2 > \frac{19n-24m}{9}$. If $n = 3$, then $G \in \{K_{1,2}, K_3\}$ and $\gamma_{sdR}(G) \geq \frac{19n-24m}{9}$ with equality only if $G = K_{1,2}$ that belongs to $\mathcal{H}$. Hence let $n \geq 4$ and assume that the statement is true for all graphs of order less than n having no isolated vertex. Let G be a graph of order n with no isolated vertex and let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(G)$-function. If $V_{-1} = \emptyset$, then $\gamma_{sdR}(G) > n > \frac{19n-24m}{9}$ since G has no isolated vertex. Hence $V_{-1} \neq \emptyset$. We consider the following cases.

Case 1. $V_3 \neq \emptyset$.

We distinguish the following.

Subcase 1.1 $V_2 \neq \emptyset$.

By the definition of a SDRDF, each vertex in V_{-1} is adjacent to at least one vertex in V_3 or to at least two vertices in V_2, and so

$$|[V_{-1}, V_{123}]| \geq |[V_{-1}, V_3]| + |[V_{-1}, V_2]| \geq |V'_{-1}| + 2|V''_{-1}| \geq n_{-1}.$$

Furthermore we have

$$2n_{-1} \leq 2|[V_{-1}, V_3]| + |[V_{-1}, V_2]| = 2\sum_{v \in V_3} d_{V_{-1}}(v) + \sum_{u \in V_2} d_{V_{-1}}(u).$$

For each vertex $v \in V_3$, we have that $f(v) + 3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 1$, and so $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) + 2$. Similarly, for each vertex $u \in V_2$, we have that $d_{V_{-1}}(u) \leq 3d_{V_3}(u) + 2d_{V_2}(u) + d_{V_1}(u) + 1$. Now, we have

$$\begin{aligned}
2n_{-1} &\leq 2\sum_{v \in V_3} d_{V_{-1}}(v) + \sum_{u \in V_2} d_{V_{-1}}(u) \\
&\leq 2\sum_{v \in V_3} (3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) + 2) + \sum_{u \in V_2} (3d_{V_3}(u) + 2d_{V_2}(u) + d_{V_1}(u) + 1) \\
&= (12m_3 + 4|[V_2, V_3]| + 2|[V_1, V_3]| + 4n_3) + (3|[V_2, V_3]| + 4m_2 + |[V_1, V_2] + n_2) \\
&= 12m_3 + 4m_2 + 7|[V_2, V_3]| + 2|[V_1, V_3]| + |[V_1, V_2]| + 4n_3 + n_2 \\
&= 12m_{123} - 12m_1 - 8m_2 - 5|[V_2, V_3]| - 10|[V_1, V_3]| - 11|[V_1, V_2]| + 4n_3 + n_2,
\end{aligned}$$

which implies that

$$m_{123} \geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| - 4n_3 - n_2).$$

Hence,

$$\begin{aligned}
m &= m_{123} + \|[V_{-1}, V_{123}]\| + m_{-1} \\
&\geq m_{123} + \|[V_{-1}, V_{123}]\| \\
&\geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\| - 4n_3 - n_2) + n_{-1} \\
&= \frac{1}{12}(14n_{-1} - 4n_{123} + 3n_2 + 4n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|) \\
&= \frac{1}{12}(14n - 18n_{123} + 3n_2 + 4n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|)
\end{aligned}$$

and so

$$n_{123} \geq \frac{1}{18}(-12m + 14n + 3n_2 + 4n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|).$$

Now, we have

$$\begin{aligned}
\gamma_{sdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\
&= 4n_3 + 3n_2 + 2n_1 - n \\
&= 4n_{123} - n - n_2 - 2n_1 \\
&\geq \frac{2}{9}(-12m + 14n + 3n_2 + 4n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + \\
&\quad 11\|[V_1, V_2]\|) - n - n_2 - 2n_1 \\
&= \frac{2}{9}(-12m + 14n - \frac{9}{2}n) + \frac{2}{9}(3n_2 + 4n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + \\
&\quad 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\| - \frac{9}{2}n_2 - 9n_1) \\
&= \frac{1}{9}(19n - 24m) + \frac{2}{9}(\frac{-3}{2}n_2 - 5n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|)
\end{aligned}$$

Let $\Theta = \frac{-3}{2}n_2 - 5n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|$. Assume first that $n_1 = 0$. Clearly then $\Theta = \frac{-3}{2}n_2 + 8m_2 + 5\|[V_2, V_3]\|$. Suppose $d_{V_{23}}(v) \geq 1$ for each $v \in V_2$. Then

$$\begin{aligned}
\Theta &= \frac{-3}{2}n_2 + 8m_2 + 5\|[V_2, V_3]\| \\
&= 4\sum_{v \in V_2} d_{V_2}(v) + 4\sum_{v \in V_2} d_{V_3}(v) + (\frac{-3}{2}n_2 + \|[V_2, V_3]\|) \\
&= 4\sum_{v \in V_2} d_{V_{23}}(v) + (\frac{-3}{2}n_2 + \|[V_2, V_3]\|) \\
&\geq 4n_2 + \frac{-3}{2}n_2 + \|[V_2, V_3]\| \\
&\geq \frac{5}{2}n_2 + \|[V_2, V_3]\| \\
&> 0.
\end{aligned}$$

Therefore $\gamma_{sdR}(G) > \frac{19n-24m}{9}$.

Now let $d_{V_{23}}(v) = 0$ for some $v \in V_2$. Since by assumption there is no isolated vertex in G and $n_1 = 0$, we have that every neighbor of v belongs to V_{-1}. Since $f[v] \geq 1$, we conclude that v is a leaf and has a neighbor, say w, such that $f(w) = -1$. Let $G' = G - v$. Then the function $g : V(G') \to \{-1, 1, 2, 3\}$ defined by $g(w) = 1$ and $g(x) = f(x)$ for $x \in V(G') - \{w\}$ is a SDRDF of G' of weight $\omega(f)$. By the induction hypothesis we have

$$\begin{aligned}\gamma_{sdR}(G) &\geq \gamma_{sdR}(G')\\ &\geq \frac{19(n-1) - 24(m-1)}{9}\\ &= \frac{19n - 24m}{9} + \frac{5}{9}\\ &> \frac{19n - 24m}{9}\end{aligned}$$

Therefore $\gamma_{sdR}(G) > \frac{19n-24m}{9}$.

Assume now that $n_1 \geq 1$ and recall that $\Theta = \frac{-3}{2}n_2 - 5n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|$. By the definition of SDRDF of G, we have $d_{V_{123}}(v) \geq 1$ for each $v \in V_1$. If $d_{V_{123}}(v) = 0$ for some $v \in V_2$, then as above v is a leaf. By considering the graph $G' = G - v$ and using a similar argument as previously, the result follows. Hence, let $d_{V_{123}}(v) \geq 1$ for each $v \in V_2$. Then

$$\begin{aligned}\Theta &= \frac{-3}{2}n_2 - 5n_1 + 12m_1 + 8m_2 + 5\|[V_2, V_3]\| + 10\|[V_1, V_3]\| + 11\|[V_1, V_2]\|\\ &= 6\sum_{v\in V_1} d_{V_1}(v) + 6\sum_{v\in V_1} d_{V_2}(v) + 6\sum_{v\in V_1} d_{V_3}(v) + 4\sum_{v\in V_2} d_{V_1}(v) + 4\sum_{v\in V_2} d_{V_2}(v) + 4\sum_{v\in V_2} d_{V_3}(v)\\ &\quad + (\frac{-3}{2}n_2 - 5n_1 + \|[V_2, V_3]\| + 4\|[V_1, V_3]\| + \|[V_1, V_2]\|)\\ &= 6\sum_{v\in V_1} d_{V_{123}}(v) + 4\sum_{v\in V_2} d_{V_{123}}(v) + (\frac{-3}{2}n_2 - 5n_1 + \|[V_2, V_3]\| + 4\|[V_1, V_3]\| + \|[V_1, V_2]\|)\\ &\geq 6n_1 + 4n_2 + \frac{-3}{2}n_2 - 5n_1 + \|[V_2, V_3]\| + 4\|[V_1, V_3]\| + \|[V_1, V_2]\|\\ &= n_1 + \frac{5}{2}n_2 + \|[V_2, V_3]\| + 4\|[V_1, V_3]\| + \|[V_1, V_2]\|\\ &> 0.\end{aligned}$$

Therefore $\gamma_{sdR}(G) > \frac{19n-24m}{9}$.

Subcase 1.2 $V_2 = \emptyset$.

By the definition of a SDRDF, each vertex in V_{-1} is adjacent to a vertex in V_3, and so

$$\|[V_{-1}, V_{13}]\| \geq \|[V_{-1}, V_3]\| \geq |V_{-1}| = n_{-1}.$$

Furthermore we have

$$n_{-1} \leq \|[V_{-1}, V_3]\| = \sum_{v\in V_3} d_{V_{-1}}(v).$$

For each vertex $v \in V_3$, we have that $f(v) + 3d_{V_3}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 1$, and so $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + d_{V_1}(v) + 2$. It follows that

$$\begin{aligned}n_{-1} &\leq \sum_{v\in V_3} d_{V_{-1}}(v)\\ &\leq \sum_{v\in V_3} (3d_{V_3}(v) + d_{V_1}(v) + 2)\\ &= 6m_3 + \|[V_1, V_3]\| + 2n_3\\ &= 6m_{13} - 6m_1 - 5\|[V_1, V_3]\| + 2n_3,\end{aligned}$$

which implies that

$$m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5\|[V_1, V_3]\| - 2n_3).$$

Hence,

$$\begin{aligned} m &\geq m_{13} + \|[V_{-1}, V_{13}]\| + m_{-1} \\ &\geq m_{13} + \|[V_{-1}, V_{13}]\| \\ &\geq \frac{1}{6}(n_{-1} + 6m_1 + 5\|[V_1, V_3]\| - 2n_3) + n_{-1} \\ &= \frac{1}{6}(7n_{-1} - 2n_{13} + 2n_1 + 6m_1 + 5\|[V_1, V_3]\|) \\ &= \frac{1}{6}(7n - 9n_{13} + 2n_1 + 6m_1 + 5\|[V_1, V_3]\|) \end{aligned}$$

and so

$$n_{13} \geq \frac{1}{9}(-6m + 7n + 2n_1 + 6m_1 + 5\|[V_1, V_3]\|).$$

Now, we have

$$\begin{aligned} \gamma_{sdR}(G) &= 3n_3 + n_1 - n_{-1} \\ &= 4n_3 + 2n_1 - n \\ &= 4n_{13} - n - 2n_1 \\ &\geq \frac{4}{9}(-6m + 7n + 2n_1 + 6m_1 + 5\|[V_1, V_3]\|) - n - 2n_1 \\ &= \frac{4}{9}(-6m + 7n - \frac{9}{4}n) + \frac{4}{9}(2n_1 + 6m_1 + 5\|[V_1, V_3]\| - \frac{9}{2}n_1) \\ &= \frac{1}{9}(19n - 24m) + \frac{4}{9}(6m_1 + 5\|[V_1, V_3]\| - \frac{5}{2}n_1) \end{aligned}$$

Let $\Theta = 6m_1 + 5\|[V_1, V_3]\| - \frac{5}{2}n_1$. We shall show that $\Theta \geq 0$. Clearly, $\Theta = 0$ if $n_1 = 0$. Thus we suppose that $n_1 \geq 1$. By the definition of a SDRDF, for each $v \in V_1$ we have $d_{V_{13}}(v) \geq 1$. Hence

$$\begin{aligned} \Theta &= \frac{-5}{2}n_1 + 6m_1 + 5\|[V_1, V_3]\| \\ &= 3\sum_{v \in V_1} d_{V_1}(v) + 3\sum_{v \in V_1} d_{V_3}(v) + (2\|[V_1, V_3]\| - \frac{5}{2}n_1) \\ &= 3\sum_{v \in V_1} d_{V_{13}}(v) + (2\|[V_1, V_3]\| - \frac{5}{2}n_1) \\ &\geq 3n_1 + 2\|[V_1, V_3]\| - \frac{5}{2}n_1 \\ &= \frac{1}{2}n_1 + 2\|[V_1, V_3]\| \\ &> 0. \end{aligned}$$

Therefore $\gamma_{sdR}(G) \geq \frac{19n-24m}{9}$.

Case 2. $V_3 = \emptyset$.

Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. By the definition of a SDRDF, each vertex in V_{-1} is adjacent to at least two vertices in V_2, and so

$$\|[V_{-1}, V_{12}]\| \geq \|[V_{-1}, V_2]\| \geq 2|V_{-1}| = 2n_{-1}.$$

Furthermore we have

$$2n_{-1} \leq |[V_{-1}, V_2]| = \sum_{v \in V_2} d_{V_{-1}}(v).$$

For each vertex $v \in V_2$, we have that $f(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 1$, and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) + 1$. It follows that

$$\begin{aligned} 2n_{-1} &\leq \sum_{v \in V_2} d_{V_{-1}}(v) \\ &\leq \sum_{v \in V_2} (2d_{V_2}(v) + d_{V_1}(v) + 1) \\ &= 4m_2 + |[V_1, V_2]| + n_2 \\ &= 4m_{12} - 4m_1 - 3|[V_1, V_2]| + n_2, \end{aligned}$$

which implies that

$$m_{12} \geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| - n_2).$$

Hence,

$$\begin{aligned} m &\geq m_{12} + |[V_{-1}, V_{12}]| + m_{-1} \\ &\geq m_{12} + |[V_{-1}, V_{12}]| \\ &\geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| - n_2) + 2n_{-1} \\ &= \frac{1}{4}(10n_{-1} - 2n_{12} + 2n_1 + n_2 + 4m_1 + 3|[V_1, V_2]|) \\ &= \frac{1}{4}(10n - 12n_{12} + 2n_1 + n_2 + 4m_1 + 3|[V_1, V_2]|) \end{aligned}$$

and so

$$n_{12} \geq \frac{1}{12}(-4m + 10n + 2n_1 + n_2 + 4m_1 + 3|[V_1, V_2]|).$$

Now, we have

$$\begin{aligned} \gamma_{sdR}(G) &= 2n_2 + n_1 - n_{-1} \\ &= 3n_2 + 2n_1 - n \\ &= 3n_{12} - n - n_1 \\ &\geq \frac{1}{4}(-4m + 10n + 2n_1 + n_2 + 4m_1 + 3|[V_1, V_2]|) - n - n_1 \\ &= \frac{1}{4}(-4m + 10n - 4n) + \frac{1}{4}(2n_1 + n_2 + 4m_1 + 3|[V_1, V_2]| - 4n_1) \\ &= \frac{1}{4}(-4m + 6n) + \frac{1}{4}(4m_1 + n_2 + 3|[V_1, V_2]| - 2n_1) \end{aligned}$$

Let $\Theta = 4m_1 + n_2 + 3|[V_1, V_2]| - 2n_1$. We will show that $\Theta > 0$. Clearly if $n_1 = 0$, then $\Theta = n_2 > 0$. Hence assume that $n_1 \geq 1$. By the definition of a SDRDF of G, we have $d_{V_{12}}(v) \geq 1$ for each $v \in V_1$. Hence,

$$\begin{aligned}
\Theta &= 4m_1 + n_2 + 3\|[V_1, V_2]\| - 2n_1 \\
&= 2\sum_{v\in V_1} d_{V_1}(v) + 2\sum_{v\in V_1} d_{V_2}(v) + (n_2 + \|[V_1, V_2]\| - 2n_1) \\
&= 2\sum_{v\in V_1} d_{V_{12}}(v) + (n_2 + \|[V_1, V_2]\| - 2n_1) \\
&\geq 2n_1 + n_2 + \|[V_1, V_2]\| - 2n_1 \\
&= n_2 + \|[V_1, V_2]\| \\
&> 0
\end{aligned}$$

Therefore $\gamma_{sdR}(G) > \frac{1}{2}(3n - 2m)$, implying that $\gamma_{sdR}(G) > \frac{1}{2}(3n - 2m) > \frac{1}{9}(19n - 24m)$, which completes the proof of the lower bound.

Assume now that $\gamma_{sdR}(G) = \frac{1}{9}(19n-24m)$. Then all the above inequalities must be equalities. In particular, $n_1 = 0$ and $n_3 = n_{13}$. Hence $V_{13} = V_3$ and $V = V_3 \cup V_{-1}$. Moreover, $m = m_{13} + \|[V_{-1}, V_3]\|$, $\|[V_{-1}, V_3]\| = n_{-1}$ and $m_{13} = \frac{1}{6}(n_{-1} - 2n_3)$. This implies that for each vertex $v \in V_{-1}$ we have $d_{V_{-1}}(v) = 0$ and $d_{V_3}(v) = 1$, that is each vertex of V_{-1} is a leaf in G. Moreover for each vertex $v \in V_3$ we have $d_{V_{-1}}(v) = 3d_{V_3}(v) + 2$. Hence, $G \in \mathcal{H}$.

On the other hand, let $G \in \mathcal{H}$. Then $G = L_k$ for some $k \geq 1$. Thus, G is obtained from a graph H (possibly disconnected) of order k by adding $3\deg_H(v) + 2$ pendant edges to each vertex v of H. Let G have order n and size m. Then,

$$n = \sum_{v\in V(H)} (3\deg_H(v) + 3) = 6m(H) + 3n(H)$$

and

$$m = m(H) + \sum_{v\in V(H)} (3\deg_H(v) + 2) = 7m(H) + 2n(H).$$

Assigning to every vertex in $V(H)$ the weight 3 and to every vertex in $V(G) - V(H)$ the weight -1 produces a SDRDF f of weight $\omega(f) = 3n(H) - (6m(H) + 2n(H)) = n(H) - 6m(H) = \frac{19n-24m}{9}$. Hence $\gamma_{sdR}(G) \leq \frac{1}{9}(19n-24m)$. It follows that $\gamma_{sdR}(G) = \frac{1}{9}(19n - 24m)$ and this completes the proof. □

For $k \geq 1$, let F_k be the graph obtained from the complete graph K_k by adding $3k - 1$ pendant edges at each vertex and let $A(F_k)$ be the family of graphs obtained from F_k by adding edges (possibly none) between the leaves of F_k so that to be independent. Let $\mathcal{F} = \cup_{k\geq 1} A(F_k)$.

Theorem 3.2. Let G be a graph of order n. Then $\gamma_{sdR}(G) \geq 4\sqrt{\frac{n}{3}} - n$, with equality if and only if $G \in \mathcal{F}$.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(G)$-function. If $|V_{-1}| = 0$, then $\gamma_{sdR}(G) \geq n + 1 \geq 4\sqrt{\frac{n}{3}} - n$. Hence, let $|V_{-1}| \geq 1$. Since each vertex in V'_{-1} is adjacent to at least one vertex in V_3, we deduce, by the Pigeonhole Principle, that at least one vertex v of V_3 is adjacent to at least $\frac{n'_{-1}}{n_3}$ vertices of V'_{-1}. It follows that $1 \leq f[v] \leq 3n_3 + 2n_2 + n_1 - \frac{n'_{-1}}{n_3}$, and thus

$$0 \leq 3n_3^2 + 2n_2n_3 + n_1n_3 - n'_{-1} - n_3. \tag{1}$$

Likewise, since each vertex in V''_{-1} is adjacent to at least two vertices in V_2, we deduce that at least one vertex u of V_2 is adjacent to at least $\frac{2n''_{-1}}{n_2}$ vertices of V''_{-1}. As above we have

$$0 \leq 3n_3n_2 + 2n_2^2 + n_1n_2 - 2n''_{-1} - n_2. \tag{2}$$

Now, by multiplying the inequality (1) by 2 and summing it with the inequality (2), we obtain

$$0 \leq 6n_3^2 + 2n_2^2 + 7n_2n_3 + 2n_1n_3 + n_1n_2 - 2n'_{-1} - 2n''_{-1} - n_2 - 2n_3.$$

Since $n = n_3 + n_2 + n_1 + n_{-1}$, we have

$$0 \leq 6n_3^2 + 2n_2^2 + 7n_2n_3 + 2n_1n_3 + n_1n_2 + 2n_1 + n_2 - 2n,$$

equivalently

$$\begin{aligned} 0 &\leq 16n_3^2 + \frac{16}{3}n_2^2 + \frac{56}{3}n_2n_3 + \frac{16}{3}n_1n_3 + \frac{16}{6}n_1n_2 + \frac{16}{3}n_1 + \frac{16}{6}n_2 - \frac{16}{3}n \\ &\leq 16n_3^2 + 9n_2^2 + 4n_1^2 + 24n_2n_3 + 16n_1n_3 + 12n_1n_2 - \frac{16}{3}n \\ &= (4n_3 + 3n_2 + 2n_1)^2 - \frac{16}{3}n \end{aligned}$$

which implies that $4\sqrt{\frac{n}{3}} \leq 4n_3 + 3n_2 + 2n_1$. Therefore

$$\begin{aligned} \gamma_{sdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\ &= 4n_3 + 3n_2 + 2n_1 - n \\ &\geq 4\sqrt{\frac{n}{3}} - n. \end{aligned}$$

Let $\gamma_{sdR}(G) = 4\sqrt{\frac{n}{3}} - n$. Then all the above inequalities must be equalities. In particular, $n_1 = n_2 = 0$, $n_3 = n_{123}$, $n = 3n_3^2$ and $n_{-1} = n_3(3n_3 - 1)$. Thus, $V_{123} = V_3$ and $V = V_3 \cup V_{-1}$. Furthermore, each vertex of V_{-1} is adjacent to exactly one vertex of V_3 and each vertex of V_3 is adjacent to all other $n_3 - 1$ vertices of V_3 and to $3n_3 - 1$ vertices of V_{-1}. Since $f[v] \geq 1$ for each vertex $v \in V_{-1}$, we conclude that $d_{V_{-1}}(v) \leq 1$ for each vertex $v \in V_{-1}$, and so $G \in \mathcal{F}$.

On the other hand, suppose $G \in \mathcal{F}$. Then $G \in A(F_k)$ for some $k \geq 1$. Then, G has order $n = 3k^2$, and so $k = \sqrt{\frac{n}{3}}$. Assigning 3 to the vertices of K_k and -1 to the remaining vertices, produces a SDRDF f of weight

$$\omega(f) = 3k - k(3k-1) = 4k - 3k^2 = 4\sqrt{\frac{n}{3}} - n.$$

Hence $\gamma_{sdR}(G) \leq 4\sqrt{\frac{n}{3}} - n$ which implies that $\gamma_{sdR}(G) = 4\sqrt{\frac{n}{3}} - n$. This completes the proof. □

Proposition 3.3. For every graph G of order n, $\gamma_{dR}(G) - \gamma_{sdR}(G) + \gamma(G) \leq n$.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(G)$-function. Recall that $\gamma_{sdR}(G) = w(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$ and $\gamma(G) \leq |V_{23}|$ since V_{23} dominates G. Define the function g on $V(G)$ by $g(x) = 0$ if $x \in V_{-1}$ and $g(x) = f(x)$ otherwise. Clearly, g is a double Roman dominating function on G, and thus $\gamma_{dR}(G) \leq |V_1| + 2|V_2| + 3|V_3| = \gamma_{sdR}(G) + |V_{-1}|$. It follows that

$$\gamma_{dR}(G) \leq \gamma_{sdR}(G) + (n - |V_{123}|) \leq \gamma_{sdR}(G) + n - \gamma(G) - |V_1| \leq \gamma_{sdR}(G) + n - \gamma(G),$$

and the result follows. □

The following two lower bounds on the double Roman domination number are given in [2] and [5], respectively.

Proposition 3.4. ([5]) For any graph G, $\gamma_{dR}(G) \geq 2\gamma(G)$.

Proposition 3.5. ([2]) For any graph G of order n with maximum degree Δ,

$$\gamma_{dR}(G) \geq \frac{2n}{\Delta} + \frac{\Delta - 2}{\Delta}\gamma(G).$$

The next results are immediate consequences of Propositions 3.3, 3.4 and 3.5

Corollary 3.6. For any graph G, $\gamma_{sdR}(G) \geq 3\gamma(G) - n$.

Proof. By Propositions 3.3 and 3.4, we have

$$\begin{aligned}\gamma_{sdR}(G) &\geq \gamma_{dR}(G) + \gamma(G) - n\\ &\geq 3\gamma(G) - n.\end{aligned}$$

□

The previous corollary gives a partial answer to Question posed in [4] concerning a characterization of graphs G for which $\gamma_{sdR}(G) \geq 0$. Clearly, by Corollary 3.6, $\gamma_{sdR}(G) \geq 0$ for all graphs G of order n with $\gamma(G) \geq n/3$.

Corollary 3.7. For any graph G of order n with maximum degree Δ,

$$\gamma_{sdR}(G) \geq \frac{(2-\Delta)}{\Delta}n + \frac{(2\Delta - 2)}{\Delta}\gamma(G).$$

Proof. By Propositions 3.3 and 3.5, we have

$$\begin{aligned}\gamma_{sdR}(G) &\geq \gamma_{dR}(G) + \gamma(G) - n\\ &\geq \frac{2n}{\Delta} + \frac{\Delta - 2}{\Delta}\gamma(G) + \gamma(G) - n\\ &= \frac{(2-\Delta)}{\Delta}n + \frac{(2\Delta - 2)}{\Delta}\gamma(G).\end{aligned}$$

□

Recall that a set S of vertices in a graph G is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G, or equivalently, for every vertex $v \in V$, $|N[v] \cap S| \leq 1$. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G. Note that for a packing S, we have $|N[S]| \geq (\delta + 1)|S|$.

Proposition 3.8. For every graph G of order n, $\gamma_{sdR}(G) \geq (\delta + 2)\rho(G) - n$. This bound is sharp for K_n, $n \neq 4$ and cycles C_{3t} $(t \geq 1)$.

Proof. Let S be a maximum packing set in G, and let f be a $\gamma_{sdR}(G)$-function. Then

$$\begin{aligned}\gamma_{sdR}(G) &= \textstyle\sum_{v\in S} f[v] + \sum_{v\in V-N[S]} f(v)\\ &\geq \textstyle\sum_{v\in S} 1 + \sum_{v\in V-N[S]}(-1)\\ &\geq |S| - |V| + |N[S]|\\ &\geq (\delta+2)\rho(G) - n.\end{aligned}$$

□

4. Special classes of graphs

In this section, we determine the signed double Roman domination number of some classes of graphs including complete graphs, cycles and complete bipartite graphs.

Proposition 4.1. For $n \geq 2$, $\gamma_{sdR}(K_n) = \begin{cases} 2 & \text{if n = 2 or 4.} \\ 1 & \text{otherwise.} \end{cases}$

Proof. The result is trivial to check for $n \leq 4$. Assume that $n \geq 5$ and let f be a $\gamma_{sdR}(G)$-function. For any vertex $v \in V(G)$, we have that $w(f) = f[v] \geq 1$ and so $\gamma_{sdR}(G) = w(f) \geq 1$. If $n \equiv 0 \pmod 3$, then assign to one vertex the weight 3, to $n/3 - 1$ vertices the weight 2, and to the remaining vertices the weight -1. Next, if $n \equiv 1 \pmod 3$, then assign to one vertex the weight 1, to $(n-1)/3$ vertices the weight 2 and to the remaining vertices the weight -1. Finally, if $n \equiv 2 \pmod 3$, then assign to $(n+1)/3$ vertices the weight 2 and to the remaining vertices the weight -1. In all cases, we produce a SDRDF of weight 1, and so $\gamma_{sdR}(G) \leq 1$. Consequently, $\gamma_{sdR}(G) = 1$. □

Proposition 4.2. For $n \geq 3$,

$$\gamma_{sdR}(C_n) = \begin{cases} n/3 & \text{if} \quad n \equiv 0 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 & \text{if} \quad n \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 1 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $C_n := (v_1 v_2 \ldots v_n)$. Define $f : V(C_n) \to \{-1, 1, 2, 3\}$ as follows. If $n \equiv 0 \pmod 3$, then let $f(v_{3i+2}) = 3$ for $0 \leq i \leq (n-3)/3$ and $f(x) = -1$ otherwise. If $n \equiv 1 \pmod 3$, then let $f(v_n) = 3$, $f(v_{3i+2}) = 3$ for $0 \leq i \leq (n-4)/3$ and $f(x) = -1$ otherwise. If $n \equiv 2 \pmod 3$, then let $f(v_{3i+2}) = 3$ for $0 \leq i \leq (n-2)/3$ and $f(x) = -1$ otherwise. Clearly, f is a SDRDF of C_n of desired weight and so

$$\gamma_{sdR}(C_n) \leq \begin{cases} n/3 & \text{if} \quad n \equiv 0 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 & \text{if} \quad n \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 1 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

To prove the inverse inequality, let f be a $\gamma_{sdR}(C_n)$-function. Assume first that $n \equiv 0 \pmod 3$. Then we have

$$\gamma_{sdR}(C_n) = \sum_{i=0}^{\frac{n}{3}-1} f(N[v_{3i+2}]) \geq \sum_{i=0}^{\frac{n}{3}-1} 1 = \frac{n}{3}.$$

Assume now that $n \equiv 1 \pmod 3$. The result is trivial for $n = 4$. Let $n \geq 5$. Obviously, the result is valid if $f(v) \geq 1$ for each $v \in V(C_n)$. Hence, without loss of generality, we assume that $f(v_2) = -1$. By definition, v_2 must have a neighbor with label 3 or two neighbors with label 2. If v_2 has a neighbor with label 3, say v_1, then we have

$$\gamma_{sdR}(C_n) = f(v_1) + \sum_{i=1}^{\frac{n-1}{3}} f(N[v_{3i}]) \geq 3 + \sum_{i=1}^{\frac{n-1}{3}} 1 = 3 + \frac{n-1}{3} = \left\lceil \frac{n}{3} \right\rceil + 2.$$

Let v_2 have two neighbors with label 2, i.e $f(v_1) = f(v_3) = 2$. Since $f(N[v_3]) \geq 1$, we must have $f(v_4) \geq 1$. It follows that

$$\gamma_{sdR}(C_n) = \sum_{i=1}^{4} f(v_i) + \sum_{i=2}^{\frac{n-1}{3}} f(N[v_{3i}]) \geq 4 + \sum_{i=2}^{\frac{n-1}{3}} 1 = 4 + \frac{n-1}{3} - 1 = \left\lceil \frac{n}{3} \right\rceil + 2.$$

Finally, let $n \equiv 2 \pmod 3$. The result holds if $f(v) \geq 1$ for all $v \in V(C_n)$. Hence, without loss of generality, assume that $f(v_2) = -1$. By definition, v_2 must have a neighbor with label 3 or two neighbors with label 2. If v_2 has a neighbor with label 3, say v_1, then we have

$$\gamma_{sdR}(C_n) = f(v_1) + f(v_2) + \sum_{i=1}^{\frac{n-2}{3}} f(N[v_{3i+1}]) \geq 2 + \sum_{i=1}^{\frac{n-2}{3}} 1 = 2 + \frac{n-2}{3} = \left\lceil \frac{n}{3} \right\rceil + 1.$$

Let v_2 have two neighbors with label 2, i.e $f(v_1) = f(v_3) = 2$. Since $f(N[v_1]) \geq 1$ and $f(N[v_3]) \geq 1$, we must have $f(v_n) \geq 1$ and $f(v_4) \geq 1$. It follows that

$$\gamma_{sdR}(C_n) = f(v_n) + \sum_{i=1}^{4} f(v_i) + \sum_{i=2}^{\frac{n-2}{3}} f(N[v_{3i}]) \geq 5 + \sum_{i=2}^{\frac{n-2}{3}} 1 = 5 + \frac{n-2}{3} - 1 \geq \left\lceil \frac{n}{3} \right\rceil + 2$$

and the proof is complete. □

Proposition 4.3. For $2 \leq m \leq n$,

$$\gamma_{sdR}(K_{m,n}) = \begin{cases} 3 & \text{if } m = 2 \text{ and } n \geq 3, \\ 4 & \text{if } m \geq 4 \text{ or } m = n = 2, \\ 5 & \text{if } m = 3. \end{cases}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{m,n}$. The result is immediate for $n = 2$. Assume that $n \geq 3$.

First let $m = 2$. Define the function $f : V(K_{2,n}) \to \{-1, 1, 2, 3\}$ by $f(x_1) = f(x_2) = 2$ and $f(y_i) = (-1)^i$ for $1 \leq i \leq n$, when n is odd, and by $f(x_1) = f(x_2) = 2$, $f(y_1) = f(y_2) = -1, f(y_3) = 2$ and $f(y_i) = (-1)^{i+1}$ for $4 \leq i \leq n$ when n is even. It is clear that f is a SDRDF of $K_{2,n}$ of weight 3 and so $\gamma_{sdR}(K_{2,n}) \leq 3$.
To prove $\gamma_{sdR}(K_{2,n}) \geq 3$, let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(K_{2,n})$-function. Without loss of generality, we assume that $f(x_2) \geq f(x_1)$. If $f(x_1) = f(x_2) = -1$ or $f(x_1) = -1$ and $f(x_2) = 1$, then $f(y_i) \geq 2$ for each i and clearly $\gamma_{sdR}(K_{2,n}) \geq 2n - 2 \geq 4$. If $f(x_1) = -1$ and $f(x_2) \geq 2$, then we have $\gamma_{sdR}(K_{2,n}) = f(x_2) + f[x_1] \geq 2 + f[x_1] \geq 3$ as desired. Assume that $f(x_2) \geq f(x_1) \geq 1$. If $f(x_2) \geq 2$, then $\gamma_{sdR}(K_{2,n}) = f(N[x_1]) + f(x_2) \geq 3$. If $f(x_2) = f(x_1) = 1$, then we must have $f(y_i) \geq 2$ for each i and hence $\gamma_{sdR}(K_{2,n}) \geq 2n + 2$. In any case, $\gamma_{sdR}(K_{2,n}) \geq 3$ and therefore $\gamma_{sdR}(K_{2,n}) = 3$.

Now let $m = 3$. Define the function $f : V(K_{m,n}) \to \{-1, 1, 2, 3\}$ by $f(x_i) = 2$ for $i = 1, 2, 3$, $f(y_1) = f(y_2) = -1$ and $f(y_i) = (-1)^{i+1}$, when n is odd, and by $f(x_i) = 2$ for $i = 1, 2, 3$, $f(y_1) = f(y_2) = -1, f(y_3) = 2$ and $f(y_i) = (-1)^{i+1}$, when n is even. Clearly f is a SDRDF of $K_{3,n}$ of weight 5 yielding $\gamma_{sdR}(K_{3,n}) \leq 5$.
To prove the inverse inequality, let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(K_{3,n})$-function. Without loss of generality, we assume that $f(x_3) \geq f(x_2) \geq f(x_1)$. If $f(x_2) + f(x_3) \geq 4$, then we have $\gamma_{sdR}(K_{3,n}) \geq f(x_2) + f(x_3) + f[x_1] \geq 5$ as desired. Suppose $f(x_2) + f(x_3) \leq 3$. It follows that $f(x_1) \leq 1$. First let $f(x_1) = -1$. Then clearly $f(x_2) + f(x_3) \geq 0$, for otherwise since $f[y_i] \leq 0$ for every i. Now if $f(x_2) + f(x_3) = 0$, then we must have $f(y_i) \geq 2$ for each i and so $\gamma_{sdR}(K_{3,n}) \geq 2n - 1 \geq 5$. Hence we assume that $1 \leq f(x_2) + f(x_3) \leq 3$. Since any vertex with label -1, must have a neighbor with label 3 or two neighbors with label 2, we conclude that $f(y_i) \geq 1$ for each i, and because of $f(x_1) = -1$, either $f(y_i) = 3$ for some i or $f(y_i) = f(y_j) = 2$ for some i, j. It follows that $\gamma_{sdR}(K_{3,n}) \geq n + 2 \geq 5$. Now let $f(x_1) = 1$. Then we must have $2 \leq f(x_2) + f(x_3) \leq 3$. As above, we have $f(y_i) \geq 1$ for each i, implying that $\gamma_{sdR}(K_{3,n}) \geq n + 4 \geq 5$. Thus $\gamma_{sdR}(K_{3,n}) = 5$.

Finally, let $m \geq 4$. To show that $\gamma_{sdR}(K_{m,n}) \geq 4$, let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(K_{m,n})$-function. If $V_{-1} = \emptyset$, then the result is trivial. Thus we assume that $V_{-1} \neq \emptyset$. If $V_{-1} \cap X = \emptyset$ (the case $V_{-1} \cap Y = \emptyset$ is similar), then we have $\gamma_{sdR}(K_{m,n}) = \sum_{i=1}^{m-1} f(x_i) + f[x_m] \geq (m-1) + 1 \geq 4$. Hence we assume that $V_{-1} \cap X \neq \emptyset$ and $V_{-1} \cap Y \neq \emptyset$. Without loss of generality, let $x_1 \in V_{-1}$ and $y_1 \in V_{-1}$. It follows from $f[x_1] \geq 1$ and $f[y_1] \geq 1$ that $\sum_{i=1}^{n} f(y_i) \geq 2$ and $\sum_{i=1}^{m} f(x_i) \geq 2$. Hence $\gamma_{sdR}(K_{m,n}) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 4$, and thus $\gamma_{sdR}(K_{m,n}) \geq 4$.
To prove the inverse inequality, define the functions $f, g, h : V(K_{m,n}) \to \{-1, 1, 2, 3\}$ as follows:
If m, n are even, then let $f(x_1) = f(y_1) = 3$, $f(x_i) = (-1)^{i+1}$ for $2 \leq i \leq m$ and $f(y_i) = (-1)^{i+1}$ for $2 \leq i \leq n$. If m, n are odd, then let $g(x_1) = g(y_1) = 3$, $g(x_2) = g(y_2) = 2$, $g(x_i) = g(y_i) = -1$ for $3 \leq i \leq 5$, $g(x_i) = (-1)^i$ for $6 \leq i \leq m$ and $g(y_i) = (-1)^i$ for $6 \leq i \leq n$. If m is even and n is odd (the case m is odd and n is even is similar), then let $h(x) = f(x)$ if $x \in X$ and $h(x) = g(x)$ if $x \in Y$. Clearly, each of these functions according to the situation for which it is defined is a SDRDF of weight 4, and thus $\gamma_{sdR}(K_{m,n}) \leq 4$. Hence $\gamma_{sdR}(K_{m,n}) = 4$ and the proof is complete. □

Acknowledgements

H. Abdollahzadeh Ahangar was supported by the Babol Noshirvani University of Technology under research Grant Number BNUT/385001/98.

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