Filomat 33:1 (2019), 135–146
https://doi.org/10.2298/FIL1901135A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Zero-Divisor Graph of Real-Valued Continuous Functions on a Frame

Mostafa Abedi[a]

[a] *Esfarayen University of Technology, Esfarayen, North Khorasan, Iran.*

Abstract. The main object of this paper is to study the zero-divisor graph $\Gamma(\mathcal{R}L)$ of the ring $\mathcal{R}L$. Using the properties of the lattice $\mathrm{Coz}\, L$, we associate the ring properties of $\mathcal{R}L$, the graph properties of $\Gamma(\mathcal{R}L)$, and the properties of a competely regular frame L. Paths in $\Gamma(\mathcal{R}L)$ are investigated, and it is shown that the diameter of $\Gamma(\mathcal{R}L)$ and the girth of $\Gamma(\mathcal{R}L)$ coincide whenever L has at least 5 elements. Cycles in $\Gamma(\mathcal{R}L)$ are surveyed, a ring-theoretic and a frame-theoretic characterizations are provided for the graph $\Gamma(\mathcal{R}L)$ to be triangulated or be hypertriangulated. We show that $\Gamma(\mathcal{R}L)$ is complemented if and only if the space of minimal prime ideals of $\mathcal{R}L$ is compact. The relation between the clique number of $\Gamma(\mathcal{R}L)$, the cellularity of L and the dominating number of $\Gamma(\mathcal{R}L)$ is given. Finally, we prove that if $\Gamma(\mathcal{R}L)$ is not triangulated, then the set of centers of $\Gamma(\mathcal{R}L)$ is a dominating set if and only if the socle of $\mathcal{R}L$ is an essential ideal.

1. Introduction

Let R be a commutative ring with identity. As in [1] and [16], by the zero-divisor graph $\Gamma(R)$ of R we mean the (simple) graph with vertices nonzero zero-divisors of R such that there is an edge between vertices x and y if and only if $x \neq y$ and $xy = 0$.

Let $C(X)$ be the ring of all real-valued continuous functions on a completely regular Hausdorff space X. The zero-divisor graph $\Gamma(C(X))$ has been studied by Azarpanah and Motamedi in [3]. They have investigated the relations between ring properties of $C(X)$, graph properties of $\Gamma(C(X))$ and topological properties of the space X.

The ring of real-valued continuous functions on a frame L is the set of all frame homomorphisms $\alpha : \mathcal{L}(\mathbb{R}) \to L$, where $\mathcal{L}(\mathbb{R})$ is the frame of reals, that is, the frame of open subsets of $\mathfrak{O}\mathbb{R}$. This ring is denoted by $\mathcal{R}L$ (see [4, 5] for details). Our main purpose in this article is to study the relations between the ring properties of $\mathcal{R}L$, the graph properties of $\Gamma(\mathcal{R}L)$ and the frame-theoretic properties of the frame L. Our characterizations extend similar ones for $\Gamma(C(X))$ given in [3]. Although, in the statements of the characterizations we give verbatim, literal translations of those in $\Gamma(C(X))$, our proofs are, of necessity, entirely different in that the proofs in [3] use points of spaces involved while our proofs rely heavily on the properties of the cozero part of frames.

Section 3 commences with a description that the concept of distance in $\Gamma(\mathcal{R}L)$ is captured in pointfree topology (Proposition 3.3). We then determine the diameter, girth and the radius of $\Gamma(\mathcal{R}L)$. It turns out that the diameter, the girth and the radius of $\Gamma(\mathcal{R}L)$ are 2 or 3, 3 or 4 and 2 or 3, respectively.

2010 *Mathematics Subject Classification*. Primary 06D22; Secondary 05C99, 13A99
Keywords. Frame, middle P-frame, cycle, triangulated and hypertriangulated, dominating set.
Received: 26 December 2017; Accepted: 05 March 2019
Communicated by Francesco Belardo
Email address: abedi@esfarayen.ac.ir, ms_abedi@yahoo.com (Mostafa Abedi)

In Section 4, we study cycles in $\Gamma(\mathcal{R}L)$. It turns out that the cycles in $\Gamma(\mathcal{R}L)$ have only length 3 or 4. Graphical characterizations of regularity and almost regularity of the ring $\mathcal{R}L$, and pointfree characterizations for the graph $\Gamma(\mathcal{R}L)$ to be triangulated, hypertriangulated and complemented are provided in this section. In the paper [15], the concept of a middle P-frame has been introduced by Ighedo. In Proposition 4.5, we show that $\Gamma(\mathcal{R}L)$ is a hypertriangulated graph if and only if L is a connected middle P-frame.

We introduce the cellularity in Definitions 5.1. This definition and the weight of a frame enable us to study the dominating number and the clique number of $\Gamma(\mathcal{R}L)$ in Section 5. A pointfree characterization and an algebraic characterization for the set of centers of $\Gamma(\mathcal{R}L)$ to be a dominating set are given in Theorem 5.7.

2. Preliminaries

2.1. *Frames*

For general facts concerning pointfree functions rings, general topology, the ring $C(X)$, and frames see [4, 5], [13], [14], and [17]. Here, we recall a few definitions and results that will be relevant for our discussion.

A frame is a complete lattice for which finite meets distribute over arbitrary joins. Let L be a frame. We denote the top element and the bottom element of L by $\top$ and $\bot$ respectively. Throughout this context L will denote a frame. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}(X)$.

The *pseudocomplement* of an element $a \in L$, denoted a^*, is the element

$$a^* = \bigvee\{x \in L \mid a \wedge x = \bot\}.$$

We recall that:

(1) if $a \leq b$, then $b^* \leq a^*$.

(2) $a \leq a^{**}$ and $a^* = a^{***}$.

(3) $(a \vee b)^* = a^* \wedge b^*$ and $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element a of L is said to be *complemented* if $a \vee a^* = \top$, and dense if $a^* = \bot$. We call the set of all complemented elements of L the *Boolean part* of L, and denote it by BL. For any frame L, we have $BL = \{x \in L : x \vee x^* = \top\}$ and BL is a sublattice of L. Notice that every element x of BL has a unique complement, which is denoted by x'.

An element $p \in L$ is said to be an *atom* if $p \neq \bot$ and there exists no element x with $\bot < x < p$.

A frame L is said to be *completely regular* if, for each $a \in L$, $a = \bigvee\{x \in L \mid x \lll a\}$, where $x \lll a$ means that there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0, 1]$ such that $c_0 = x$, $c_1 = a$, and $c_p \prec c_q$ for $p < q$. Note that $x \prec a$ means that there is an element b such that $x \wedge b = \bot$ and $b \vee a = \top$, or equivalently, $x^* \vee a = \top$. Throughout, all frames under consideration are assumed to be completely regular.

2.2. *The ring $\mathcal{R}L$*

Regarding the frame of reals $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of continuous real functions on L, we use the notation of [5]. We freely use the properties of the cozero map $\operatorname{coz}\colon \mathcal{R}L \to L$, given by

$$\operatorname{coz}\alpha = \bigvee\{\alpha(p, q) \mid q < 0 \text{ or } p > 0\},$$

and those of $\operatorname{Coz}L = \{\operatorname{coz}\alpha \mid \alpha \in \mathcal{R}L\}$, the *cozero part* of L. Note that $\operatorname{Coz}L$ is a regular sub-σ-frame of L; and a frame is completely regular if and only if it is generated by its cozero part. We refer to [4–6] for general properties of cozero elements and cozero parts of frames.

3. Paths in $\Gamma(\mathcal{R}L)$

To begin with, we note that $\alpha \in \mathcal{R}L$ is a zero-divisor if and only if $\operatorname{coz}\alpha$ is not dense (see [9, Corollary 4.2] for details). Hence, $\mathbf{0} \neq \alpha \in \Gamma(\mathcal{R}L)$ if and only if $(\operatorname{coz}\alpha)^* \neq \bot$. Also if $a \in L$ is different from the top or the bottom, then there is $\alpha \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\alpha \lll a$. To see this, by complete regularity, take $\alpha \in \mathcal{R}L$ such that $\bot \neq \operatorname{coz}\alpha \lll a$. That is to say $(\operatorname{coz}\alpha)^* \vee a = \top$, implying that $(\operatorname{coz}\alpha)^* \neq \bot$. Thus $\alpha \in \Gamma(\mathcal{R}L)$.

Remark 3.1. *If $|L| = 2$, that is, $L = \mathbf{2}$, then $\mathcal{R}L$ is isomorphic with the field of real numbers, that is, $\mathcal{R}L \cong \mathbb{R}$. On the other hand, the three-element chain $\mathbf{3} = \{\perp, m, \top\}$ is not completely regular. Thus for studying the zero-divisor graph of $\mathcal{R}L$, we should consider $|L| \geq 4$. Next, by [1, Theorem 2.2], it is easy to see that $\Gamma(\mathcal{R}L)$ is always infinite.*

Recall that for two vertices α and β of $\Gamma(\mathcal{R}L)$, $d(\alpha, \beta)$ is the length of the shortest path from α to β. The diameter of $\Gamma(\mathcal{R}L)$ is denoted by $\operatorname{diam}\Gamma(\mathcal{R}L)$ and is defined by $\operatorname{diam}\Gamma(\mathcal{R}L) = \sup\{d(\alpha,\beta) \mid \alpha, \beta \in \Gamma(\mathcal{R}L)\}$. The girth of $\Gamma(\mathcal{R}L)$, denoted $\operatorname{gr}\Gamma(\mathcal{R}L)$, is defined as the length of the shortest cycle in $\Gamma(\mathcal{R}L)$.

The following proposition characterizes the concept of distance in $\Gamma(\mathcal{R}L)$ using cozero elements of L. First, we need the following lemma. In order to state this lemma, we need some background. As in [7], if $\alpha : \mathcal{L}(\mathbb{R}) \to L$ and $a \in L$, then $\alpha|a$ denotes the composite $\alpha : \mathcal{L}(\mathbb{R}) \to L \to \downarrow a$. Recall also from [7, Lemma 1] that if $a \ll b$ in L, then there exists $\varphi \in \mathcal{R}L$ such that $\mathbf{0} \leq \varphi \leq \mathbf{1}$, $\varphi|a = \mathbf{1}_a$, and $\varphi|b^* = \mathbf{0}$.

Lemma 3.2. *For every $\alpha, \beta \in \Gamma(\mathcal{R}L)$, there exists a vertex $\varphi \in \Gamma(\mathcal{R}L)$ adjacent to both α and β if and only if $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$.*

Proof. We begin with the sufficiency. Let $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$. Then, by complete regularity, there exists $\gamma \in \mathcal{R}L$ such that $\perp \neq \operatorname{coz}\gamma \ll (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*$. Now, take $\varphi \in \mathcal{R}L$ such that $\varphi|\operatorname{coz}\gamma = \mathbf{1}$ and $\varphi|\big((\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*\big)^* = \mathbf{0}$. The latter implies that $\operatorname{coz}\varphi \wedge \big((\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*\big)^* = \perp$ and hence

$$\operatorname{coz}\varphi \leq \big((\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*\big)^{**} = (\operatorname{coz}\alpha)^{***} \wedge (\operatorname{coz}\beta)^{***} = (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*.$$

In consequence,

$$\begin{aligned}\operatorname{coz}(\varphi\alpha) = \operatorname{coz}\varphi \wedge \operatorname{coz}\alpha &\leq \big((\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*\big) \wedge \operatorname{coz}\alpha \\ &= \big((\operatorname{coz}\alpha)^* \wedge \operatorname{coz}\alpha\big) \wedge (\operatorname{coz}\beta)^* = \perp.\end{aligned}$$

Therefore $\varphi\alpha = \mathbf{0}$, similarly $\varphi\beta = \mathbf{0}$. Consequently, $\varphi \in \Gamma(\mathcal{R}L)$ and φ adjacent to both α and β. Conversely, if there exists $\varphi \in \Gamma(\mathcal{R}L)$ adjacent to both α and β, then $\varphi\alpha = \varphi\beta = \mathbf{0}$. This implies that $\operatorname{coz}\varphi \leq (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^*$ and therefore $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$, since $\operatorname{coz}\varphi \neq \perp$. □

Proposition 3.3. *Let $\alpha, \beta \in \Gamma(\mathcal{R}L)$. Then the following statements hold.*

1. $d(\alpha, \beta) = 1$ *if and only if* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \perp$.
2. $d(\alpha, \beta) = 2$ *if and only if* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ *and* $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$.
3. $d(\alpha, \beta) = 3$ *if and only if* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ *and* $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp$.

Proof. (1). Trivial.

To prove (2), first suppose that $d(\alpha, \beta) = 2$. Then, by part (1), $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ and there exists $\varphi \in \Gamma(\mathcal{R}L)$ such that φ is adjacent to both α and β. Therefore, by Lemma 3.2, $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$. Conversely, let $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$. Then, by part (1), $d(\alpha, \beta) > 1$ and, by Lemma 3.2, there is a vertex adjacent to both α and β. These imply $d(\alpha, \beta) = 2$.

To show (3), let $d(\alpha, \beta) = 3$. Clearly, by parts (1) and (2), $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp$. Conversely, suppose that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp$. By parts (1) and (2), $d(\alpha, \beta) > 2$. Now, if a vertex δ is adjacent to α and a vertex γ is adjacent to β, then $\alpha\delta = \beta\gamma = \mathbf{0}$. In consequence, $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma \leq (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp$, implying that $\operatorname{coz}(\delta\gamma) = \perp$ and hence $\delta\gamma = \mathbf{0}$, this means that δ is adjacent to γ. Therefore $d(\alpha, \beta) = 3$. □

Note that if $\alpha \in \Gamma(\mathcal{R}L)$, then $\operatorname{coz}\alpha \wedge \operatorname{coz}\mathbf{2}\alpha \neq \perp$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\mathbf{2}\alpha)^* \neq \perp$. Now as a consequence, by part (2) of Proposition 3.3, we have the following.

Corollary 3.4. *Whenever $|L| \geq 4$, then* $\operatorname{diam}\Gamma(\mathcal{R}L) \geq 2$.

We intend to show that $\operatorname{diam}\Gamma(\mathcal{R}L) = \operatorname{gr}\Gamma(\mathcal{R}L) = 3$ for when $|L| \neq 4$. For this we shall need a series of results. We begin with a lemma. Before the following lemma is presented, let us recall that a graph G is connected if there is a path between any two distinct vertices. Note that $\Gamma(\mathcal{R}L)$ is always connected (see [1, Theorem 2.3]).

Lemma 3.5. *Let $\alpha, \beta \in \Gamma(\mathcal{R}L)$ be such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \perp$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \perp$. Then $\operatorname{gr}\Gamma(\mathcal{R}L) = 3$.*

Proof. By hypothesis, we have $\alpha\beta = \mathbf{0}$ and $\alpha^2+\beta^2$ is a nonzero zero-divisor. Hence, there exists $\gamma \in \Gamma(\mathcal{R}L)$ such that $\gamma(\alpha^2 + \beta^2) = \mathbf{0}$, implying that $\gamma^2(\alpha^2 + \beta^2) = \mathbf{0}$. This shows that $\gamma\alpha = \gamma\beta = \mathbf{0}$. Therefore $\operatorname{gr}\Gamma(\mathcal{R}L) = 3$. □

Recall that if a is an atom of L, then, by complete regularity, it is complemented and so $a \in \operatorname{Coz} L$. For the proof of the next corollary, we shall use the fact that if a and b are two atoms of L such that $a' \neq b$, then

$$a \wedge b = \perp \Rightarrow a \vee b \neq \top \Rightarrow a' \wedge b' \neq \perp.$$

Corollary 3.6. *Whenever L has at least 3 atoms, then $\operatorname{diam}\Gamma(\mathcal{R}L) = \operatorname{gr}\Gamma(\mathcal{R}L) = 3$.*

Proof. Whenever L has at least 3 atoms, then there exist $\alpha, \beta \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp$. Now, part (3) of Proposition 3.3 implies that $\operatorname{diam}\Gamma(\mathcal{R}L) = 3$. To prove the second part it is easy to see that there exist $\delta, \rho \in \mathcal{R}L$ such that $\operatorname{coz}\delta \wedge \operatorname{coz}\rho = \perp$ and $(\operatorname{coz}\delta)^* \wedge (\operatorname{coz}\rho)^* \neq \perp$. Now, Lemma 3.5 shows that $\operatorname{gr}\Gamma(\mathcal{R}L) = 3$. □

Recall from [8, Lemma 3.3] that if $\operatorname{coz}\alpha \lll \operatorname{coz}\beta$ for some $\alpha, \beta \in \mathcal{R}L$, then there exists $\delta \in \mathcal{R}L$ such that $\alpha = \delta\beta$.

Example 3.7. *Suppose that $|L| = 4$. Since the four-element chain $\mathbf{4} = \{\perp, m < n, \top\}$ is not completely regular, we can conclude that $L = \{\perp, a, b, \top\}$, where $b = a'$. Now, let $\alpha, \beta \in \mathcal{R}L$ with $\operatorname{coz}\alpha = a, \operatorname{coz}\beta = b$. We put*

$$A = \{\delta \in \mathcal{R}L \mid \operatorname{coz}\delta = \operatorname{coz}\alpha\} \quad \text{and} \quad B = \{\gamma \in \mathcal{R}L \mid \operatorname{coz}\gamma = \operatorname{coz}\beta\}.$$

It is easy to see that the zero-divisor graph of $\mathcal{R}L$ is a graph where its vertices are two disjoint nonempty sets A and B such that two vertices δ and γ are adjacent if and only if $\delta \in A$ and $\gamma \in B$. This means that $\Gamma(\mathcal{R}L)$ is a bipartite complete graph. Consequently, $\operatorname{diam}\Gamma(\mathcal{R}L) = 2$ and $\operatorname{gr}\Gamma(\mathcal{R}L) = 4$.

Before proving the next result let us notice the following about cozero elements. If $a \lll b$ in L, then there is $c \in \operatorname{Coz} L$ such that $a \lll c \lll b$ (see [6, Corollary 1]). As a consequence, there exists $d \in \operatorname{Coz} L$ such that $a \wedge d = \perp$ and $d \vee b = \top$.

Lemma 3.8. *Let $\alpha \in \Gamma(\mathcal{R}L)$. Then the following statements hold.*

1. *If $\operatorname{coz}\alpha \notin BL$, then there exists $\beta \in \Gamma(\mathcal{R}L)$ such that $d(\alpha, \beta) = 3$.*
2. *Let $\operatorname{coz}\alpha \in BL$. If there exists $\gamma \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\gamma \lneqq \operatorname{coz}\alpha$ and $\operatorname{coz}\gamma \in BL$, then there exists $\beta \in \Gamma(\mathcal{R}L)$ such that $d(\alpha, \beta) = 3$.*

Proof. (1). Since $\operatorname{coz}\alpha \notin BL$, there exists $\delta \in \mathcal{R}L$ such that $\perp \neq \operatorname{coz}\delta \lll \operatorname{coz}\alpha$. This means that $\operatorname{coz}\delta \wedge \operatorname{coz}\beta = \perp$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = \top$ for some $\beta \in \mathcal{R}L$. In consequence,

$$(\operatorname{coz}\beta)^* \neq \perp, \ \operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp \text{ and } (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp.$$

Therefore, part (3) of Proposition 3.3 implies that $d(\alpha, \beta) = 3$.

(2). Putting $\operatorname{coz}\beta = (\operatorname{coz}\gamma)'$, we then have

$$\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \perp \text{ and } (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \perp.$$

Therefore, part (3) of Proposition 3.3 implies that $d(\alpha, \beta) = 3$. □

Proposition 3.9. *Whenever L has at least 5 elements, then $\operatorname{diam}\Gamma(\mathcal{R}L) = 3$.*

Proof. We consider two cases.

Case 1: Suppose that $\operatorname{Coz} L = BL$ (note that L can be finite or infinite). Then $\operatorname{Coz} L$ has at least 3 elements other than the top or the bottom. When $\operatorname{Coz} L$ has at least 3 atoms, Corollary 3.6 implies that $\operatorname{diam}\Gamma(\mathcal{R}L) = 3$. Now, suppose $\operatorname{coz}\alpha$ is not an atom for some $\alpha \in \mathcal{R}L$. Then there exists $\delta \in \mathcal{R}L$ such that $\perp \neq \operatorname{coz}\delta \lneqq \operatorname{coz}\alpha$. Therefore, part (2) of Lemma 3.8 implies that $\operatorname{diam}\Gamma(\mathcal{R}L) = 3$.

Case 2: Suppose $BL \subsetneq \mathrm{Coz}\, L$ (note that L and $\mathrm{Coz}\, L$ are infinite). If there exists $\alpha \in \Gamma(\mathcal{R}L)$ such that $\mathrm{coz}\, \alpha \in \mathrm{Coz}\, L \setminus BL$, then part (1) of Lemma 3.8 implies that $\mathrm{diam}\, \Gamma(\mathcal{R}L) = 3$. Otherwise, for every $\alpha \in \mathcal{R}L$ with $(\mathrm{coz}\, \alpha)^* \neq \bot$, we have $\mathrm{coz}\, \alpha \in BL$. This means that we can choose $\delta \in \Gamma(\mathcal{R}L)$ with $\mathrm{coz}\, \delta \in \mathrm{Coz}\, L$. So, take $\rho \in \mathcal{R}L$ such that $\mathrm{coz}\, \rho = (\mathrm{coz}\, \delta)'$. In case $\mathrm{coz}\, \delta$ or $\mathrm{coz}\, \rho$ is not an atom, part (2) of Lemma 3.8 implies that $\mathrm{diam}\, \Gamma(\mathcal{R}L) = 3$. Now, let $\mathrm{coz}\, \delta$ and $\mathrm{coz}\, \rho$ be two atoms. If there is $\gamma \in \Gamma(\mathcal{R}L)$ such that $\mathrm{coz}\, \gamma \neq \mathrm{coz}\, \delta$ and $\mathrm{coz}\, \gamma \neq \mathrm{coz}\, \rho$, then, by either Corollary 3.6 or part (1) of Lemma 3.8, we can conclude that $\mathrm{diam}\, \Gamma(\mathcal{R}L) = 3$. Otherwise, by complete regularity, it is easy to show $L = \{\bot, \mathrm{coz}\, \delta, \mathrm{coz}\, \rho, \top\}$ which is a contradiction. Therefore, the proof is complete. □

The combination of this proposition with Example 3.7 imply the following corollary.

Corollary 3.10. *The* $\mathrm{diam}\, \Gamma(\mathcal{R}L) = 2$ *if and only if* $L = \{\bot, a, b, \top\}$, *where* $b = a'$.

Next, we are going to discuss the girth of $\Gamma(\mathcal{R}L)$. We begin with the following lemma. For the proof of this lemma, we shall use the following fact: If $a, b \in L$ and $a \ll b$, then $b^* \ll a^*$.

Lemma 3.11. *Let* $\alpha \in \Gamma(\mathcal{R}L)$. *Then the following statements hold.*

1. *Let* $\mathrm{coz}\, \alpha \in BL$. *If there exists* $\gamma \in \Gamma(\mathcal{R}L)$ *such that* $\mathrm{coz}\, \gamma \lneqq \mathrm{coz}\, \alpha$ *and* $\mathrm{coz}\, \gamma \in BL$, *then* $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$.
2. *If* $\mathrm{coz}\, \alpha \notin BL$, *then* $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$.

Proof. (1). Putting $\mathrm{coz}\, \beta = (\mathrm{coz}\, \alpha)'$, we then have

$$\mathrm{coz}\, \gamma \wedge \mathrm{coz}\, \beta \leq \mathrm{coz}\, \alpha \wedge \mathrm{coz}\, \beta = \bot, \text{ in consequence } \mathrm{coz}\, \gamma \wedge \mathrm{coz}\, \beta = \bot;$$

and also we claim that $(\mathrm{coz}\, \gamma)^* \wedge (\mathrm{coz}\, \beta)^* \neq \bot$. To see this, suppose, by way of contradiction, that $(\mathrm{coz}\, \gamma)^* \wedge (\mathrm{coz}\, \beta)^* = \bot$. Then $(\mathrm{coz}\, \gamma)^* \wedge \mathrm{coz}\, \alpha = \bot$, implying that $\mathrm{coz}\, \alpha \leq \mathrm{coz}\, \gamma \leq \mathrm{coz}\, \alpha$ and hence $\mathrm{coz}\, \alpha = \mathrm{coz}\, \gamma$ which is a contradiction. Therefore, by Lemma 3.5, $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$.

(2). Since $\mathrm{coz}\, \alpha \notin BL$, there exists $\delta \in \mathcal{R}L$ such that $\bot \neq \mathrm{coz}\, \delta \ll \mathrm{coz}\, \alpha$ and so $(\mathrm{coz}\, \alpha)^* \ll (\mathrm{coz}\, \delta)^*$. Take $\rho \in \mathcal{R}L$ such that $(\mathrm{coz}\, \alpha)^* \ll \mathrm{coz}\, \rho \ll (\mathrm{coz}\, \delta)^*$. This show that $(\mathrm{coz}\, \rho)^* \neq \bot$ and $\mathrm{coz}\, \rho \wedge \mathrm{coz}\, \delta \leq \mathrm{coz}\, \rho \wedge (\mathrm{coz}\, \delta)^{**} = \bot$, that is, $\mathrm{coz}\, \rho \wedge \mathrm{coz}\, \delta = \bot$. Now, if $(\mathrm{coz}\, \rho)^* \wedge (\mathrm{coz}\, \delta)^* \neq \bot$, then Lemma 3.5 shows that $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$. Otherwise, let $(\mathrm{coz}\, \rho)^* \wedge (\mathrm{coz}\, \delta)^* = \bot$. On the other hand, $\mathrm{coz}\, \rho \ll (\mathrm{coz}\, \delta)^*$ implies that $(\mathrm{coz}\, \rho)^* \vee (\mathrm{coz}\, \delta)^* = \top$. Therefore $(\mathrm{coz}\, \delta)^* \in BL$. Now, we consider two cases.

Case 1: Suppose $(\mathrm{coz}\, \alpha)^* \in BL$. Then since $\mathrm{coz}\, \delta \ll \mathrm{coz}\, \alpha$, $(\mathrm{coz}\, \delta)^* \vee \mathrm{coz}\, \alpha = \top$, showing that $(\mathrm{coz}\, \delta)^* \neq (\mathrm{coz}\, \alpha)^*$. Therefore, by (1), $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$ since $(\mathrm{coz}\, \alpha)^* \lneqq (\mathrm{coz}\, \delta)^*$, $(\mathrm{coz}\, \alpha)^{**} \neq \bot$, and $(\mathrm{coz}\, \delta)^{**} \neq \bot$.

Case 2: Suppose $(\mathrm{coz}\, \alpha)^* \notin BL$. Pick $\varphi \in \mathcal{R}L$ such that $(\mathrm{coz}\, \varphi)^* \neq \bot$ and $\bot \neq \mathrm{coz}\, \varphi \ll (\mathrm{coz}\, \alpha)^* \ll (\mathrm{coz}\, \delta)^*$, implying that $\mathrm{coz}\, \varphi \wedge \mathrm{coz}\, \delta = \bot$. Now, if $(\mathrm{coz}\, \varphi)^* \wedge (\mathrm{coz}\, \delta)^* \neq \bot$, then, by Lemma 3.5, $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$. Otherwise, $(\mathrm{coz}\, \varphi)^* \wedge (\mathrm{coz}\, \delta)^* = \bot$ implies that $(\mathrm{coz}\, \varphi)^* \leq (\mathrm{coz}\, \delta)^{**}$. This shows that $(\mathrm{coz}\, \varphi)^* = (\mathrm{coz}\, \delta)^{**}$ since $\mathrm{coz}\, \varphi \ll (\mathrm{coz}\, \delta)^*$. Consequently, $(\mathrm{coz}\, \delta)^{**} = (\mathrm{coz}\, \alpha)^{**}$, implying $(\mathrm{coz}\, \delta)^* = (\mathrm{coz}\, \alpha)^*$, a contradiction because $(\mathrm{coz}\, \delta)^* \in BL$ implies $(\mathrm{coz}\, \alpha)^* \in BL$. □

Proposition 3.12. *Whenever L has at least* 5 *elements, then* $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$.

Proof. We consider two cases.

Case 1: Suppose $\mathrm{Coz}\, L = BL$. Then $\mathrm{Coz}\, L$ has at least 3 elements other than the top or bottom. Whenever $\mathrm{Coz}\, L$ has at least 3 atoms, then Corollary 3.6 implies $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$. Now, suppose $\mathrm{coz}\, \alpha$ is not an atom for some $\alpha \in \mathcal{R}L$. Then there exists $\tau \in \mathcal{R}L$ such that $\bot \neq \mathrm{coz}\, \tau \lneqq \mathrm{coz}\, \alpha$. Therefore, part (1) of Lemma 3.11 implies that $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$.

Case 2: Suppose $BL \subsetneq \mathrm{Coz}\, L$. If there exists $\alpha \in \Gamma(\mathcal{R}L)$ such that $\mathrm{coz}\, \alpha \in \mathrm{Coz}\, L \setminus BL$, Then part (2) of Lemma 3.11 implies that $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$. Otherwise, for every $\alpha \in \mathcal{R}L$ with $(\mathrm{coz}\, \alpha)^* \neq \bot$, we have $\mathrm{coz}\, \alpha \in BL$. This means that we can choose $\delta \in \Gamma(\mathcal{R}L)$ with $\mathrm{coz}\, \delta \in \mathrm{Coz}\, L$. Hence, take $\rho \in \mathcal{R}L$ such that $\mathrm{coz}\, \rho = (\mathrm{coz}\, \delta)'$. In case $\mathrm{coz}\, \delta$ or $\mathrm{coz}\, \rho$ is not an atom, part (1) of Lemma 3.11 implies that $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$. Now, suppose $\mathrm{coz}\, \delta$ and $\mathrm{coz}\, \rho$ are atoms. If there is $\gamma \in \Gamma(\mathcal{R}L)$ such that $\mathrm{coz}\, \gamma \neq \mathrm{coz}\, \delta$ and $\mathrm{coz}\, \gamma \neq \mathrm{coz}\, \rho$, then, by either Corollary 3.6 or part (1) of Lemma 3.11, we can conclude that $\mathrm{gr}\, \Gamma(\mathcal{R}L) = 3$. Otherwise, by complete regularity, it is easy to show that $L = \{\bot, \mathrm{coz}\, \delta, \mathrm{coz}\, \rho, \top\}$ which is a contradiction. Therefore, the proof is complete. □

An immediate consequence of Example 3.7 and the previous proposition, is the following corollary.

Corollary 3.13. *The* $\operatorname{gr}\Gamma(\mathcal{R}L) = 4$ *if and only if* $L = \{\bot, a, b, \top\}$ *where* $b = a'$.

Combining Propositions 3.9 and 3.12, we have the following theorem.

Theorem 3.14. *The diameter of* $\Gamma(\mathcal{R}L)$ *and the girth of* $\Gamma(\mathcal{R}L)$ *coincide whenever* L *has at least* 5 *elements.*

In what follows, we intend to study the radius of the zero-divisor graph of $\Gamma(\mathcal{R}L)$. Let us recall the definition of the radius of a graph G. The *associated number* of a vertex x of a graph G denoted by $e(x)$ is defined as $e(x) = \sup\{d(x, y) \mid x \neq y \in G\}$. A *center* of G is defined to be a vertex t with the smallest associated number. The associated number $e(t)$ of any center t is said to be the *radius* of G and is denoted by $\varrho(G)$.

Proposition 3.15. *Suppose* $\alpha \in \Gamma(\mathcal{R}L)$, *then*

$$e(\alpha) = \begin{cases} 2 & \text{if } \operatorname{coz}\alpha \text{ is an atom} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. First, let $\operatorname{coz}\alpha$ be an atom. Consider $\beta \in \Gamma(\mathcal{R}L)$. Then $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \operatorname{coz}\alpha$ or $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$. The latter implies that $d(\alpha, \beta) = 1$. By the former case, we have $\operatorname{coz}\alpha \leq \operatorname{coz}\beta$, showing that $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = (\operatorname{coz}\beta)^* \neq \bot$. Hence, by part (2) of Proposition 3.3, $d(\alpha, \beta) = 2$. Therefore $e(\alpha) = 2$ since $\operatorname{coz}\alpha \wedge \operatorname{coz}\mathbf{2}\alpha = \operatorname{coz}\alpha$. Now, suppose $\operatorname{coz}\alpha$ is not an atom. Then we consider two cases.

Case 1: If $\operatorname{coz}\alpha \notin BL$, then part (1) of Lemma 3.8 implies that $e(\alpha) = 3$.

Case 2. Suppose $\operatorname{coz}\alpha \in BL$. Since $\operatorname{coz}\alpha$ is not an atom, there exists $\delta \in \mathcal{R}L$ such that $\bot \neq \operatorname{coz}\delta \ll \operatorname{coz}\alpha$ and $(\operatorname{coz}\delta)^* \neq \bot$. In case $\operatorname{coz}\delta \in BL$, part (2) of Lemma 3.8 implies that $e(\alpha) = 3$. Otherwise, by Case 1, $e(\delta) = 3$. Take $\beta \in \Gamma(\mathcal{R}L)$ such that $d(\delta, \beta) = 3$, and so, by part (2) of Proposition 3.3, we have $\operatorname{coz}\delta \wedge \operatorname{coz}\beta \neq \bot$ and $(\operatorname{coz}\delta)^* \wedge (\operatorname{coz}\beta)^* = \bot$. This implies quickly that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \bot$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$. Again, by part (2) of Proposition 3.3, $d(\alpha, \beta) = 3$, which shows that $e(\alpha) = 3$. □

As an immediate consequence, we now have the following corollary.

Corollary 3.16. *If* L *has an atom, then* $\varrho(\Gamma(\mathcal{R}L)) = 2$; *otherwise* $\varrho(\Gamma(\mathcal{R}L)) = 3$.

By this corollary and the definition of center, if L has no atoms, that is, $\varrho(\Gamma(\mathcal{R}L)) = 3$; then every vertex is a center. But whenever L has at least one atom, that is, if $\varrho(\Gamma(\mathcal{R}L)) = 2$, then the set of centers of $\Gamma(\mathcal{R}L)$ is the set of all vertices $\alpha \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\alpha$ is an atom.

4. Cycles in $\Gamma(\mathcal{R}L)$

A graph G is called *triangulated* (*hypertriangulated*) if each vertex (edge) of G is a vertex (edge) of a triangle. In the next proposition, we show that every vertex of $\Gamma(\mathcal{R}L)$ is a cycle vertex, that is, every vertex of $\Gamma(\mathcal{R}L)$ belongs to a cycle. In fact, it turns out that for every vertex α in $\Gamma(\mathcal{R}L)$, there exists a 4-cycle (quadrangle) containing α and whenever L has no atoms, then for every vertex α of $\Gamma(\mathcal{R}L)$, there exists a 3-cycle (triangle) containing α.

Proposition 4.1. *For a frame* L, *every vertex of* $\Gamma(\mathcal{R}L)$ *is a* 4*-cycle-vertex.*

Proof. For every vertex α, there exists a vertex β such that $\alpha\beta = \mathbf{0}$ since $\Gamma(\mathcal{R}L)$ is always connected. Therefore $\alpha\beta = (\mathbf{2}\alpha)\beta = (\mathbf{2}\alpha)(\mathbf{2}\beta) = \alpha(\mathbf{2}\beta) = \mathbf{0}$, that is, the path with vertices α, β, $\mathbf{2}\alpha$ and $\mathbf{2}\beta$ is a cycle with length 4 containing α. □

By the above proposition, every vertex in $\Gamma(\mathcal{R}L)$ is a vertex of a cycle. It is also easy to see that every edge in $\Gamma(\mathcal{R}L)$ is an edge of a cycle.

In the following theorem, we give the frame-theoretic property of L and the ring-theoretic property of $\mathcal{R}L$ for which the graph $\Gamma(\mathcal{R}L)$ is triangulated. We begin with the following lemma.

Lemma 4.2. *If $\alpha, \beta \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\alpha \lneqq \operatorname{coz}\beta$ and $\operatorname{coz}\beta \in BL$, then α is a vertex of a triangle.*

Proof. Putting $\operatorname{coz}\gamma = (\operatorname{coz}\beta)'$, we then have $\operatorname{coz}\alpha \wedge \operatorname{coz}\gamma = \bot$. Now, $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\gamma)^* = \bot$ implies that $\operatorname{coz}\alpha = \operatorname{coz}\beta$ which is a contradiction. In consequence, $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\gamma)^* \neq \bot$. Therefore, by Lemma 3.2, there exists a vertex δ adjacent to both α and γ, showing α is a vertex of the triangle with vertices α, γ and δ. □

Recall that an ideal of a ring is *essential* if it meets every nonzero ideal non-trivially. By Lemma 4.3 in [9], an ideal I in $\mathcal{R}L$ is essential if and only if $\bigvee_{\delta\in I} \operatorname{coz}\delta$ is dense.

Theorem 4.3. *The following are equivalent for a frame L.*

1. $\Gamma(\mathcal{R}L)$ *is a triangulated graph.*
2. *L has no atoms.*
3. *There is no maximal ideal in $\mathcal{R}L$ generated by an idempotent.*

Proof. (1) ⇒ (2). Let $\Gamma(\mathcal{R}L)$ be a triangulated graph and suppose L has an atom a. Consider $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}\alpha = a'$, clearly $\alpha \in \Gamma(\mathcal{R}L)$. Since $\Gamma(\mathcal{R}L)$ is triangulated, then there are $\gamma, \delta \in \Gamma(\mathcal{R}L)$ such that $\alpha\gamma = \alpha\delta = \gamma\delta = \mathbf{0}$. This shows that $\operatorname{coz}\gamma \leq a$, $\operatorname{coz}\delta \leq a$, and $\operatorname{coz}\gamma \wedge \operatorname{coz}\delta = \bot$. Therefore $\gamma = \mathbf{0}$ or $\delta = \mathbf{0}$, which is a contradiction.

(2) ⇒ (1). Suppose that L does not contain atoms and take $\alpha \in \Gamma(\mathcal{R}L)$. Then there exists $\mathbf{0} \neq \beta \in \mathcal{R}L$ such that $\operatorname{coz}\beta \lll (\operatorname{coz}\alpha)^*$ and $\operatorname{coz}\beta \neq (\operatorname{coz}\alpha)^*$. Now, we consider two cases.

Case 1: Suppose $(\operatorname{coz}\beta)^* \wedge (\operatorname{coz}\alpha)^* \neq \bot$. Then, by Lemma 3.2, there exists a vertex γ adjacent to both α and β, showing α is a vertex of the triangle with vertices α, β and γ since $\operatorname{coz}\beta \wedge \operatorname{coz}\alpha = \bot$.

Case 2: Assume $(\operatorname{coz}\beta)^* \wedge (\operatorname{coz}\alpha)^* = \bot$. Pick $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\delta \lll \operatorname{coz}\beta$. If $(\operatorname{coz}\delta)^* \wedge (\operatorname{coz}\alpha)^* \neq \bot$, then, similar to Case 1, α is a vertex of a triangle. Otherwise, let $(\operatorname{coz}\delta)^* \wedge (\operatorname{coz}\alpha)^* = \bot$. Then since

$$(\operatorname{coz}\beta)^* \vee (\operatorname{coz}\alpha)^* = \top \text{ and } (\operatorname{coz}\delta)^* \vee (\operatorname{coz}\alpha)^* = \top,$$

we can conclude that $(\operatorname{coz}\alpha)^{**} = (\operatorname{coz}\beta)^* = (\operatorname{coz}\delta)^*$. This shows that $\operatorname{coz}\beta \vee (\operatorname{coz}\beta)^* = \top$ and so $\operatorname{coz}\beta \in BL$. On the other hand, $\operatorname{coz}\beta \lll (\operatorname{coz}\alpha)^*$ implies that $\operatorname{coz}\alpha \leq (\operatorname{coz}\alpha)^{**} \lll (\operatorname{coz}\beta)^*$, showing $\operatorname{coz}\alpha \leq (\operatorname{coz}\beta)^*$. Therefore if $\operatorname{coz}\alpha \neq (\operatorname{coz}\beta)^*$, then, by Lemma 4.2, α is a vertex of a triangle. Otherwise, $\operatorname{coz}\alpha = (\operatorname{coz}\beta)^*$ shows that $(\operatorname{coz}\alpha)^* = \operatorname{coz}\beta$ which is a contradiction.

(2) ⇒ (3). Let M be a maximal ideal of $\mathcal{R}L$ generated by an idempotent. Take an idempotent η in $\mathcal{R}L$ such that $M = \langle\eta\rangle$. Then, $\langle\mathbf{1} - \eta\rangle$ is a minimal ideal generated by the idempotent $\mathbf{1} - \eta$, and hence, by the proof of Lemma 3.4 in [11], $\operatorname{coz}(\mathbf{1} - \eta)$ is an atom.

(3) ⇒ (2). Let a be an atom of L. Again, by Lemma 3.4 in [11], the ideal $M_a = \{\delta \in \mathcal{R}L | \operatorname{coz}\delta \leq a\}$ is a minimal ideal and $\bigvee M_a = a$. Hence, Lemma 4.3 in [9] shows that M_a is a non-essential ideal of $\mathcal{R}L$. Thus, Lemma 4.5 in [11] implies that $M_a = \langle\eta\rangle$ for some idempotent η in $\mathcal{R}L$. Now, since $\mathcal{R}L$ is a reduced ring, $M = \langle\mathbf{1} - \eta\rangle$ is a maximal ideal of $\mathcal{R}L$ generated by an idempotent. □

Recall from [3] that a zeroset Z in X is said to be a *middle zeroset* if there exist two proper zerosets E and F such that $Z = E \cap F$ and $E \cup F = X$. A space X is called a *middle P-space* if every nonempty middle zeroset in X has a nonempty interior. Clearly, every almost P-space is a middle P-space but not conversely (see [3] for details). Now, adapting this to frames, Ighedo [15] has introduced the following definition.

Definition 4.4. *(1) A cozero element c in L is said to be middle cozero element if there exist two cozero elements a and b other than the bottom such that $c = a \vee b$ and $a \wedge b = \bot$.*

(2) A frame L is called a middle P-frame if every non-top middle cozero element in L is not dense. This is equivalent to saying L is a middle P-frame if and only if the only dense middle cozero element of L is $\top$.

Clearly, a topological space X is a middle P-space if and only if the frame $\mathfrak{O}X$ is a middle P-frame. For more details about middle P-frames see [15].

In the following proposition, we consider frame-theoretic properties of L for which the graph $\Gamma(\mathcal{R}L)$ is hypertriangulated. Recall from [4] that a frame is *disconnected* if there is at least one non-trivial complemented element. A frame is connected if it is not disconnected, or equivalently, if $a \wedge b = \bot$ and $a \vee b = \top$ imply $a = \top$ or $b = \top$.

Proposition 4.5. *For a frame L, $\Gamma(\mathcal{R}L)$ is a hypertriangulated graph if and only if L is a connected middle P-frame.*

Proof. Let $\Gamma(\mathcal{R}L)$ be a hypertriangulated graph. If L is not connected, then there exists a complemented element $\bot \neq a$ in L. Take α, β in $\mathcal{R}L$ such that $\operatorname{coz}\alpha = a$ and $\operatorname{coz}\beta = a'$. Now, $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$ implies that α is adjacent to β and since $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$, then by Lemma 3.2, there is no vertex adjacent to both α and β. So the edge $\alpha - \beta$ does not belong to a triangle, which is a contradiction; therefore L is connected. Now, let $\operatorname{coz}\alpha$ be a middle cozero element. Then $\operatorname{coz}\alpha = \operatorname{coz}\beta \vee \operatorname{coz}\gamma$ and $\operatorname{coz}\beta \wedge \operatorname{coz}\gamma = \bot$ for some cozero elements $\operatorname{coz}\beta$ and $\operatorname{coz}\gamma$. Since $\Gamma(\mathcal{R}L)$ is hypertriangulated, then $\beta - \gamma$ is an edge of a triangle, that is, there exists a vertex δ such that $\beta\delta = \gamma\delta = \mathbf{0}$. This implies that $\operatorname{coz}\alpha = \operatorname{coz}\beta \vee \operatorname{coz}\gamma \leq (\operatorname{coz}\delta)^*$, showing $\bot \neq \operatorname{coz}\delta \leq (\operatorname{coz}\delta)^{**} \leq (\operatorname{coz}\alpha)^*$ which means that $(\operatorname{coz}\alpha)^* \neq \bot$. Consequently, L is a middle P-frame.

Conversely, let L be a connected middle P-frame and $\alpha - \beta$ be an edge in $\Gamma(\mathcal{R}L)$. Since $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$ and L is connected, then $\operatorname{coz}\alpha \vee \operatorname{coz}\beta \neq \top$. This shows that $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \bot$, because L is a middle P-frame. Now, by Lemma 3.2 there exists a vertex adjacent to both α and β. This means that $\Gamma(\mathcal{R}L)$ is a hypertriangulated graph. □

If α and β are two vertices in $\Gamma(\mathcal{R}L)$, by $c(\alpha, \beta)$, we mean the length of the smallest cycle containing α and β. For every two vertices α and β, all possible cases for $c(\alpha, \beta)$ are provided in the next proposition.

Proposition 4.6. *Let $\alpha, \beta \in \Gamma(\mathcal{R}L)$. Then the following statements hold.*

1. $c(\alpha, \beta) = 3$ *if and only if* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$ *and* $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \bot$.
2. $c(\alpha, \beta) = 4$ *if and only if either* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \bot$ *and* $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* \neq \bot$ *or* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$ *and* $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$.
3. $c(\alpha, \beta) = 6$ *if and only if* $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \bot$ *and* $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$.

Proof. First of all, by Lemma 3.2 and Proposition 3.3, it is easily checked that parts (1) and (2) are true. To prove part (3), if $c(\alpha, \beta) = 6$, then parts (1) and (2) imply that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \bot$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$. Conversely, since $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \neq \bot$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$, then by part (3) of Proposition 3.3, we have $d(\alpha, \beta) = 3$. Thus, there exist vertices γ and δ such that $\alpha\gamma = \gamma\delta = \delta\beta = \mathbf{0}$. Now, if some vertex φ is adjacent to β, then $\varphi\beta = \mathbf{0}$. In consequence, $\operatorname{coz}\varphi \leq (\operatorname{coz}\beta)^*$ and $\operatorname{coz}\gamma \leq (\operatorname{coz}\alpha)^*$ implying that $\operatorname{coz}\gamma \wedge \operatorname{coz}\varphi \leq (\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$, and so φ is adjacent to γ. But this simply means that $c(\alpha, \beta) \geq 5$. On the other hand, $d(\alpha, \beta) = 3$ implies that α is not adjacent to φ. Therefore, $c(\alpha, \beta) \geq 6$. Now, if we consider the vertices $\mathbf{2}\gamma$ and $\mathbf{2}\delta$, then we have a cycle with vertices α, β, γ, δ, $\mathbf{2}\gamma$ and $\mathbf{2}\delta$, that is to say $c(\alpha, \beta) = 6$. □

As an immediate consequence, we now have the following result.

Corollary 4.7. *For a frame L, the following statements hold.*

1. *Every cycle in* $\Gamma(\mathcal{R}L)$ *has length* 3 *or* 4.
2. *Every edge of* $\Gamma(\mathcal{R}L)$ *is an edge of a cycle with length* 3 *or* 4.

A frame L is said to be a *P-frame* if for every $\alpha \in \mathcal{R}L$, $\operatorname{coz}\alpha$ is complemented. It is well known that L is a P-frame if and only if the ring $\mathcal{R}L$ is regular (that is, for every $\alpha \in \mathcal{R}L$, there exists $\beta \in \mathcal{R}L$ such that $\alpha^2\beta = \alpha$). A frame L is called *almost P-frame* if every cozero element in L is regular (or equivalently, every nonunit element of $\mathcal{R}L$ is zero-divisor). Whenever L is an almost P-frame, we call the ring $\mathcal{R}L$ almost regular. We refer the reader to [8] and [9] for more details and properties of P-frames and almost P-frames.

Proposition 4.8. *For a frame L, the ring $\mathcal{R}L$ is almost regular if and only if for every nonunit $\alpha \in \mathcal{R}L$, there exists $\mathbf{1} \neq \beta \in \mathcal{R}L$ such that $\alpha = \alpha\beta$.*

Proof. To prove the 'if' part, let $\alpha \in \mathcal{R}L$ be a nonunit element. We can assume that α is a nonzero-nonunit element since otherwise there is nothing to prove. Then $\bot \neq (\operatorname{coz}\alpha)^* \neq \top$, and so there exists $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\gamma \ll (\operatorname{coz}\alpha)^*$, implying that $\operatorname{coz}\alpha \leq (\operatorname{coz}\alpha)^{**} \ll (\operatorname{coz}\gamma)^*$. It follows that $\operatorname{coz}\alpha \ll \operatorname{coz}\rho \ll (\operatorname{coz}\gamma)^*$ for some $\rho \in \mathcal{R}L$. Therefore, by [8, Lemma 3.3], there exists $\mathbf{1} \neq \beta \in \mathcal{R}L$ such that $\alpha = \alpha\beta$.

To prove the 'only if' part, it is enough to show that every nonunit element of $\mathcal{R}L$ is zero-divisor. Let $\alpha \in \mathcal{R}L$ be a nonunit element. Then, by the hypothesis, there exists $\mathbf{1} \neq \beta \in \mathcal{R}L$ such that $\alpha = \alpha\beta$. This shows that $\operatorname{coz}\alpha \wedge \operatorname{coz}(\mathbf{1} - \beta) = \bot$, implying that $\operatorname{coz}(\mathbf{1} - \beta) \leq (\operatorname{coz}\alpha)^*$. In consequence $(\operatorname{coz}\alpha)^* \neq \bot$, since $\bot \neq \operatorname{coz}(\mathbf{1} - \beta)$. Therefore α is zero-divisor. □

In the following theorem, the almost regularity of the ring $\mathcal{R}L$ is characterized graphically. We begin with the following lemma.

Lemma 4.9. *Let L be an infinite frame. Then there exist $\delta, \gamma \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\delta \vee \operatorname{coz}\gamma = \top$ and $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma \neq \bot$.*

Proof. We consider two cases.

Case 1. Suppose that for every $\alpha \in \Gamma(\mathcal{R}L)$, $\operatorname{coz}\alpha \in BL$. Now, if there exists $\alpha \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\alpha$ is not an atom, then we have $\operatorname{coz}\beta \ll \operatorname{coz}\alpha$ for some $\beta \in \Gamma(\mathcal{R}L)$ with $\operatorname{coz}\beta \neq \operatorname{coz}\alpha$. Putting $\operatorname{coz}\delta = \operatorname{coz}\alpha$ and $\operatorname{coz}\gamma = (\operatorname{coz}\beta)^*$, we then have $\operatorname{coz}\delta \vee \operatorname{coz}\gamma = \top$ and $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma \neq \bot$. Otherwise, for every $\alpha \in \Gamma(\mathcal{R}L)$, $\operatorname{coz}\alpha$ is an atom. Then it is easy to see that $|L| = 4$ which is a contradiction.

Case 2. Let $\operatorname{coz}\alpha \notin BL$ for some $\alpha \in \Gamma(\mathcal{R}L)$. Then there exists $\beta \in \Gamma(\mathcal{R}L)$ with $\operatorname{coz}\beta \ll \operatorname{coz}\alpha$. Take $\operatorname{coz}\delta \in \operatorname{Coz}L$ such that $\operatorname{coz}\beta \wedge \operatorname{coz}\delta = \bot$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\delta = \top$. Putting $\gamma = \alpha$, we then have $\operatorname{coz}\delta \vee \operatorname{coz}\gamma = \top$ and $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma \neq \bot$. □

Theorem 4.10. *For a frame L, $\mathcal{R}L$ is an almost regular ring if and only if for every pair of vertices β and γ of $\Gamma(\mathcal{R}L)$ and every nonunit $\alpha \in \mathcal{R}L$, $c(\alpha\beta, \alpha\gamma) \leq 4$.*

Proof. Necessity. Let $\mathcal{R}L$ be an almost regular ring. Then for any nonunit $\alpha \in \mathcal{R}L$, we have $(\operatorname{coz}\alpha)^* \neq \bot$. Since $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\alpha\beta)^* \wedge (\operatorname{coz}\alpha\gamma)^*$, then by parts (1) and (2) of Proposition 4.6 $c(\alpha\beta, \alpha\gamma) \leq 4$.

Sufficiency. Suppose that $\alpha \in \mathcal{R}L$ is a nonunit and $c(\alpha\beta, \alpha\gamma) \leq 4$ for all vertices β and γ of $\Gamma(\mathcal{R}L)$. If L is finite, then it is easy to show that $\mathcal{R}L$ is almost regular. Otherwise, suppose that L is infinite. Then, by Lemma 4.9, there exist $\delta, \rho \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\delta \vee \operatorname{coz}\rho = \top$ and $\operatorname{coz}\delta \wedge \operatorname{coz}\rho \neq \bot$. If $\operatorname{coz}\alpha\delta \wedge \operatorname{coz}\alpha\rho = \bot$, then $\operatorname{coz}\alpha \wedge \operatorname{coz}\delta \wedge \operatorname{coz}\rho = \bot$. This implies that $\bot \neq \operatorname{coz}\delta \wedge \operatorname{coz}\rho \leq (\operatorname{coz}\alpha)^*$. In consequence, $(\operatorname{coz}\alpha)^* \neq \bot$. Now, suppose that $\operatorname{coz}\alpha\delta \wedge \operatorname{coz}\alpha\rho \neq \bot$. Then since $c(\operatorname{coz}\alpha\delta, \operatorname{coz}\alpha\rho) \leq 4$, by part (2) of Proposition 4.6, we have $(\operatorname{coz}\alpha\delta)^* \wedge (\operatorname{coz}\alpha\rho)^* \neq \bot$. On the other hand,

$$\operatorname{coz}\alpha = \operatorname{coz}\alpha \wedge \top = \operatorname{coz}\alpha \wedge (\operatorname{coz}\delta \vee \operatorname{coz}\rho) = \operatorname{coz}\alpha\delta \vee \operatorname{coz}\alpha\rho,$$

and so $(\operatorname{coz}\alpha)^* = (\operatorname{coz}\alpha\delta)^* \wedge (\operatorname{coz}\alpha\rho)^*$. Consequently, $(\operatorname{coz}\alpha)^* \neq \bot$. Therefore α is zero-divisor and the proof is complete. □

In the next theorem, the regularity of the ring $\mathcal{R}L$ is characterized graphically.

Theorem 4.11. *For a frame L, $\mathcal{R}L$ is a regular ring if and only if $\mathcal{R}L$ is an almost regular ring and for every vertex α of $\Gamma(\mathcal{R}L)$, there exists a vertex β of $\Gamma(\mathcal{R}L)$ adjacent to α such that $c(\alpha, \beta) = 4$.*

Proof. Necessity. If $\mathcal{R}L$ is regular, then clearly it is almost regular. Since for every vertex α, $\operatorname{coz}\alpha$ is complemented, then $(\operatorname{coz}\alpha)^*$ is also a cozero element. Putting $\operatorname{coz}\beta = (\operatorname{coz}\alpha)^*$, we then have $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$. Now, part (2) of Proposition 4.6 implies that $c(\alpha, \beta) = 4$.

Sufficiency. Let $\mathbf{0} \neq \alpha \in \mathcal{R}L$ be a nonunit. Then $\alpha \in \Gamma(\mathcal{R}L)$ since $\mathcal{R}L$ is an almost regular ring. Now, the hypothesis implies that α is adjacent to β for some $\beta \in \Gamma(\mathcal{R}L)$, such that $c(\alpha, \beta) = 4$. Hence, by part (2) of Proposition 4.6, we have $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$. This shows that $(\operatorname{coz}\alpha \vee \operatorname{coz}\beta)^* = \bot$ and hence $(\operatorname{coz}\alpha \vee \operatorname{coz}\beta)^{**} = \top$, implying that $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = \top$ since $\mathcal{R}L$ is almost regular. Therefore $\operatorname{coz}\alpha$ is complemented, that is, $\mathcal{R}L$ is regular. □

As defined in [16], for distinct vertices x and y in a graph G, we say that x and y are *orthogonal*, written $x \perp\!\!\!\perp y$, if x and y are adjacent and there is no vertex z of G which is adjacent to both x and y. A graph G is said to be *complemented* if for each vertex x of G, there is a vertex y of G, called a complement of x, such that $x \perp\!\!\!\perp y$, and that G is *uniquely complemented* if G is complemented and whenever $x \perp\!\!\!\perp y$ and $x \perp\!\!\!\perp z$, then $y \sim z$, this means that y and z are adjacent to exactly the same vertices.

By Lemma 3.2, for every two vertices α and β in $\Gamma(\mathcal{R}L)$, $\alpha \perp\!\!\!\perp \beta$ if and only if $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \bot$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\beta)^* = \bot$. Hence, $\Gamma(\mathcal{R}L)$ is complemented if and only if for every vertex α, there exists a vertex δ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\delta = \bot$ and $(\operatorname{coz}\alpha)^* \wedge (\operatorname{coz}\delta)^* = \bot$.

For our next characterization, we let $\mathrm{Min}(\mathcal{R}L)$ be the space of minimal prime ideals of $\mathcal{R}L$. A frame L has been defined in [9] to be *cozero-complemented* if for each $a \in \mathrm{Coz}\, L$ there exists $b \in \mathrm{Coz}\, L$ such that $a \wedge b = \perp$ and $a \vee b$ is dense. The space $\mathrm{Min}(\mathcal{R}L)$ is compact if and only if L is cozero-complemented (see [9, Proposition 4.7]). Now, the following corollary is obvious.

Corollary 4.12. *For a frame L, $\Gamma(\mathcal{R}L)$ is complemented if and only if L is cozero-complemented if and only if the space* $\mathrm{Min}(\mathcal{R}L)$ *is compact.*

We close this section by giving a direct proof for the next proposition which is also true for every reduced ring, see [2, Theorem 3.5].

Proposition 4.13. *For a frame L, $\Gamma(\mathcal{R}L)$ is complemented if and only if it is uniquely complemented.*

Proof. To prove the nontrivial part of the proposition, let α, β and γ be vertices of $\Gamma(\mathcal{R}L)$ such that $\alpha \perp\!\!\!\perp \beta$ and $\alpha \perp\!\!\!\perp \gamma$. So $\mathrm{coz}\,\alpha \wedge \mathrm{coz}\,\beta = \perp$, $(\mathrm{coz}\,\alpha)^* \wedge (\mathrm{coz}\,\beta)^* = \perp$, $\mathrm{coz}\,\alpha \wedge \mathrm{coz}\,\gamma = \perp$ and $(\mathrm{coz}\,\alpha)^* \wedge (\mathrm{coz}\,\gamma)^* = \perp$. In consequence, $\mathrm{coz}\,\beta \leq \mathrm{coz}\,\alpha)^* \leq (\mathrm{coz}\,\gamma)^{**}$ and $\mathrm{coz}\,\gamma \leq \mathrm{coz}\,\alpha)^* \leq (\mathrm{coz}\,\beta)^{**}$, showing that $(\mathrm{coz}\,\beta)^* = (\mathrm{coz}\,\gamma)^*$. This means that β and γ are adjacent to exactly the same vertices, that is, $\beta \sim \gamma$. Therefore $\Gamma(\mathcal{R}L)$ is uniquely complemented. □

5. Dominating sets

A *dominating set* of a graph G is a set of vertices V such that every vertex outside V is adjacent to at least one vertex in V. The dominating number of G denoted by dtG is the smallest cardinal number of the form $|V|$, where V is a dominating set. A complete subgraph of G is a subgraph in which every vertex is adjacent to every other vertex. The smallest cardinal number a such that every complete subgraph G has cardinality $\leq a$, denoted by ωG, is called the *clique number* of G.

The cellularity of the space X is denoted by $c(X)$ and is the smallest cardinal number $n \geq \aleph_0$ such that every family of pairwise disjoint of nonempty open subsets of X has cardinality $\leq n$. This motivates the following definition.

Definition 5.1. *The cellularity of a frame L is denoted by $c(L)$ and is the smallest cardinal number $n \geq \aleph_0$ such that every family of pairwise disjoint of nonzero elements of L has cardinality $\leq n$.*

Clearly, for a topological space X, we have $c(X) = c(\mathfrak{O}X)$.

Proposition 5.2. *For a frame L, $\omega\Gamma(\mathcal{R}L) = c(L)$. In particular, whenever L is a boolean frame, then $\omega\Gamma(\mathcal{R}L) = |L|$.*

Proof. We first show that $\omega\Gamma(\mathcal{R}L) \leq c(L)$. Let $A \subseteq \Gamma(\mathcal{R}L)$ be a complete subgraph. Then for every $\alpha, \beta \in A$, $\alpha\beta = \mathbf{0}$ implies that $\mathrm{coz}\,\alpha \wedge \mathrm{coz}\,\beta = \perp$. Thus $S = \{\mathrm{coz}\,\gamma \mid \gamma \in A\}$ is a family of pairwise disjoint nonzero elements of L. This establishes that $\omega\Gamma(\mathcal{R}L) \leq c(L)$. To show the other containment, suppose that S is a collection of pairwise disjoint nonzero elements of L. For every $s \in S$, pick $\alpha_s \in \mathcal{R}L$ such that $\mathrm{coz}\,\alpha_s \ll s$ and $(\mathrm{coz}\,\alpha_s)^* \neq \perp$. Now, for every $s, t \in S$, we have $\alpha_s\alpha_t = \mathbf{0}$. So $A = \{\alpha_s \mid s \in S\}$ is a complete subgraph of $\Gamma(\mathcal{R}L)$. This shows that $\omega\Gamma(\mathcal{R}L) \geq c(L)$ and therefore $\omega\Gamma(\mathcal{R}L) = c(L)$. The second part is obvious. □

The set of all cardinal numbers of the form $|S|$, where S is a base for a frame L, has a smallest element; this cardinal number is called the weight of the frame L and is denoted by $\omega(L)$ (see [12]). Clearly, for a topological space X, we have $\omega(X) = \omega(\mathfrak{O}X)$.

Proposition 5.3. *For a frame L, $dt\Gamma(\mathcal{R}L) \leq \omega(L)$.*

Proof. Suppose S is a base for L. Then for every $s \in S$, take $\alpha_s \in \mathcal{R}L$ such that $\mathrm{coz}\,\alpha_s \ll s$ and $(\mathrm{coz}\,\alpha_s)^* \neq \perp$. We claim that $A = \{\alpha_s \mid s \in S\}$ is a dominating set of $\Gamma(\mathcal{R}L)$. To see this, let $\beta \in \Gamma(\mathcal{R}L)$, then there exists $s_0 \in S$ such that $s_0 \leq (\mathrm{coz}\,\beta)^*$. This implies that $\mathrm{coz}\,\alpha_{s_0} \leq (\mathrm{coz}\,\beta)^*$ and hence $\mathrm{coz}\,\alpha_{s_0} \wedge (\mathrm{coz}\,\beta)^{**} = \perp$. Therefore $\mathrm{coz}\,\alpha_{s_0} \wedge \mathrm{coz}\,\beta = \perp$, that is, $\alpha_{s_0}\beta = \mathbf{0}$ and consequently A is a dominating set. Now, $dt\Gamma(\mathcal{R}L) \leq |A| \leq |S|$ for every base S of L. But this means that $dt\Gamma(\mathcal{R}L) \leq \omega(L)$. □

In the following example, we show that there is a frame L for which the dominating number of $\Gamma(\mathcal{R}L)$ is strictly less than the weight of L.

Example 5.4. *Consider $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. By Example 3.3 in [3], $dt\Gamma(C(\beta\mathbb{N})) = \aleph_0 \lneqq c = \omega(\beta\mathbb{N})$. Putting $L = \mathfrak{O}(\beta\mathbb{N})$, we then have $dt\Gamma(\mathcal{R}L) = dt\Gamma(C(\beta\mathbb{N})) \lneqq \omega(\beta\mathbb{N}) = \omega(L)$, since $C(\beta\mathbb{N}) \cong \mathcal{R}(\mathfrak{O}(\beta\mathbb{N}))$.*

Next, we intend to give a frame-theoretic characterization and an algebraic characterization for the set of centers of $\Gamma(\mathcal{R}L)$ to be a dominating set. Before we give these characterizations, we first give the two propositions below. Recall that a frame is *atomic* if below every nonzero element there is an atom. We denote the set of centers of $\Gamma(\mathcal{R}L)$ by $C(\Gamma(\mathcal{R}L))$.

Proposition 5.5. *For a frame L, $\Gamma(\mathcal{R}L)$ is not triangulated and the set of centers of $\Gamma(\mathcal{R}L)$ is a dominating set containing at least two elements if and only if L is atomic.*

Proof. ($\Rightarrow$) Since $\Gamma(\mathcal{R}L)$ is not triangulated, then by Theorem 4.3, L has at least one atom and so Proposition 5.5 implies that $C(\Gamma(\mathcal{R}L)) = \{\varphi \in \Gamma(\mathcal{R}L) \mid \operatorname{coz}\varphi \text{ is an atom}\}$. Now, let $a \in L$ be different from the top and the bottom, and let $\alpha \in \Gamma(\mathcal{R}L)$ be such that $\operatorname{coz}\alpha \ll a$. If $\operatorname{coz}\alpha$ is an atom, then there is nothing to prove. Otherwise, pick $\beta \in \mathcal{R}L$ such that $\operatorname{coz}\beta \ll \operatorname{coz}\alpha$ and so $(\operatorname{coz}\alpha)^* \ll (\operatorname{coz}\beta)^*$. Now, we consider two cases.

Case 1. Suppose that $(\operatorname{coz}\alpha)^*$ is an atom. Since, by our supposition, $C(\Gamma(\mathcal{R}L))$ has at least two elements, let $c \in L$ be an atom such that $c \neq (\operatorname{coz}\alpha)^*$. Then $c \wedge \operatorname{coz}\alpha = c$ or $c \wedge \operatorname{coz}\alpha = \bot$. The latter is not possible, lest we have $c \leq (\operatorname{coz}\alpha)^*$, implying $(\operatorname{coz}\alpha)^* = c$. Therefore $c \wedge \operatorname{coz}\alpha = c$, showing that $c \leq \operatorname{coz}\alpha \leq a$.

Case 2. Suppose that $(\operatorname{coz}\alpha)^*$ is not an atom. Take $\gamma \in \mathcal{R}L$ such that $(\operatorname{coz}\alpha)^* \ll \operatorname{coz}\gamma \ll (\operatorname{coz}\beta)^*$. Clearly $\gamma \in \Gamma(\mathcal{R}L) \setminus C(\Gamma(\mathcal{R}L))$, so we can choose $\delta \in C(\Gamma(\mathcal{R}L))$ such that $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma = \bot$ because $C(\Gamma(\mathcal{R}L))$ is a dominating set. This shows that $\operatorname{coz}\delta \wedge (\operatorname{coz}\alpha)^* = \bot$, that is, $\operatorname{coz}\delta \leq (\operatorname{coz}\alpha)^{**}$. Next, since $\operatorname{coz}\delta$ is an atom, we have $\operatorname{coz}\alpha \wedge \operatorname{coz}\delta = \operatorname{coz}\delta$ or $\operatorname{coz}\alpha \wedge \operatorname{coz}\delta = \bot$. The latter is not possible, lest we have $\operatorname{coz}\delta \leq (\operatorname{coz}\alpha)^*$, implying $\operatorname{coz}\delta = \bot$ which is a contradiction. Therefore $\operatorname{coz}\delta \leq \operatorname{coz}\alpha \leq a$.

($\Leftarrow$) Suppose that L is atomic. Then by Theorem 4.3 $\Gamma(\mathcal{R}L)$ is not triangulated. For the second part, first note that if L has exactly one atom, then the present hypothesis implies that $L = \mathbf{2}$, a contradiction since $|L| \geq 4$. Next, let $\alpha \in \Gamma(\mathcal{R}L) \setminus C(\Gamma(\mathcal{R}L))$. Then there exists an atom a shch that $a \leq (\operatorname{coz}\alpha)^*$. Putting $a = \operatorname{coz}\beta$, we then have $\beta \in C(\Gamma(\mathcal{R}L))$ with $\operatorname{coz}\beta \leq (\operatorname{coz}\alpha)^*$. Thus $\operatorname{coz}\beta \wedge (\operatorname{coz}\alpha)^{**} = \bot$, implying that $\operatorname{coz}\beta \wedge \operatorname{coz}\alpha = \bot$, that is, α adjacent to β. □

Before proving the next proposition, we recall some definitions. A frame L is *basically disconnected* if $c^* \vee c^{**} = \top$ for all $c \in \operatorname{Coz} L$, and L is *zero dimensional* if it has a base of complemented elements (see [4] for details).

Proposition 5.6. *Let $\Gamma(\mathcal{R}L)$ not be triangulated. If the set of centers of $\Gamma(\mathcal{R}L)$ is a dominating set, then it has at least two elements.*

Proof. Since $\Gamma(\mathcal{R}L)$ is not triangulated, then by Theorem 4.3, L has at least one atom. Then Proposition 5.5 implies that $C(\Gamma(\mathcal{R}L)) = \{\varphi \in \Gamma(\mathcal{R}L) \mid \operatorname{coz}\varphi \text{ is an atom}\}$. Now suppose, by way of contradiction, that $|C(\Gamma(\mathcal{R}L))| = 1$, that is, L has exactly one atom, say a. Take $\alpha, \beta \in \mathcal{R}L$ such that $a = \operatorname{coz}\alpha$ and $a' = \operatorname{coz}\beta$. Now, we continue the proof in three stages.

The first stage: We show that if $\gamma \in \Gamma(\mathcal{R}L)$, then $(\operatorname{coz}\gamma)^* = \operatorname{coz}\alpha$ or $(\operatorname{coz}\gamma)^* = \operatorname{coz}\beta$. We claim that for every $\gamma \in \Gamma(\mathcal{R}L)$ with $\operatorname{coz}\gamma \neq \operatorname{coz}\alpha$, $(\operatorname{coz}\gamma)^* = \operatorname{coz}\alpha$. It suffices to prove that for every $\gamma \in \Gamma(\mathcal{R}L)$ with $\operatorname{coz}\gamma \neq \operatorname{coz}\alpha$ and $\operatorname{coz}\gamma \neq \operatorname{coz}\beta$, $(\operatorname{coz}\gamma)^* = \operatorname{coz}\alpha$. To see this, take $\gamma \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\gamma \neq \operatorname{coz}\alpha$ and $\operatorname{coz}\gamma \neq \operatorname{coz}\beta$, Then $\operatorname{coz}\alpha \wedge \operatorname{coz}\gamma = \bot$ since $C(\Gamma(\mathcal{R}L)) = \{\alpha\}$ is a dominating set. This shows that $\operatorname{coz}\gamma \vee \operatorname{coz}\alpha \neq \top$ because $\operatorname{coz}\gamma \neq \operatorname{coz}\beta$. Pick $\varphi \in \mathcal{R}L$ such that $\operatorname{coz}\varphi = \operatorname{coz}\gamma \vee \operatorname{coz}\alpha$. If $(\operatorname{coz}\varphi)^* \neq \bot$, then $\operatorname{coz}\varphi \wedge \operatorname{coz}\alpha = \bot$, showing $\operatorname{coz}\varphi \leq \operatorname{coz}\beta$, that is, $\operatorname{coz}\alpha \leq \operatorname{coz}\beta$ which is a contradiction. Consequently, $(\operatorname{coz}\varphi)^* = \bot$, this means that $(\operatorname{coz}\gamma)^* \wedge \operatorname{coz}\beta = \bot$, implying that $(\operatorname{coz}\gamma)^* \leq \operatorname{coz}\alpha$. Therefore $(\operatorname{coz}\gamma)^* = \operatorname{coz}\alpha$ since $\operatorname{coz}\alpha \leq (\operatorname{coz}\gamma)^*$.

The second stage: We show that L is a zero dimensional frame. By [4, Proposition 8.4.4], it suffices to prove that L is a basically disconnected frame. To see this, let $c \in \operatorname{Coz} L$. Then, by the first stage, we have $c^* = \bot$, $c^* = \operatorname{coz}\alpha$ or $c^* = \operatorname{coz}\beta$. This shows that $c^* \vee c^{**} = \top$, that is, L is basically disconnected.

The third stage: We argue to arrive at a contradiction. By the second stage, L has a base of complemented elements, say S. If $S = \{\operatorname{coz}\alpha, \operatorname{coz}\beta, \}$, then $\operatorname{coz}\beta$ is an atom which is a contradiction. Otherwise, there exists

$\delta \in \Gamma(\mathcal{R}L)$ such that $\operatorname{coz}\delta \in BL$, $\operatorname{coz}\delta \neq \operatorname{coz}\alpha$ and $\operatorname{coz}\delta \neq \operatorname{coz}\beta$. Since $C(\Gamma(\mathcal{R}L))$ is a dominating set, $\operatorname{coz}\delta \wedge \operatorname{coz}\alpha = \perp$ and $(\operatorname{coz}\delta)^* \wedge \operatorname{coz}\alpha = \perp$, showing that $\operatorname{coz}\delta \leq \operatorname{coz}\beta$ and $(\operatorname{coz}\delta)^* \leq \operatorname{coz}\beta$. In consequence, $\operatorname{coz}\beta = \top$ which is a contradiction. □

We conclude the article with the following theorem. Before the theorem is presented, let us recall that the *socle* of a ring R is the ideal generated by minimal ideals of R. In [10, 11], the socle of $\mathcal{R}L$ is characterized as the ideal consisting of functions each of which has cozero equal to a join of finitely many atoms. The equivalence of parts (2) and (3) of the following theorem is shown in [10]. Now combining Propositions 5.5 and 5.6, we obtain the following result.

Theorem 5.7. *The following statements are equivalent for a frame L.*

1. $\Gamma(\mathcal{R}L)$ *is not triangulated and the set of centers of* $\Gamma(\mathcal{R}L)$ *is a dominating set.*
2. L *is atomic.*
3. *The socle of* $\mathcal{R}L$ *is an essential ideal.*

Acknowledgement

Thanks are due to Professor Themba Dube for reading the paper and giving advise. We are also grateful to Professor Ali Akbar Estaji for his support. We appreciate Dr Oghenetega Ighedo for sending her paper [16] to make sure that the results in the paper do not overlap with ours. We would like to express our deep gratitude to the referee for the comments which have improved the paper.

References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, Journal of Algebra 217 (1999) 434–447.
[2] D. F. Anderson, R. Levy and J. Shapiro, Zero-divisor graphs, von neumann regular rings, and boolean algebras, Journal of Pure and Applied Algebra 180 (2003) 221–241.
[3] F. Azarpanah and M. Motamedi, Zero-divisor graph of $C(X)$, Acta Mathematica Hungarica 108 (2005) 25–36.
[4] R. N. Ball and J. Walters-Wayland, C and C^*-quotients in pointfree topology, Dissertationes Mathematicae (Rozprawy Matematyczne) 412 (2002), 1–61.
[5] B. Banaschewski, The real numbers in pointfree topology, Textos de Mathemática (Séies B), No. 12, Departamento de Mathemática da Universidade de Coimbra, Coimbra (1997).
[6] B. Banaschewski and C. Gilmour, Pseudocompactness and the cozero part of a frame, Commentationes Mathematicae Universitatis Carolinae 37 (1996) 577–587.
[7] B. Banaschewski and S. S. Hong, Completeness properties of function rings in pointfree topology, Commentationes Mathematicae Universitatis Carolinae 44 (2003) 245–259.
[8] T. Dube, Concerning P-frames, essential P-frames, and strongly zero-dimensional frames, Algebra Universalis 61 (2009) 115–138.
[9] T. Dube, Some ring-theoretic properties of almost P-frames, Algebra Universalis 60 (2009) 145–162.
[10] T. Dube, Contracting the socle in rings of continuous functions, Rendiconti del Seminario Matematico della Università di Padova 123 (2010) 37–53.
[11] T. Dube, A note on the socle of certain types of f-rings, Bulletin of the Iranian Mathematical Society 38 (2012) 517–528.
[12] T. Dube, S. Iliadis, J. van Mill, and I. Naidoo, Universal frames, Topology and its Applications 160 (2013), 2454–2464.
[13] R. Engelking, General topology, Heldermann-Verlag, Berlin, 1989.
[14] L. Gillman and M. Jerison, Rings of continuous functions, Springer, Berlin, 1976.
[15] O. Ighedo, A note on middle P-spaces and related, Bulletin of the Iranian Mathematical Society 44 (2018) 1195-1209.
[16] R. Levy and J. Shapiro, The zero-divisor graph of von neumann regular rings, Communications in Algebra 30 (2002) 745–750.
[17] J. Picado and A. Pultr, Frames and locales: Topology without points, Frontiers in Mathematics, Springer, Basel, 2012.