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Protection of Graphs with Emphasis on Cartesian Product Graphs

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Abstract. In this paper we study the weak Roman domination number and the secure domination number of a graph. In particular, we obtain general bounds on these two parameters and, as a consequence of the study, we derive new inequalities of Nordhaus-Gaddum type involving secure domination and weak Roman domination. Furthermore, the particular case of Cartesian product graphs is considered.

1. Introduction

The following approach to protection of a graph was described by Cockayne et al. [7]. Suppose that one or more guards are stationed at some of the vertices of a simple graph G and that a guard at a vertex can deal with a problem at any vertex in its closed neighbourhood. Consider a function $f : V(G) \longrightarrow \{0, 1, 2, \dots\}$ where $f(v)$ is the number of guards at v, and let $V_i = \{v \in V(G) : f(v) = i\}$ for every $i \in \{0, 1, 2, \dots\}$. We will identify f with the partition of $V(G)$ induced by f and write $f(V_0, V_1, \dots)$. The weight of f is defined to be $w(f) = \sum_{v\in V(G)} f(v) = \sum_i i|V_i|$. A vertex $v \in V(G)$ is *undefended* with respect to f if $f(v) = 0$ and $f(u) = 0$ for every vertex u adjacent to v. We say that G is *protected* under the function f if f has no undefended vertices, *i.e.*, G is protected if there is at least one guard available to handle a problem at any vertex. We now define the four particular subclasses of protected graphs considered in [7]. The functions in each subclass protect the graph according to a certain strategy.

- We say that $f(V_0, V_1)$ is a *dominating function* (DF) if G is protected under f. Obviously, $f(V_0, V_1)$ is a DF if and only if V_1 is a dominating set. The *domination number*, denoted by $\gamma(G)$ is the minimum cardinality among all dominating sets of G. This method of protection has been studied extensively [11, 12].

- A *Roman dominating function* (RDF) is a function $f(V_0, V_1, V_2)$ such that for every $v \in V_0$ there exists a vertex $u \in V_2$ which is adjacent to v. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight among all Roman dominating functions on G. This concept of protection has historical motivation [18] and was formally proposed by Cockayne et al. in [8].

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- A *weak Roman dominating function* (WRDF) is a function $f(V_0, V_1, V_2)$ such that for every v with $f(v) = 0$ there exists a vertex u adjacent to v such that $f(u) \in \{1, 2\}$ and the function $f' : V(G) \longrightarrow \{0, 1, 2\}$ defined by $f'(v) = 1$, $f'(u) = f(u) - 1$ and $f'(z) = f(z)$ for every $z \in V(G) \setminus \{u, v\}$, has no undefended vertices. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight among all weak Roman dominating functions on G. A WRDF of weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. For instance, for the tree shown in Figure 1, on the left, a $\gamma_r(G)$-function can place 2 guards at the vertex of degree three and one guard at the other black-coloured vertex. This concept of protection was introduced by Henning and Hedetniemi [13] and studied further in [5, 6, 19].

- A *secure dominating function* is a WRDF function $f(V_0, V_1, V_2)$ in which $V_2 = \emptyset$. In this case, it is convenient to define this concept of save graph by the properties of V_1. Obviously $f(V_0, V_1)$ is a secure dominating function if and only if V_1 is a dominating set and for every $v \in V_0$ there exists $u \in V_1$ which is adjacent to v and $(V_1 \setminus \{u\}) \cup \{v\}$ is a dominating set. In such a case, V_1 is said to be a *secure dominating set*. The *secure domination number*, denoted by $\gamma_s(G)$, is the minimum cardinality among all secure dominating sets. A secure dominating function of weight $\gamma_s(G)$ is called a $\gamma_s(G)$-function. Analogously, a secure dominating set of cardinality $\gamma_s(G)$ is called a $\gamma_s(G)$-set. This concept of protection was introduced by Cockayne et al. in [7], and studied further in [2, 4–6, 16].

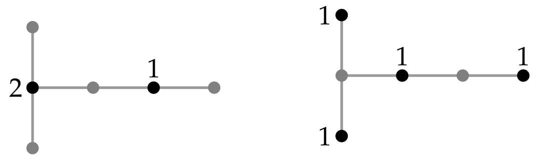

Figure 1: Two placements of guards which correspond to two different weak Roman dominating functions on the same tree. Notice that $2 = \gamma(G) < \gamma_r(G) < \gamma_s(G) = 4$.

The problem of computing $\gamma_r(G)$ is NP-hard, even when restricted to bipartite or chordal graphs [13], and the problem of computing $\gamma_s(G)$ is also NP-hard, even when restricted to split graphs [2]. This suggests finding the weak Roman domination number and the secure domination number for special classes of graphs or obtaining good bounds on these invariants. This is precisely the aim of this work. The remainder of the paper is structured as follows. Section 2 is devoted to obtain general bound on $\gamma_r(G)$ and $\gamma_s(G)$ in terms of several invariants of G. As a consequence of the study we derive new inequalities of Nordhaus-Gaddum type involving secure domination and weak Roman domination. Finally, in Section 3 we restrict our study to the particular case of Cartesian product graphs.

Throughout the paper, we will use the notation K_t, $K_{1,t-1}$, C_t, N_t and P_t for complete graphs, star graphs, cycle graphs, empty graphs and path graphs of order t, respectively. We use the notation $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G, $N(v)$ will denote the set of neighbours or *open neighbourhood* of v in G. The *closed neighbourhood*, denoted by $N[v]$, equals $N(v) \cup \{v\}$. We denote by $\delta(v) = |N(v)|$ the degree of vertex v, as well as $\delta(G) = \min_{v \in V(G)}\{\delta(v)\}$, $\Delta(G) = \max_{v \in V(G)}\{\delta(v)\}$ and $n(G) = |V(G)|$. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2. General bounds

To begin this section we would emphasize the following inequality chains.

Proposition 2.1. [7] *The following inequalities hold for any graph G.*

(i) $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

(ii) $\gamma(G) \leq \gamma_r(G) \leq \gamma_s(G)$.

The problem of characterizing the graphs with $\gamma_r(G) = \gamma(G)$ was solved by Henning and Hedetniemi [13]. The inequality chain (ii) has motivated us to obtain the following result, which shows that the problem of characterizing the graphs with $\gamma_s(G) = \gamma(G)$ is already solved.

Theorem 2.2. *Let G be a graph. The following statements are equivalent.*

(i) $\gamma_r(G) = \gamma(G)$.

(ii) $\gamma_s(G) = \gamma(G)$.

Proof. By Proposition 2.1 (ii), $\gamma_s(G) = \gamma(G)$ leads to $\gamma_r(G) = \gamma(G)$. Now, if $\gamma_r(G) = \gamma(G)$, then for any $\gamma_r(G)$-function $f(V_0, V_1, V_2)$ we have $V_2 = \emptyset$, as $V_1 \cup V_2$ is a dominating set and $\gamma(G) = \gamma_r(G) = |V_1| + 2|V_2| \geq |V_1| + |V_2| \geq \gamma(G)$. Hence, V_1 is a secure dominating set, which implies that $\gamma(G) = |V_1| \geq \gamma_s(G) \geq \gamma(G)$. Therefore, $\gamma_s(G) = \gamma(G)$. □

Given a graph G and an edge $e \in E(G)$, the graph obtained from G by removing e will be denoted by $G - e$, *i.e.*, $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. As observed in [13], any $\gamma_r(G - e)$-function is a WRDF for G. Similarly, any $\gamma_s(G - e)$-set is a secure dominating set for G. Therefore, the following basic result follows.

Proposition 2.3. *The following statement hold for any spanning subgraph H of a graph G.*

(i) [13] $\gamma_r(G) \leq \gamma_r(H)$.

(ii) $\gamma_s(G) \leq \gamma_s(H)$.

Proposition 2.4. *For any integer $t \geq 4$,*

(i) [13] $\gamma_r(C_t) = \gamma_r(P_t) = \left\lceil \frac{3t}{7} \right\rceil$.

(ii) [7] $\gamma_s(C_t) = \gamma_s(P_t) = \left\lceil \frac{3t}{7} \right\rceil$.

By Proposition 2.3 (ii) and Proposition 2.4 (ii) we deduce the following result.

Theorem 2.5. *For any Hamiltonian graph G with $n(G) \geq 4$,*

$$\gamma_s(G) \leq \left\lceil \frac{3n(G)}{7} \right\rceil.$$

Obviously, the bound above is tight, as it is achieved by any cycle graph of order at least four.

A set $S \subseteq V(G)$ is a *k-dominating set* if $|N(v) \cap S| \geq k$ for every $v \in \overline{S}$. The minimum cardinality among all k-dominating sets is called the *k-domination number* of G and it is denoted by $\gamma_k(G)$. It is readily seen that any 2-dominating set is a secure dominating set. Therefore, we can state the following result.

Theorem 2.6. [5] *For any graph G,*

$$\gamma_s(G) \leq \gamma_2(G).$$

Theorem 2.7. [4] *Let $G \not\cong C_5$ be a connected graph. If $\delta(G) \geq 2$, then*

$$\gamma_s(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor.$$

An example of a graph with $\delta(G) = 3$ and $\gamma_s(G) = \gamma_2(G) = \left\lfloor \frac{n(G)}{2} \right\rfloor$ is the 3-cube graph. Notice that from the result above and the fact that $\gamma_r(G) \leq \gamma_s(G)$ we can conclude that if $G \not\cong C_5$ is connected and $\delta(G) \geq 2$, then $\gamma_r(G) \leq \left\lfloor \dfrac{n(G)}{2} \right\rfloor$. With the aim of providing a general upper bound on the weak Roman domination number of any graph in terms of $n(G)$, we need to introduce some additional notation. For any support vertex v of a tree T, the set of leaves adjacent to v in T will be denoted by $L_T(v)$. Let $S(T)$ be the set of support vertices $v \in V(T)$ of degree $\delta(v) \leq |L_T(v)| + 1$ and define

$$X(T) = \bigcup_{v \in S(T)} (\{v\} \cup L_T(v)).$$

Let $T_0, T_1, \ldots, T_k$ be the sequence of all embedded subtrees of T, of order greater than or equal to three, defined as follows: $T_0 = T$ and T_i is the subtree of T_{i-1} induced by $V(T_{i-1}) \setminus X(T_{i-1})$, for every $i \in \{1, \ldots, k\}$. Notice that the smallest subtree T_k satisfies $|V(T_k) \setminus X(T_k)| \leq 2$. With this notation in mind we proceed to prove the two following results.

Theorem 2.8. *For any connected nontrivial graph G,*

$$\gamma_r(G) \leq \left\lfloor \frac{2n(G)}{3} \right\rfloor.$$

Proof. Since the case $n(G) = 2$ is straightforward, we can assume that $n(G) \geq 3$. Let T be a spanning tree of G and $T_0, T_1, \ldots, T_k$ the sequence of all embedded subtrees of T of order greater than or equal to three defined previously. By Proposition 2.3, $\gamma_r(G) \leq \gamma_r(T)$. It remains to show that $\gamma_r(T) \leq \frac{2n(G)}{3}$. To this end, we proceed to construct a WRDF f such that $w(f) \leq \frac{2n(G)}{3}$.

For every $v \in X(T_i)$ and $i \in \{0, \ldots, k\}$ we set

$$f(v) = \begin{cases} 2 & \text{if } v \in S(T_i) \text{ and } |L_{T_i}(v)| \geq 2, \\ 1 & \text{if } v \in S(T_i) \text{ and } |L_{T_i}(v)| = 1, \\ 0 & \text{if } v \in X(T_i) \setminus S(T_i). \end{cases}$$

Notice that $V(G) = \bigcup_{i=0}^{k} X(T_i) \cup (V(T_k) \setminus X(T_k))$ and $X(T_i) \cap X(T_j) = \emptyset$ for every $i \neq j$. Hence, it remains to define $f(x)$ for every $x \in V(T_k) \setminus X(T_k)$, if any. Notice that for any $i \in \{0, \ldots, k\}$,

$$\sum_{v \in X(T_i)} f(v) = \sum_{v \in S(T_i)} f(v) \leq \frac{2}{3}|X(T_i)| \tag{1}$$

and, if there is a support vertex v of T_i with $|L_{T_i}(v)| = 1$, then

$$\sum_{v \in X(T_i)} f(v) = \sum_{v \in S(T_i)} f(v) < \frac{2}{3}|X(T_i)|. \tag{2}$$

Hence, if $V(T_k) = X_k$ then $\sum_{i=0}^{k} |X(T_i)| = n(G)$, which implies that

$$w(f) = \sum_{i=0}^{k} \left(\sum_{v \in X(T_i)} f(v) \right) \leq \frac{2}{3} \sum_{i=0}^{k} |X(T_i)| \leq \frac{2n(G)}{3}.$$

Suppose that $V(T_k) \setminus X_k = \{x\}$. In this case, we set $f(x) = 0$ whenever $f(v) = 2$ for some neighbour v of x, otherwise we set $f(x) = 1$. Obviously, if $f(x) = 0$, then

$$w(f) = \sum_{i=0}^{k} \left(\sum_{v \in X(T_i)} f(v) \right) + f(x) \leq \frac{2}{3} \sum_{i=0}^{k} |X(T_i)| \leq \frac{2(n(G)-1)}{3} < \frac{2n(G)}{3}.$$

Now, if $f(x) = 1$, then (2) leads to $\sum_{v \in X_k} f(v) \leq \frac{2}{3}|X_k| - 1$, which implies that

$$\begin{aligned} w(f) &= \sum_{i=0}^{k-1} \left(\sum_{v \in X(T_i)} f(v) \right) + \sum_{v \in X_k} f(v) + f(x) \\ &\leq \frac{2}{3} \sum_{i=0}^{k-1} |X(T_i)| + \left(\frac{2}{3}|X(T_k)| - 1 \right) + 1 \\ &= \frac{2(n(G)-1)}{3} < \frac{2n(G)}{3}. \end{aligned}$$

Finally, if $V(T_k) \setminus X_k = \{a, b\}$, then we set $f(a) = 0$ and $f(b) = 1$. Thus,

$$\begin{aligned} w(f) &= \sum_{i=0}^{k} \left(\sum_{v \in X(T_i)} f(v) \right) + f(a) + f(b) \\ &\leq \frac{2}{3} \sum_{i=0}^{k} |X(T_i)| + 1 \\ &= \frac{2(n(G)-2)}{3} + 1 < \frac{2n(G)}{3}. \end{aligned}$$

In summary, we can conclude that $w(f) \leq \frac{2n(G)}{3}$, and it is readily seen that f is a WRDF. Therefore, the result follows. □

To see that the bound above is tight we can take any graph G_1 and construct the corona graph $G \cong G_1 \odot N_2$ by considering one copy of G_1 and $n(G_1)$ copies of N_2 and joining, by an edge, each vertex of G_1 with the vertices in the corresponding copy of N_2. In this case we have $\gamma_r(G) = 2n(G_1)$ and $n(G) = 3n(G_1)$.

Theorem 2.9. *Let T be a spanning tree of a connected graph G such that $n(G) \geq 3$. If $T_0, T_1, \ldots, T_k$ is the sequence of all embedded subtrees of T of order greater than or equal to three defined above, then*

$$\gamma_s(G) \leq \sum_{i=0}^{k} \sum_{v \in S(T_i)} |L_{T_i}(v)| + \varrho(T),$$

where $\varrho(T) = 0$ if $V(T_k) = X(T_k)$ and $\varrho(T) = 1$ otherwise.

Proof. Notice that Proposition 2.3 leads to $\gamma_s(G) \leq \gamma_s(T)$. Let

$$W = \bigcup_{i=0}^{k} \left(\bigcup_{v \in S(T_i)} L_{T_i}(v) \right) \cup W_k,$$

where W_k is defined as follows. If $V(T_k) = X(T_k)$, then we set $W_k = \emptyset$, otherwise we fix $x_k \in V(T_k) \setminus X(T_k)$ and we set $W_k = \{x_k\}$. To conclude that W is a secure dominating set for T we only need to observe that W is a dominating set and the movement of a guard from $L_{T_i}(v)$ to v does not produce undefended vertices, as well as, the movement of a guard from x_k to a vertex in $V(T_k) \setminus X(T_k)$ (if any) does not produce undefended vertices. Therefore, the result follows. □

The bound above is achieved, for instance, by the family of corona graphs $G \cong G_1 \odot N_t$. Obviously, for any spanning tree T of G we have $\varrho(T) = 0$ and $tn(G_1) \leq \gamma_r(G) = \sum_{i=0}^{k} \sum_{v \in S(T_i)} |L_{T_i}(v)| + \varrho(T) = tn(G_1)$. Notice that the lower bound $\gamma_r(G) \geq tn(G_1)$ is deduced from the fact that every secure dominating set contains at least one guard per each vertex of degree one in G. In general, we can state the following tight bound in terms of the number of vertices of degree one, denoted by $\ell(G)$.

Remark 2.10. *For any graph G,*

$$\gamma_s(G) \geq \ell(G).$$

In particular, for any graph G',

$$\gamma_s(G' \odot N_t) = \ell(G' \odot N_t) = n(G')t.$$

Two edges in a graph G are independent if they are not adjacent in G. The *matching number* $\alpha'(G)$ of graph G, sometimes known as the edge independence number, is the cardinality of a maximum independent edge set.

Theorem 2.11. [6] *If a graph G does not have isolated vertices, then*

$$\gamma_s(G) \leq n(G) - \alpha'(G).$$

It is known that for every graph G with no isolated vertex $\alpha'(G) \geq \gamma(G)$ [12]. Hence, Theorem 2.11 leads to the following corollary.

Corollary 2.12. *If a graph G does not have isolated vertices, then*

$$\gamma_s(G) \leq n(G) - \gamma(G).$$

Recall that a graph without isolated vertices satisfies $\gamma(G) = n(G)/2$ if and only if its components are isomorphic to C_4 or to corona graphs of the form $H \odot K_1$. If $\gamma(G) = n(G)/2$, then Corollary 2.12 leads to $\frac{n(G)}{2} = \gamma(G) \leq \gamma_r(G) \leq \gamma_s(G) \leq \frac{n(G)}{2}$. Thus, we deduce the following result.

Remark 2.13. *If $\gamma(G) = \frac{n(G)}{2}$, then $\gamma_r(G) = \gamma_s(G) = \frac{n(G)}{2}$.*

As we will show in Theorem 2.15, in some cases the bound provided by Theorem 2.11 can be improved. To this end, we need to introduce some additional notation. Let $\mathcal{D}(G)$ be the set of all $\gamma(G)$-sets. For every $S \in \mathcal{D}(G)$ we define

$$T(S) = \{v \in V(G) \setminus S : \ N[v] = N[s] \text{ for some } s \in S\}.$$

Finally, we define

$$\tau(G) = \max\{|T(S)| : \ S \in \mathcal{D}(G)\}.$$

Recall that two vertices u, v are called *true twins* if $N[u] = N[v]$.

Lemma 2.14. *Let G be a graph such that no component of G is a complete graph. If S is a $\gamma(G)$-set, then $V(G) \setminus (S \cup T(S))$ is a dominating set.*

Proof. Since every vertex in $T(S)$ has a true twin in S, we only need to show that every vertex in S has a neighbour in $S' = V(G) \setminus (S \cup T(S))$.

Notice that, since G has no isolated vertices and S is a $\gamma(G)$-set, every vertex in S has at least one neighbour outside of S. Suppose that there exists $s \in S$ such that $N(s) \cap S' = \emptyset$. In such a case, $N(s) \cap T(S) \neq \emptyset$ and, if $N(s) \cap S = \emptyset$, then the subgraph induced by $N[s]$ is a component of G, which is a contradiction. Thus, $N(s) \cap S \neq \emptyset$. Now, let $x \in N(s) \cap T(S)$. If s and x are true twins, then every neighbour of s belonging to S is a neighbour of x, while if s and x are not true twins, then there exists $s'' \in S \setminus \{s\}$ which is twin with x. Therefore, $S \setminus \{s\}$ is a dominating set, which is a contradiction. $\square$

Theorem 2.15. *If no component of G is a complete graph, then*

$$\gamma_s(G) \leq n(G) - \gamma(G) - \tau(G).$$

Proof. Let S be a $\gamma(G)$-set such that $|T(S)| = \tau(G)$. We will show that $S' = V(G) \setminus (S \cup T(S))$ is a secure dominating set. We already know from Lemma 2.14 that S' is a dominating set. It remains to show that for every $v \in S \cup T(S)$ there exists $u \in S' \cap N(v)$ such that $S'_{uv} = (S' \setminus \{u\}) \cup \{v\}$ is a dominating set. To this end, for every $u \in S'$ we define $P(u)$ as follows:

$$P(u) = \{v \in S : \ N(v) \cap S' = \{u\}\}.$$

If there exists $u \in S'$ such that $|P(u)| \geq 2$, then $S_1 = (S \setminus P(u)) \cup \{u\}$ is a dominating set and $|S_1| < |S| = \gamma(G)$, which is a contradiction. Hence, $|P(u)| \leq 1$ for every $u \in S'$. With this fact in mind, we differentiate two cases for $v \in V(G) \setminus S'$.

Case 1: $v \in S$. Suppose that $P(u) = \{v\}$ for some $u \in S'$. In this case, for every $w \in N(u) \cap (S \setminus \{v\})$ we have $|N(w) \cap S'| \geq 2$. So that, if there exists $y \in (N(u) \cap T(S)) \setminus N(v)$, then $|N(y) \cap S'| \geq 2$, as y has a twin in $S \setminus \{v\}$. Hence, S'_{uv} is a dominating set. From now on we assume that $|N(v) \cap S'| \geq 2$. Now, if there exists $u' \in N(v) \cap S'$ such that $P(u') = \emptyset$, then $|N(w) \cap S'| \geq 2$ for every $w \in N(u') \cap (S \setminus \{v\})$, and also for every $w \in (N(u') \cap T(S)) \setminus N(v)$, which implies $S'_{u'v}$ is a dominating set. Finally, suppose that $P(u) \neq \emptyset$ for every $u \in N(v) \cap S'$. Let

$$X = \{v\} \cup \left(\bigcup_{u \in N(v) \cap S'} P(u) \right).$$

Notice that $|X| = 1 + |N(v) \cap S'|$. Hence, $S_2 = (S \setminus X) \cup (N(v) \cap S')$ is a dominating set of G and $|S_2| < |S|$, which is a contradiction.

Case 2: $v \in T(S)$. Let $v' \in S$ such that $N[v] = N[v']$. As discussed in Case 1, there exists $u \in S'$ such that $S'_{uv'}$ is a dominating set. Since v and v' are true twins, we can conclude that S'_{uv} is also a dominating set.

According to the two cases above, S' is a secure dominating set. Therefore, $\gamma_s(G) \leq |S'| = n(G) - \gamma(G) - \tau(G)$. □

To show an example where Theorem 2.15 improves the bound given by Theorem 2.11, we take the graph $G \cong K_3 + N_2 \cong K_5 - e$. In this case $\gamma(G) = 1$, $\tau(G) = 2$ and $\alpha'(G) = 2$, which implies that $\gamma_s(G) \leq n(G) - \gamma(G) - \tau(G) = 2 < 3 = n(G) - \alpha'(G)$.

A set $X \subseteq V(G)$ is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$. The 2-*packing number* $\rho(G)$ is the cardinality of any largest 2-packing of G. A 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-set. It is well known that for any graph G, $\gamma(G) \geq \rho(G)$, [12]. Meir and Moon [17] showed in 1975 that $\gamma(T) = \rho(T)$ for any tree T. We remark that in general, these $\gamma(T)$-sets and $\rho(T)$-sets are not identical. The following result is a direct consequence of Theorem 2.15.

Corollary 2.16. *If no component of G is a complete graph, then*

$$\gamma_s(G) \leq n(G) - \rho(G) - \tau(G).$$

To see the sharpness of the bound above, consider the corona graph $G_1 \odot N_p$, where G_1 is an arbitrary graph. In this case, $n(G_1 \odot N_p) = n(G_1)(p+1)$, $\rho(G_1 \odot N_p) = n(G_1)$ and $\gamma_s(G_1 \odot N_p) = n(G_1)p = n(G_1 \odot N_p) - \rho(G_1 \odot N_p) = n(G_1 \odot N_p) - \gamma(G_1 \odot N_p)$. From $G' \cong G_1 \odot N_2$ we can construct a family of graphs G of order $n(G) = 3n(G_1) + l_1 + \cdots + l_{n(G_1)}$ with $\gamma_s(G) = n(G) - \gamma(G) - \tau(G)$. We construct G from G' and a $\gamma(G')$-set $S = \{v_1, \ldots, v_{n(G_1)}\}$ by replacing every $v_j \in S$ with a copy of K_{l_j} and joining by an edge each vertex of K_{l_j} with each neighbour of v_j in G'.

As shown in [21], the domination number of any graph G is bounded below by $\frac{n(G)}{\Delta(G)+1}$. Therefore, the following result is deduced from Theorem 2.15.

Corollary 2.17. *If no component of G is a complete graph, then*

$$\gamma_s(G) \leq \left\lfloor \frac{n(G)\Delta(G)}{\Delta(G)+1} \right\rfloor - \tau(G).$$

The bound above is tight. For instance, it is achieved for any graph isomorphic to $K_n - e$. In this case $\tau(G) = n(G) - 3$ and $\Delta(G) = n(G) - 1$ so $\gamma_s(G) = 2$.

Since $\gamma_r(G) \leq 2\gamma(G)$ and $\gamma_r(G) \leq \gamma_s(G)$, Theorem 2.15 leads to the following upper bounds on the weak Roman domination number.

Corollary 2.18. *If no component of G is a complete graph, then the following assertions hold.*

(i) $\gamma_r(G) \leq \left\lfloor \dfrac{n(G)+\gamma(G)-\tau(G)}{2} \right\rfloor$.

(ii) *If $\gamma(G) \geq \frac{n(G)}{3}$, then $\gamma_r(G) \leq 2\gamma(G) - \tau(G)$.*

To see the sharpness of the bounds above, consider the corona graph $G \cong G_1 \odot N_2$, where G_1 is an arbitrary graph. In this case, $n(G) = 3n(G_1)$, $\gamma(G) = n(G_1)$, $\tau(G) = 0$ and $\gamma_r(G) = 2n(G_1)$. Another example of equality for bound (i) is $G \cong K_n - e$, where $\gamma_r(G) = 2$, $\tau(G) = n(G) - 3$ and $\gamma(G) = 1$.

The minimum number of cliques of a given graph G needed to cover the vertex set $V(G)$ is called the *clique covering number* of G and denoted by $\theta(G)$. Before stating our next result we need to recall the following theorem, which states a Nordhaus-Gaddum inequality for the chromatic number of a graph.

Theorem 2.19. [1] *For any graph G,*

$$\chi(G) + \chi(\overline{G}) \leq n(G) + 1 \text{ and } \chi(G)\chi(\overline{G}) \leq \frac{(n(G)+1)^2}{4}.$$

Theorem 2.20. *The following statements hold for any graph G.*

(i) $\gamma_s(G) \leq \theta(G)$.

(ii) $\gamma_r(G) + \gamma_r(\overline{G}) \leq \gamma_s(G) + \gamma_s(\overline{G}) \leq n(G) + 1$.

(iii) $\gamma_r(G)\gamma_r(\overline{G}) \leq \gamma_s(G)\gamma_s(\overline{G}) \leq \dfrac{(n(G)+1)^2}{4}$.

Furthermore, if $G \not\cong C_5$ is a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \leq n(G) - 3$, then the following statement hold.

(iv) $\gamma_r(G) + \gamma_r(\overline{G}) \leq \gamma_s(G) + \gamma_s(\overline{G}) \leq n(G) - 1$ *for $n(G)$ odd and*
$\gamma_r(G) + \gamma_r(\overline{G}) \leq \gamma_s(G) + \gamma_s(\overline{G}) \leq n(G)$ *for $n(G)$ even.*

(iiv) $\gamma_r(G)\gamma_r(\overline{G}) \leq \gamma_s(G)\gamma_s(\overline{G}) \leq \frac{(n(G)-1)^2}{4}$ *for $n(G)$ odd and*
$\gamma_r(G)\gamma_r(\overline{G}) \leq \gamma_s(G)\gamma_s(\overline{G}) \leq \frac{(n(G))^2}{4}$ *for $n(G)$ even.*

Proof. Let Π be a partition of $V(G)$ into cliques such that $|\Pi| = \theta(G)$. The proof of (i) directly follows from the fact that any set formed by one representative of each clique in Π is a secure dominating set.

Since $\chi(G) = \theta(\overline{G})$, (i) and Theorem 2.19 lead to

$$\gamma_s(G) + \gamma_s(\overline{G}) \leq \theta(G) + \theta(\overline{G}) = \chi(\overline{G}) + \chi(G) \leq n(G) + 1$$

and

$$\gamma_s(G)\gamma_s(\overline{G}) \leq \theta(G)\theta(\overline{G}) = \chi(\overline{G})\chi(G) \leq \frac{(n(G)+1)^2}{4},$$

as required. Finally, (iv) and (v) are a direct consequence of Theorem 2.7. □

The inequalities above are tight. For instance, (i) is achieved by the graphs shown in Figure 2, (ii) and (iii) are achieved by the self-complementary graph shown in Figure 2 (on the left) and also by C_5. In both cases we have $n(G) = 5$ and $\gamma_r(G) = \gamma_s(G) = 3$. Finally, (iv) and (v) are achieved by the self-complementary graph shown in Figure 2 (on the right), in this case we have $n(G) = 8$ and $\gamma_r(G) = \gamma_s(G) = 4$.

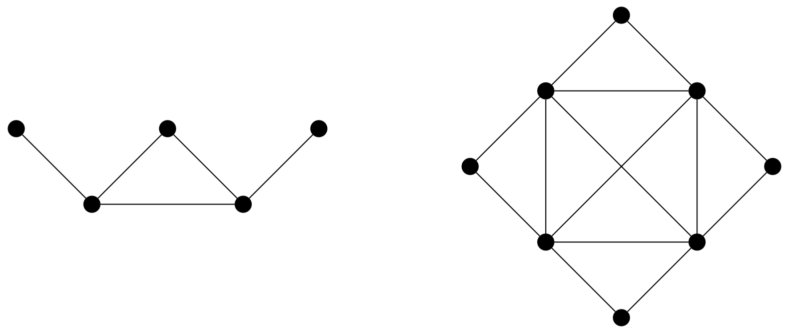

Figure 2: Two self-complementary graphs.

3. Results on Cartesian product graphs

The *Cartesian product* of two graphs G and H is the graph $G\Box H$, such that $V(G\Box H) = V(G) \times V(H)$ and two vertices $(g, h), (g', h') \in V(G\Box H)$ are adjacent in $G\Box H$ if and only if either

- $g = g'$ and $hh' \in E(H)$, or
- $gg' \in E(G)$ and $h = h'$.

The Cartesian product is a straightforward and natural construction, and is in many respects the simplest graph product [10, 14]. Hypercubes, Hamming graphs, grid graphs, cylinder graphs and torus graphs are some particular cases of this product. The *Hamming graph* $H_{k,t}$ is the Cartesian product of k copies of the complete graph K_t. The *hypercube* Q_t is defined as $H_{t,2}$. Moreover, the *grid graph* $P_k\Box P_t$ is the Cartesian product of the paths P_k and P_t, the *cylinder graph* $C_k\Box P_t$ is the Cartesian product of the cycle C_k and the path P_t, and the *torus graph* $C_k\Box C_t$ is the Cartesian product of the cycles C_k and C_t.

This operation is commutative in the sense that $G\Box H \cong H\Box G$, and is also associative, as the graphs $(F\Box G)\Box H$ and $F\Box(G\Box H)$ are naturally isomorphic. A Cartesian product graph is connected if and only if both of its factors are connected.

Notice that for any $u \in V(G)$ and $v \in V(H)$ the subgraph of $G\Box H$ induced by $\{u\} \times V(H)$ is isomorphic to H and the subgraph of $G\Box H$ induced by $V(G) \times \{v\}$ is isomorphic to G.

This product has been extensively investigated from various perspectives. For instance, the most popular open problem in the area of domination theory is known as Vizing's conjecture. Vizing [20] suggested that for any graphs G and H,

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H).$$

Several researchers have worked on it, for instance, some partial results appears in [3, 10]. For more information on structure and properties of the Cartesian product of graphs we refer the reader to [10, 14].

The study of the secure domination number of Cartesian product graphs was initiated by Cockayne et al. in [7], where they obtained bounds on $\gamma_s(C_k\Box C_t)$ and $\gamma_s(P_k\Box P_t)$ in terms of k and t. Before stating our first result we need to recall the following well known lower bound on the domination number of any Cartesian product graph.

Lemma 3.1. [9] *For any pair of graphs G and H,*

$$\gamma(G\Box H) \geq \min\{n(G), n(H)\}.$$

Theorem 3.2. *For any graphs G and H, the following statements hold.*

(i) $\min\{n(G), n(H)\} \le \gamma_r(G\Box H) \le \min\{n(G)\gamma_r(H), n(H)\gamma_r(G)\}$.

(ii) $\min\{n(G), n(H)\} \le \gamma_s(G\Box H) \le \min\{n(G)\gamma_s(H), n(H)\gamma_s(G)\}$.

Proof. Let $f(U_0, U_1, U_2)$ be a $\gamma_r(G)$-function. In order to prove the upper bound, we claim that the function $g : V(G\Box H) \longrightarrow \{0, 1, 2\}$ defined by $g(x, y) = f(x)$ is a WRDF on $G\Box H$, where

$$\{W_0 = U_0 \times V(H), W_1 = U_1 \times V(H), W_2 = U_2 \times V(H)\}$$

is the partition of $V(G\Box H)$ associated to g. To see this we only need to observe the following two facts.
Fact (a): Since every $x \in U_0$ is dominated by some $x' \in U_1 \cup U_2$, every $(x, y) \in W_0$ is dominated by $(x', y) \in W_1 \cup W_2$.
Fact (b): Since for every $x \in U_0$ there exists $x' \in N(x) \cap (U_1 \cup U_2)$ such that the movement of a guard from x' to x does not produce undefended vertices in G, the movement of a guard from $(x', y) \in W_1 \cup W_2$ to $(x, y) \in W_0$ does not produce undefended vertices in the subgraph of $G\Box H$ induced by $V(G) \times \{y\}$, which is isomorphic to G.

According to Facts (a) and (b) we can conclude that g is a WRDF on $G\Box H$, which implies that $\gamma_r(G\Box H) \le w(g) = n(H)w(f) = n(H)\gamma_r(G)$, as required. By analogy we deduce that $\gamma_r(G\Box H) \le n(G)\gamma_r(H)$. Therefore, the upper bound of (i) follows. The proof of the upper bound of (ii) is deduced by analogy to the previous one by taking a WRDF $f(U_0, U_1, U_2)$ such that $U_2 = \emptyset$ and $|U_1| = \gamma_s(G)$.

Finally, the lower bounds are deduced from Lemma 3.1, as $\gamma_s(G\Box H) \ge \gamma_r(G\Box H) \ge \gamma(G\Box H) \ge \min\{n(G), n(H)\}$. □

As we will show in the following results, the bounds above are tight.

Corollary 3.3. *Let t be an integer. If* $2 \le n(H) \le t$, *then* $\gamma_r(K_t\Box H) = \gamma_s(K_t\Box H) = n(H)$.

According to the result above, it remains to study the weak Roman domination number and the secure domination number of $K_t\Box H$ for $n(H) > t$. Our next result covers two particular cases.

Proposition 3.4. *For any integers* $t \ge 3$ *and* $t' \ge 3$,

$$\gamma_r(K_t\Box C_{t'}) = \gamma_r(K_t\Box P_{t'}) = \gamma_s(K_t\Box P_{t'}) = \gamma_s(K_t\Box C_{t'}) = t'.$$

Proof. By Theorem 3.2 and Propositions 2.1 and 2.3 we have that

$$\min\{t, t'\} \le \gamma_r(K_t\Box C_{t'}) \le \gamma_r(K_t\Box P_{t'}) \le t'$$

and

$$\min\{t, t'\} \le \gamma_r(K_t\Box C_{t'}) \le \gamma_s(K_t\Box C_{t'}) \le \gamma_s(K_t\Box P_{t'}) \le t'.$$

It remains to show that $\gamma_r(K_t\Box C_{t'}) \ge t'$ for $t' > t \ge 3$. Let $f(W_0, W_1, W_2)$ be a $\gamma_r(K_t\Box C_{t'})$-function and $V(C_{t'}) = \{v_1, \dots, v_{t'}\}$, where the subscripts are taken modulo t' and $v_iv_{i+1} \in E(C_{t'})$ for any $i \le t'$. Let $A_i = (V(K_t) \times \{v_i\})$ and $\alpha_i = f(A_i)$ for every $i \in \{1, \dots, t'\}$. We differentiate the following cases in which $\alpha_i = 0$ for some i. Symmetric cases are omitted.

Case 1: $\alpha_i = 0$. Since $W_1 \cup W_2$ is a dominating set, we can conclude that

$$\alpha_{i-1} + \alpha_i + \alpha_{i+1} \ge t \ge 3.$$

Case 2: $\alpha_{i-1} = \alpha_{i+1} = 0$ and $\alpha_i = 1$. In this case, no guard can move from A_i to A_{i+1} (or to A_{i-1}), which implies that $\alpha_{i-2} \ge t$ and $\alpha_{i+2} \ge t$. Hence, we can conclude that

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} \ge t + 1 \ge 4 \text{ and } \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \alpha_{i+2} \ge 1 + t \ge 4.$$

In this case, if $t' \geq 6$, then

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \alpha_{i+2} \geq 2t + 1 \geq 7.$$

Case 3: $\alpha_i = 2$ and $\alpha_{i-1} = \alpha_{i+1} = 0$. From Case 1 we know that $\alpha_{i-2} \geq t-2$ and $\alpha_{i+2} \geq t-2$. Suppose that $\alpha_{i-2} = t-2$ and $\alpha_{i+2} < t$. Notice that $W_2 \cap (A_{i-2} \cup A_i) = \emptyset$, as every vertex in A_{i-1} has to be dominated by some vertex in $W_1 \cup W_2$. Hence, for $(u, v_i), (u', v_i) \in V_1$ we have that $(u, v_{i-2}), (u', v_{i-2}) \in V_0$ and $(u, v_{i+2}) \in V_0$ or $(u', v_{i+2}) \in V_0$, as $\alpha_{i-2} = t-2$ and $\alpha_{i+2} < t$. We can assume that $(u, v_{i+2}) \in V_0$. Thus, the movement of a guard form (u, v_i) to (u, v_{i-1}) produces undefended vertices in A_{i+1}, which is a contradiction. Hence, $\alpha_{i-2} + \alpha_{i+2} \geq 2(t-1)$ and so we can conclude that

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \alpha_{i+2} \geq 2t \geq 6.$$

According to the conclusions derived from the cases above we can deduce that,

$$\gamma_r(K_t \square C_{t'}) = w(f) = \sum_{i=1}^{t'} \alpha_i \geq t'.$$

Therefore, the result follows. $\square$

Notice that the result above does not include the case of complete graphs of order two. For this case we propose the following conjecture.

Conjecture 3.5. *For any integer $t \geq 2$*

$$\gamma_s(P_t \square K_2) = \left\lceil \frac{3t+1}{4} \right\rceil.$$

Furthermore, for $t \geq 3$,

$$\gamma_s(C_t \square K_2) = \begin{cases} \left\lceil \frac{3t}{4} \right\rceil + 1, & \text{if } t \equiv 4 \pmod 8 \\ \left\lceil \frac{3t}{4} \right\rceil, & \text{otherwise.} \end{cases}$$

Regarding the conjecture above, we would emphasize that it is known from [15] that $\gamma(P_t \square K_2) = \left\lceil \frac{t+1}{2} \right\rceil$ and from [8] that $\gamma_R(P_t \square K_2) = t + 1$.

Proposition 3.6. *Let $t \geq 2$ and $t' \geq 2$ be two integers. The following statements hold.*

(i) $\gamma_r(K_t \square K_{1,t'-1}) = \min\{2t, t'\}$.

(ii) $\gamma_s(K_t \square K_{1,t'-1}) = t'$.

Proof. From Theorem 3.2 we have that $\gamma_r(K_t \square K_{1,t'-1}) \leq \min\{2t, t'\}$. We proceed to show that $\gamma_r(K_t \square K_{1,t'-1}) \geq \min\{2t, t'\}$. Let $f(W_0, W_1, W_2)$ be a $\gamma_r(K_t \square K_{1,t'-1})$-function and let y_0 be the universal vertex of $K_{1,t'-1}$. Suppose that $\gamma_r(K_t \square K_{1,t'-1}) < \min\{2t, t'\}$. Now, since $\gamma_r(K_t \square K_{1,t'-1}) < 2t$, there exists $x \in V(K_t)$ such that $f(\{x\} \times V(K_{1,t'-1})) \leq 1$ and, since $\gamma_r(K_t \square K_{1,t'-1}) < t'$, there exist $y \in V(K_{1,t'-1})$ such that $V(K_t) \times \{y\} \subseteq W_0$. If $y = y_0$, then there is exactly one guard for each copy of K_t different from the one associated to y_0 (as every vertex has to be defended), which implies that the movement of any guard to a vertex in $V(K_t) \times \{y_0\}$ produces undefended vertices, so that $y \neq y_0$. Notice that $f(V(K_t) \times \{y_0\}) \geq t$, otherwise there are undefended vertices in $V(K_t) \times \{y\}$. Now, suppose that $V(K_t) \times \{y'\} \subseteq W_0$, for some $y' \in V(K_{1,t'-1}) \setminus \{y_0, y\}$. In such a case, (x, y') and (x, y) are only defended by a guard located at (x, y_0), but (x, y) will become undefended after the movement of that guard to (x, y'), which is a contradiction. Hence, $\sum_{v \neq y_0} f(V(K_t) \times \{v\}) \geq t' - 2$, and so $w(f) \geq t + t' - 2 \geq t'$, which is a contradiction again. Thus, $\gamma_r(K_t \square K_{1,t'-1}) \geq \min\{2t, t'\}$, as required. Therefore, (i) follows.

We now proceed to prove (ii). As above, let y_0 be the universal vertex of $K_{1,t'-1}$, W a $\gamma_s(K_t \square K_{1,t'-1})$-set and $u \in V(K_t)$. Suppose that $|W| \leq t' - 1$. In such a case, there exists $v \in V(K_{1,t'-1})$ such that $W \cap (V(K_t) \times \{v\}) = \emptyset$.

Notice that $N(u,v) \cap W \neq \emptyset$. We differentiate two cases.
Case 1: $v \neq y_0$. Since W is a dominating set, $V(K_t) \times \{y_0\} \subseteq W$. Thus, there exists $v_1 \in V(K_{1,t'-1}) \setminus \{v, y_0\}$ such that $V(K_t) \times \{v_1\} \subseteq \overline{W}$. Hence, $N[(u,v)] \cap W = \{(u,y_0)\} = N[(u,v_1)] \cap W$, and so $(W \setminus \{(u,y_0)\}) \cup \{(u,v_1)\}$ is not a dominating set, which is a contradiction.
Case 2: $v = y_0$. Since W is a dominating set and $|W| < t'$, for every $v' \in V(K_{1,t'-1}) \setminus \{y_0\}$ we have that $|(V(K_t) \times \{v'\}) \cap W| = 1$. Hence, for $u \in V(K_t)$ such that $(u,v') \in W$ and $u' \in V(K_t) \setminus \{u\}$ we have that $N[(u',v')] \cap W = \{(u,v')\}$. Thus, for every $v' \in V(K_{1,t'-1}) \setminus \{y_0\}$ and $u \in V(K_t)$ such that $(u,v') \in W$, we have that $(W \setminus \{(u,v')\}) \cup \{(u,y_0)\}$ is not a dominating set, which is a contradiction.
According to the two cases above we can conclude that $\gamma_s(K_t \Box K_{1,t'-1}) = |W| \geq t'$. Finally, Theorem 3.2 leads to $\gamma_s(K_t \Box K_{1,t'-1}) = t'$. □

Proposition 3.7. *For any graph G and any integer $t > 2n(G) \geq 4$,*

$$\gamma_r(G \Box K_{1,t-1}) = 2n(G).$$

Proof. By Theorem 3.2 we have $\gamma_r(G \Box K_{1,t-1}) \leq 2n(G)$. To conclude the proof we only need to observe that Propositions 2.3 and 3.6 lead to $\gamma_r(G \Box K_{1,t-1}) \geq \gamma_r(K_{n(G)} \Box K_{1,t-1}) = 2n(G)$. □

Theorem 3.8. *If no component of a graph H is a complete graph, then for any nontrivial graph G,*

$$\gamma_s(G \Box H) \leq n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H) - \gamma(G)\tau(H).$$

Proof. In this proof we use the set $T(S)$ as defined prior to Lemma 2.14. Let S_1 be a $\gamma(G)$-set and S_2 a $\gamma(H)$-set such that $|T(S_2)| = \tau(H)$. We will show that $W = (S_1 \times S_2') \cup (\overline{S_1} \times S_2)$ is a secure dominating set of $G \Box H$, where $S_2' = V(H) \setminus (S_2 \cup T(S_2))$. First of all, notice that W is a dominating set of $G \Box H$ as S_2 and S_2' are dominating sets in H (by Lemma 2.14). We differentiate the following three cases for $(x,y) \in \overline{W}$.
Case 1: $(x,y) \in S_1 \times \overline{S_2'}$. In the proof of Theorem 2.15 we have shown that S_2' is a secure dominating set. Hence, for each vertex $(x,y) \in S_1 \times \overline{S_2'}$ there exists $(x,y') \in S_1 \times S_2'$ such that the movement of a guard from (x,y') to (x,y) does not produce undefended vertices in $\{x\} \times \overline{S_2'}$. Such a movement of guards does not produce undefended vertices in $\overline{S_1} \times \{y'\}$, as these vertices are dominated by the ones in $\overline{S_1} \times S_2$.
Case 2: $(x,y) \in \overline{S_1} \times S_2'$. For any $y' \in S_2 \cap N(y)$ the movement of a guard from (x,y') to (x,y) does not produce undefended vertices in $S_1 \times \{y'\}$, as these vertices are dominated by the ones in $S_1 \times \{y\}$. Such a movement of guards does not produce undefended vertices in $\{x\} \times S_2'$, as these vertices are dominated by the ones in $\{x'\} \times S_2'$, for every $x' \in S_1 \cap N(x)$. Now, suppose that $y'' \in N(y') \cap T(S_2)$. If $|N(y'') \cap S_2| \geq 2$, then (x,y'') remains defended after the above mentioned movement of guards. If $|N(y'') \cap S_2| = \{y'\}$, then y' and y'' are twins, which implies $(x,y'') \in N(x,y)$, so that (x,y'') remains defended after the movement of a guard form (x,y') to (x,y).
Case 3: $(x,y) \in \overline{S_1} \times T(S_2)$. Let $y' \in S_2$ such that $N[y] = N[y']$. As in the previous case, the movement of a guard from (x,y') to (x,y) does not produce undefended vertices in $S_1 \times \{y'\}$. On the other hand, since y and y' are twins, the movement of a guard from (x,y') to (x,y) does not produce undefended vertices in $\{x\} \times \overline{S_2}$.

According to the three cases above, W is a secure dominating set of $G \Box H$. Therefore,

$$\gamma_s(G \Box H) \leq |W| = n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H) - \gamma(G)\tau(H)$$

as desired. □

According to the result above, for any noncomplete graph H,

$$\gamma_s(K_t \Box H) \leq (t-2)\gamma(H) + n(H) - \tau(H).$$

It is not difficult to check that the bound above is tight. For instance, it is achieved by $H \cong K_l + N_3$ for $l \geq 2$, as $\gamma_s(K_3 \Box (K_l + N_3)) = 5$, $\gamma(H) = 1$ and $\tau(H) = l - 1$. Notice that, in this case, Theorem 3.8 gives a better result than Theorem 3.2.

We learned from Theorem 2.7 that $\gamma_s(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$ for every graph $G \not\cong C_5$ having minimum degree $\delta(G) \geq 2$. If G and H have no isolated vertices, then $\gamma(G) \in \{1, \ldots, \lfloor n(G)/2 \rfloor\}$ and $\gamma(H) \in \{1, \ldots, \lfloor n(H)/2 \rfloor\}$. Hence, we can state the following remark which shows that the bound provide by Theorem 3.8 is never worse that the bound $\gamma_s(G \Box H) \leq \left\lfloor \frac{n(G)n(H)}{2} \right\rfloor$ deduced from Theorem 2.7.

Remark 3.9. *If G and H have no isolated vertices, then*

$$n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H) \leq \left\lfloor \frac{n(G)n(H)}{2} \right\rfloor.$$

The inequality chain

$$\gamma_r(G \Box H) \leq \gamma_s(G \Box H) \leq n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H)$$

is tight. It is achieved for $P_3 \Box P_3$ and $K_2 \Box K_2 \cong C_4$, as $\gamma_r(P_3 \Box P_3) = 4$ and $\gamma_r(C_4) = 2$. Proposition 3.10 provides another example of graphs for which this inequality chain is achieved.

Proposition 3.10. *For any integer $t \geq 3$,*

$$\gamma_r(K_{1,t-1} \Box K_{1,t-1}) = \gamma_s(K_{1,t-1} \Box K_{1,t-1}) = 2(t-1).$$

Proof. According to Theorem 3.8, we only need to prove the lower bound $\gamma_r(K_{1,t-1} \Box K_{1,t-1}) \geq 2(t-1)$. Let $f(W_0, W_1, W_2)$ be a $\gamma_r(K_{1,t-1} \Box K_{1,t-1})$-function and, for simplicity, set $V = V(K_{1,t-1})$. Let $x \in V$ be the vertex of degree $t-1$. From now on, we suppose that $w(f) \leq 2t-3$. We proceed to show the following claim.
Claim 1. $f(\{u\} \times V) \geq 1$, for every $u \in V \setminus \{x\}$.

In order to prove Claim 1, we suppose that there exists $u \in V \setminus \{x\}$ such that $f(\{u\} \times V) = 0$. In such a case, $f(x, y) \geq 1$, for every $y \in V$. Now, since $w(f) \leq 2t-3$, there exist $u' \in V \setminus \{x, u\}$ and $v \in V$ such that $f(\{u'\} \times V) = 0$ and $f(x, v) = 1$, which is a contradiction as (u', v) is undefended after the movement of the guard located in (x, v) to (u, v). Thus, Claim 1 follows.

Since $w(f) \leq 2t-3$, Claim 1 leads to the following ones.
Claim 2. There exists $u^* \in V \setminus \{x\}$ such that $f(\{u^*\} \times V) = 1$.
Claim 3. There exists $v^* \in V \setminus \{x\}$ such that $f(x, v^*) = 0$.

We differentiate the following two cases for $f(u^*, x)$.
Case 1: $f(u^*, x) = 0$. By Claims 2 and 3 we can conclude that $f(u^*, v^*) = 1$, otherwise (u^*, v^*) is not dominated by the elements in $W_1 \cap W_2$. Since every vertex in $\{u^*\} \times V \setminus \{(u^*, x), (u^*, v^*)\}$ has to be dominated by some vertex in $W_1 \cup W_2$, from $w(f) \leq 2t-3$ and Claim 1 we deduce that $f(x, v) = 1$ for every $v \in V \setminus \{x, v^*\}$, $f(\{u\} \times V) = 1$ for every $u \in V \setminus \{x, u^*\}$, and $f(x, x) = 0$. Hence, the movement of any guard from a vertex in $\{x\} \times V$ to (x, x) produces undefended vertices in $\{u^*\} \times V$, and the movement of a guard from a vertex of the form (a, x) to (x, x) leaves vertex (a, v^*) undefended. In both cases we have a contradiction.

Case 2: $f(u^*, x) = 1$. In this case, (u^*, x) is the only vertex in $W_1 \cup W_2$ which is adjacent to (u^*, v^*). Hence, the movement of a guard from (u^*, x) to (u^*, v^*) does not produce undefended vertices, and so from $w(f) \leq 2t-3$ and Claim 1 we deduce that $f(x, v) = 1$ for every $v \in V \setminus \{x, v^*\}$, $f(\{u\} \times V) = 1$ for every $u \in V \setminus \{x, u^*\}$, and $f(x, x) = 0$. Thus, the movement of a guard from a vertex of the form (a, x) to (x, x) leaves vertex (a, v^*) undefended, which is a contradiction.

According to the two cases above we can conclude that, $w(f) \geq 2(t-1)$, as required. □

As usual in domination theory, when studying a domination parameter, we can ask if a Vizing-like conjecture can be proved or formulated. By Proposition 3.10 we can claim that there are graphs with

$$\gamma_s(G \Box H) \not\geq \gamma_s(G)\gamma_s(H),$$

i.e., for any $p \geq 3$ we have $\gamma_s(K_{1,p} \Box K_{1,p}) = 2p < p^2 = \gamma_s(K_{1,p})\gamma_s(K_{1,p})$.

Theorem 3.11. *Let $f_H = (V_0, V_1, V_2)$ be a $\gamma_r(H)$-function of a graph H such that $V_2 \neq \emptyset$, and let $Y = V(H) \setminus N[V_2]$. For any graph G,*

$$\gamma_r(G \Box H) \leq 2n(G)|V_2| + |Y|\gamma_r(G).$$

Proof. Let $f_G = (U_0, U_1, U_2)$ be a $\gamma_r(G)$-function, $W_1 = U_1 \times Y$ and $W_2 = (V(G) \times V_2) \cup (U_2 \times Y)$. In order to show that $f = (W_0, W_1, W_2)$ is a WRDF of $G \Box H$, we differentiate the following two cases for $(x, y) \in W_0$.
Case 1: $(x, y) \in V(G) \times (N(V_2) \setminus V_2)$. Since there exists $y' \in V_2 \cap N(y)$, the movement of a guard from (x, y') to (x, y) does not produce undefended vertices.
Case 2: $(x, y) \in U_0 \times Y$. Since f_G is a $\gamma_r(G)$-function, there exists $x' \in U_1 \cup U_2$ such that the movement of a guard from x' to x does not produce undefended vertices. Which implies that the movement of a guard from (x', y) to (x, y) does not produce undefended vertices in $V(G) \times Y$. □

Notice that for any graph with $\gamma_r(H) = 2\gamma(H)$, Theorems 3.2 and 3.11 lead to the same result $\gamma_r(G \Box H) \leq 2n(G)\gamma(H)$. In order to show an example where Theorem 3.11 gives a better result we take $G \cong K_3$ and the graph H shown in Figure 3. In this case, an optimum solution consists of two guards at each vertex of the copy of K_3 corresponding to the vertex $v \in V(H)$ of maximum degree and one guard at each copy of K_3 corresponding to the vertices of H nonajacent to v.

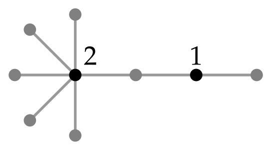

Figure 3: A graph with $\gamma_r(H) = 3$, $|Y| = 2$ and $\gamma_r(K_3 \Box H) = 2n(G)|V_2| + |Y|\gamma_r(G) = 8$.

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