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Topologies on Normed Spaces Generated by Porosity

Stanisław Kowalczyk[a], **Małgorzata Turowska**[a]

[a]*Pomeranian University in Słupsk, Kozietulskiego 6-7, 76-200 Słupsk, Poland*

Abstract. In the present paper we study properties of porouscontinuous functions defined by J. Borsík and J. Holos in 2014. We find maximal additive classes for different families of these functions. Furthermore, we define new families of topologies generated by the notion of porosity, which are used to study maximal multiplicative classes for porouscontinuous functions. Some relevant properties of defined topologies are considered.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. For $f\colon Y \to Z$ and $A \subset Y$, by $f_{\restriction A}$ we mean the restriction of f to A. The symbol $(X, \|\ \|)$ always stands for a normed space, $\operatorname{cl} A$ and $\operatorname{int} A$ denote a closure and an interior of $A \subset X$ with respect to a topology generated by the norm. The aim of our paper is to describe topologies on a normed space generated by the notion of porosity and to study their connections with families of porouscontinuous functions.

The open ball in $(X, \|\ \|)$ with the center $x \in X$ and the radius R will be denoted by $B(x, R)$. Similarly, by $S(x, R)$ and $\overline{B}(x, R)$ we will denote a sphere and a closed ball with the center x and the radius R.

First, we recall the definition of porosity. Let $M \subset X$, $x \in X$ and $R > 0$. Then, according to [3, 9], we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$ by $\gamma(x, R, M)$. The number $p(M, x) = 2 \limsup_{R \to 0^+} \frac{\gamma(x,R,M)}{R}$ is called the porosity of M at x. Obviously, $p(M, x) = p(\operatorname{cl} M, x)$ for $M \subset X$ and $x \in X$. In a normed space one have $p(M, x) \le 2$ and if $x \in M$, then $p(M, x) \le 1$.

We say that the set M is porous at $x \in X$ if $p(M, x) > 0$. The set M is called porous if M is porous at each point $x \in M$. We say that M is strongly porous at x if $p(M, x) \ge 1$ and M is called strongly porous if M is strongly porous at each $x \in M$. Obviously every strongly porous set is porous and every porous set is nowhere dense. Moreover, none of reverse inclusions is true.

Remark 1.1. Let $(X, \|\ \|)$ be a normed space, $A \subset M \subset X$ and $x \in X$. Then $p(M, x) \le p(A, x)$. In particular, if $p(A, x) = 0$, then $p(M, x) = 0$.

In some applications we will use notions of porosities for subsets of $\mathbb{R}$. Due to L. Zajíček, J. Borsík and J. Holos [1, 9] we give another definitions of porosities of subsets of the real line. For a set $A \subset \mathbb{R}$ and an

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Email addresses: stanislaw.kowalczyk@apsl.edu.pl (Stanisław Kowalczyk), malgorzata.turowska@apsl.edu.pl (Małgorzata Turowska)

interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A. Let $x \in \mathbb{R}$. Then, according to [1, 9], the right-porosity of the set A at x is defined as

$$p^+(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x, x+h))}{h},$$

the left-porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x-h, x))}{h},$$

and the porosity of A at x is defined as

$$p(A, x) = \max\{p^-(A, x), p^+(A, x)\}.$$

It is easy to see that for any $A \subset \mathbb{R}$ if $x \in A$, then the both definitions of $p(A, x)$ are equivalent.

Theorem 1.2. *Let $(X, \|\ \|)$ be a normed space, $x_0 \in X$ and $A \subset X$ be such that $x_0 \in \mathrm{cl}(\mathrm{int}\, A) \setminus \mathrm{int}\, A$. Then there exists a sequence $(\overline{B}(x_n, r_n))_{n\in\mathbb{N}}$ of pairwise disjoint closed balls such that $\bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n) \subset A \setminus \{x_0\}$, $\lim_{n\to\infty} x_n = x_0$ and*

$$p(X \setminus A, x_0) = p\Big(X \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n), x_0\Big) = \lim_{n\to\infty} \frac{2r_n}{r_n + \|x_0 - x_n\|}.$$

Proof. Fix $R > 0$. Since $x_0 \in \mathrm{cl}(\mathrm{int}\, A)$, we obtain $\gamma(x_0, R, X \setminus A) > 0$. For every $\varepsilon \in (0, \gamma(x_0, R, X \setminus A))$ we can find a closed ball $\overline{B}(y, \eta)$ such that $\eta > \gamma(x_0, R, X \setminus A) - \varepsilon$ and $\overline{B}(y, \eta) \cap (X \setminus A) = \emptyset$ i.e. $\overline{B}(y, \eta) \subset A$. Since $x_0 \notin \mathrm{int}\, A$, we have $\|y - x_0\| \geq \eta$. Take any $\eta_1 \in (\gamma(x_0, R, X \setminus A) - \varepsilon, \eta)$. Then $\eta_1 > \gamma(x_0, R, X \setminus A) - \varepsilon$, $B(y, \eta_1) \subset A$ and $x_0 \notin \overline{B}(y, \eta_1)$. Therefore we can find by induction a sequence of closed balls $\left(\overline{B}(x_n, r_n)\right)_{n\in\mathbb{N}}$ such that $\bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n) \subset A \setminus \{x_0\}$, $\lim_{n\to\infty} x_n = x_0$ and $p(X \setminus A, x_0) = p(X \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n), x_0)$.

Since $\inf\{R > 0 \colon \overline{B}(x_n, r_n) \subset B(x_0, R)\} = r_n + \|x_n - x_0\|$, we obtain the equality $p(X \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n), x_0) = \lim_{n\to\infty} \frac{2r_n}{r_n + \|x_0 - x_n\|}$. □

In [1] J. Borsík and J. Holos defined families of porouscontinuous functions $f \colon \mathbb{R} \to \mathbb{R}$. Applying their ideas we transfer this concept for real functions defined on a normed space.

Definition 1.3. Let $(X, \|\ \|)$ be a normed space, $r \in (0, 1)$, $f \colon X \to \mathbb{R}$ and $x \in X$. The function f will be called:

- $\mathcal{P}_r$-continuous at x if there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A, x) > r$ and $f_{\restriction A}$ is continuous at x;
- $\mathcal{S}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A, x) > r$ and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$;
- $\mathcal{M}_r$-continuous at x if there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A, x) \geq r$ and $f_{\restriction A}$ is continuous at x;
- $\mathcal{N}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset X$ such that $x \in A$, $p(X \setminus A, x) \geq r$ and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

By $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$ and $\mathcal{N}_r(f)$ we denote the sets of points at which f is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous and $\mathcal{N}_r$-continuous, respectively.

Proposition 1.4. *Let $(X, \|\ \|)$ be a normed space, $f \colon X \to \mathbb{R}$, $x_0 \in X$ and $r \in (0, 1)$. Then*

1. *$x_0 \in \mathcal{S}_r(f)$ if and only if $p(X \setminus \{x \colon |f(x) - f(x_0)| < \varepsilon\}, x_0) > r$ for every $\varepsilon > 0$;*
2. *$x_0 \in \mathcal{N}_r(f)$ if and only if $p(X \setminus \{x \colon |f(x) - f(x_0)| < \varepsilon\}, x_0) \geq r$ for every $\varepsilon > 0$.*

Proposition 1.5. *Let $(X, \|\ \|)$ be a normed space, $f \colon X \to \mathbb{R}$, $x_0 \in X$ and $r, r_1, r_2 \in (0, 1)$, $r_1 < r_2$. Then*

1. *if* $x_0 \in \mathcal{P}_r(f)$*, then* $x_0 \in \mathcal{S}_r(f)$*;*
2. *if* $x_0 \in \mathcal{S}_r(f)$*, then* $x_0 \in \mathcal{N}_r(f)$*;*
3. *if* $x_0 \in \mathcal{M}_{r_2}(f)$*, then* $x_0 \in \mathcal{P}_{r_1}(f)$*.*

In [1] is proved that f is $\mathcal{M}_r$-continuous at x if and only if it is $\mathcal{N}_r$-continuous at x for functions defined on $\mathbb{R}$. It is easily seen that this remains true for functions defined on a normed space. Some other properties of porouscontinuous functions can be found in [5, 7].

If f is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous at every point of X for some $r \in (0,1)$, then we say that f is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous, respectively.

All of these functions are called porouscontinuous functions.

Obviously, if f is continuous in the norm $\|\ \|$ at some x, then f is porouscontinuous (in each sense) at x. Moreover, by $C(f)$ we denote the sets of points at which f is continuous.

Following [1], we introduce for $r \in (0,1)$ the following notations:

- $\mathcal{M}_r = \mathcal{N}_r = \{f : \mathcal{M}_r(f) = X\}$;
- $\mathcal{P}_r = \{f : \mathcal{P}_r(f) = X\}$;
- $\mathcal{S}_r = \{f : \mathcal{S}_r(f) = X\}$.

Proposition 1.6. *Let* $(X, \|\ \|)$ *be a normed space,* $x \in X$*,* $R > 0$ *and* $f : X \to \mathbb{R}$*. If* $f_{\restriction \overline{B}(x,R)}$ *is continuous, then* f *is porouscontinuous (in each considered sense) at every* $y \in S(x,R)$*.*

Proof. Fix $y \in S(x,R)$. Take any $\delta \in (0,R)$ and $h \in (0, \frac{\delta}{2})$. Then

$$B\Big(y + \tfrac{h}{R}(x-y), h\Big) \subset B(x,R) \cap B(y,\delta).$$

Therefore $p(X \setminus B(x,R), y) \geq 1$. Since $f_{\restriction \overline{B}(x,R)}$ is continuous, f is porouscontinuous (in each considered sense) at y. □

In the sequel we will consider X with several different topologies and $f : X \to \mathbb{R}$. Let τ be a topology on X (in particular $\mathcal{T}_{\|\ \|}$ is a topology generated by the norm $\|\ \|$). Then we will say that f is τ-continuous at $x \in X$ if it is continuous at x as $f : (X,\tau) \to (\mathbb{R}, \tau_N)$, where τ_N is the natural topology on $\mathbb{R}$. Thus τ-continuity of f at x means that for each $\varepsilon > 0$ there exists τ-open set U such that $x \in U$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. We will say that f is τ-continuous if it is τ-continuous at each point.

Denote $C_\tau = \{f : X \to \mathbb{R} : f \text{ is } \tau\text{-continuous}\}$, $C = \{f : X \to \mathbb{R} : f \text{ is } \mathcal{T}_{\|\ \|} - \text{continuous}\}$. Finally, for any $f : X \to \mathbb{R}$ let $N_f = \{x \in X : f(x) = 0\}$.

2. Maximal Additive Families for Porouscontinuous Functions

It is easily seen that result of addition and multiplication of functions from discussed classes of functions, in general, need not belong to these classes. Therefore we studied the following notion.

Definition 2.1. ([2]) Let $\mathcal{F}$ be a family of real functions defined on a normed space $(X, \|\ \|)$. A set $\mathfrak{M}_a(\mathcal{F}) = \{g : X \to \mathbb{R} : \forall_{f \in \mathcal{F}}\, (f + g \in \mathcal{F})\}$ is called the maximal additive class for $\mathcal{F}$.

Remark 2.2. Let $f : X \to \mathbb{R}$, $f(x) = 0$ for $x \in X$, be a constant function. Clearly, if $f \in \mathcal{F}$, then $\mathfrak{M}_a(\mathcal{F}) \subset \mathcal{F}$.

Lemma 2.3. *Let* $(X, \|\ \|)$ *be a normed space,* $x, x_0 \in X$*,* $x \neq x_0$ *and* $r \in (0, \|x - x_0\|)$*. Denote*

$$\mathcal{A} = \{(B(y,c), R) : B(y,c) \subset (B(x,r) \setminus \{x\}) \cap B(x_0,R)\}.$$

Then

$$\sup\left\{\frac{2c}{R} : \ (B(y,c),R) \in \mathcal{A}\right\} = \frac{r}{\|x - x_0\|}.$$

Proof. Denote $K = \sup\left\{\frac{2c}{R}\colon\ (B(y,c),R) \in \mathcal{A}\right\}$. For each ball $B(y,c)$, with $c < \|x_0 - y\|$, we obtain

$$\inf\{R > 0\colon B(y,c) \subset B(x_0,R)\} = \|x_0 - y\| + c.$$

Moreover, for each $(B(y,c),R) \in \mathcal{A}$ let $y_1 \in \left\{x + t\frac{x_0-x}{\|x_0-x\|}\colon t \in (0,r)\right\}$ be such that $\|y - x\| = \|y_1 - x\|$. Then $(B(y_1,c), \|y_1 - x_0\| + c) \in \mathcal{A}$ and $\|y_1 - x_0\| \le \|y - x_0\|$. Therefore

$$K = \sup\left\{\frac{2c}{\|y - x_0\| + c}\colon y \in \left\{x + t\frac{x_0 - x}{\|x_0 - x\|}\colon t \in (0,r)\right\}, B(y,c) \subset B(x,r) \setminus \{x\}\right\}.$$

Let us consider two cases.

1. If $t_0 \in \left(0, \frac{r}{2}\right)$ and $y_0 = x + t_0\frac{x_0-x}{\|x_0-x\|}$, then

$$\sup\{c > 0\colon B(y_0,c) \subset B(x,r) \setminus \{x\}\} = t_0.$$

   Hence,

$$\sup\left\{\frac{2c}{\|y_0 - x_0\| + c}\colon B(y_0,c) \subset B(x,r) \setminus \{x\}\right\} = \frac{2t_0}{\|y_0 - x_0\| + t_0} = \frac{2t_0}{\|x_0 - x\| - t_0 + t_0} = \frac{2t_0}{\|x_0 - x\|} \le \frac{r}{\|x - x_0\|}.$$

2. If $t_0 \in \left[\frac{r}{2}, r\right)$ and $y_0 = x + t_0\frac{x_0-x}{\|x_0-x\|}$, then

$$\sup\{c > 0\colon B(y_0,c) \subset B(x,r) \setminus \{x\}\} = r - t_0$$

   and

$$\sup\left\{\frac{2c}{\|y_0 - x_0\| + c}\colon B(y_0,c) \subset B(x,r) \setminus \{x\}\right\} = \frac{2(r - t_0)}{(\|x - x_0\| - t_0) + r - t_0} = \frac{2\,(r - t_0)}{\|x_0 - x\| - r + 2(r - t_0)} \le \frac{2\frac{r}{2}}{\|x_0 - x\| - r + 2\frac{r}{2}} = \frac{r}{\|x - x_0\|},$$

   because the function $f(x) = \frac{x}{a+x}$, where $a > 0$ is increasing on $[0,\infty)$.

On the other hand, if $t_0 = \frac{r}{2}$ and $y_0 = x + \frac{r}{2}\frac{x_0-x}{\|x_0-x\|}$, then $\frac{2\frac{r}{2}}{\|y_0-x_0\|+\frac{r}{2}} = \frac{r}{\|x_0-x\|}$. Finally, $K = \frac{r}{\|x-x_0\|}$. □

Theorem 2.4. *Let $(X, \|\,\|)$ be a normed space, $f\colon X \to \mathbb{R}$ and $x_0 \in X$. The following conditions are equivalent:*

(1) *f is continuous at x_0;*

(2) $\forall_{r\in(0,1)}\forall_{g\in\mathcal{M}_r}\ (x_0 \in \mathcal{M}_r(f + g))$;

(3) $\exists_{r\in(0,1)}\forall_{g\in\mathcal{M}_r}\ (x_0 \in \mathcal{M}_r(f + g))$;

(4) $\forall_{r\in(0,1)}\forall_{g\in\mathcal{S}_r}\ (x_0 \in \mathcal{S}_r(f + g))$;

(5) $\exists_{r\in(0,1)}\forall_{g\in\mathcal{S}_r}\ (x_0 \in \mathcal{S}_r(f + g))$;

(6) $\forall_{r\in(0,1)}\forall_{g\in\mathcal{P}_r}\ (x_0 \in \mathcal{P}_r(f + g))$;

(7) $\exists_{r\in(0,1)}\forall_{g\in\mathcal{P}_r}\ (x_0 \in \mathcal{P}_r(f + g))$.

Proof. Implications (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) are obvious.

(1) $\Rightarrow$ (2). Assume that f is continuous at x_0. Let $r \in (0,1)$, $g \in \mathcal{M}_r$ and $\varepsilon > 0$. Then there exists $E \subset X$ such that $x_0 \in E$, $g(E) \subset \left(g(x_0) - \frac{\varepsilon}{2}, g(x_0) + \frac{\varepsilon}{2}\right)$ and $p(X \setminus E, x_0) \geq r$. On the other hand, we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ for each $x \in B(x_0, \delta)$. Let $F = E \cap B(x_0, \delta)$. Then $\left|(f+g)(x) - (f+g)(x_0)\right| \leq \left|f(x) - f(x_0)\right| + \left|g(x) - g(x_0)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $x \in F$. Moreover, $p(X \setminus F, x_0) \geq r$. This means that $x_0 \in \mathcal{M}_r(f+g)$.

Proofs of implications (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (6) are very similar to the proof of (1) $\Rightarrow$ (2) and we omit them.

(3) $\Rightarrow$ (1). Assume that there exists $r \in (0,1)$ such that for each $g \in \mathcal{M}_r$ we have $x_0 \in \mathcal{M}_r(f+g)$. Suppose that f is not continuous at x_0. Then there exist $\varepsilon > 0$ and a sequence $(x_n)_{n\geq 1}$ convergent to x_0 such that $|f(x_n) - f(x_0)| \geq \varepsilon$ for each $n \geq 1$. Denote $R_n = \frac{r\|x_0 - x_n\|}{2-r}$ for each $n \geq 1$. Clearly, for each $n \geq 1$ we have $R_n < \|x_0 - x_n\|$ and $B(x_n, R_n) \subset B(x_0, \|x_0 - x_n\| + R_n)$, because if $y \in B(x_n, R_n)$, then

$$\|x_0 - y\| \leq \|x_0 - x_n\| + \|x_n - y\| < \|x_0 - x_n\| + R_n.$$

Without loss of generality we may assume that balls $\overline{B}(x_n, R_n)$ are pairwise disjoint. Moreover,

$$\frac{2R_n}{\|x_0 - x_n\| + R_n} = \frac{\frac{2r\|x_0 - x_n\|}{2-r}}{\|x_0 - x_n\| + \frac{r\|x_0 - x_n\|}{2-r}} = \frac{\frac{2r}{2-r}}{1 + \frac{r}{2-r}} = \frac{2r}{2-r+r} = r.$$

Therefore

$$p\left(X \setminus \bigcup_{n\geq 1} \overline{B}(x_n, R_n), x_0\right) \geq r. \tag{1}$$

Define $g\colon X \to \mathbb{R}$ letting

$$g(x) = \begin{cases} -f(x_0), & x \in \{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n), \\ -f(x) + \varepsilon, & x \in X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right). \end{cases}$$

Observe that g is continuous at $x \in \bigcup_{n\geq 1} B(x_n, R_n)$ and g is $\mathcal{M}_r$-continuous at $x \in X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right)$, because $X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right)$ is open. By (1) we conclude that g is $\mathcal{M}_r$-continuous at x_0. Applying Proposition 1.6, we obtain that g is $\mathcal{M}_r$-continuous at $x \in \bigcup_{n\geq 1} S(x_n, R_n)$. It follows that $g \in \mathcal{M}_r$. On the other hand,

$$\begin{aligned} &(f+g)(x_0) = 0, \\ &(f+g)(x) = \varepsilon \quad \text{for} \quad x \in X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right), \\ &\left|(f+g)(x_n)\right| = \left|f(x_n) - f(x_0)\right| \geq \varepsilon \quad \text{for} \quad n \geq 1. \end{aligned} \tag{2}$$

Take any $R_0 > 0$. Let $B(y, R) \subset B(x_0, R_0)$ be any open ball disjoint from $X \setminus \left\{x\colon \left|(f+g)(x) - (f+g)(x_0)\right| < \varepsilon\right\}$. By (2), we obtain

$$B(y, R) \subset \bigcup_{n\geq 1} \left(B(x_n, R_n) \setminus \{x_n\}\right).$$

Therefore there exists n_0 such that $B(y, R) \subset B(x_{n_0}, R_{n_0}) \setminus \{x_{n_0}\}$. By Lemma 2.3, we get

$$\frac{2R}{R_0} \leq \frac{R_{n_0}}{\|x_0 - x_{n_0}\|} = \frac{r\|x_0 - x_{n_0}\|}{(2-r)\|x_0 - x_{n_0}\|} = \frac{r}{2-r}.$$

Therefore,

$$p\left(X \setminus \left\{x\colon \left|(f+g)(x) - (f+g)(x_0)\right| < \varepsilon\right\}, x_0\right) \leq \frac{r}{2-r} < r.$$

This means that $x_0 \notin \mathcal{M}_r(f+g)$, a contradiction.

(5) ⇒ (1) and (7) ⇒ (1). Assume that there exists $r \in (0,1)$ such that for each $g \in \mathcal{S}_r$ ($g \in \mathcal{P}_r$, respectively) we have $x_0 \in \mathcal{S}_r(f+g)$ ($x_0 \in \mathcal{P}_r(f+g)$, respectively). Choose $r_1 \in (r,1)$ such that $\frac{r_1}{2-r_1} < r$. Suppose that f is not continuous at x_0. Then there exist $\varepsilon > 0$ and a sequence $(x_n)_{n\geq 1}$ convergent to x_0 such that $|f(x_n) - f(x_0)| \geq \varepsilon$ for each $n \geq 1$. Denote $R_n = \frac{r_1\|x_0 - x_n\|}{2-r_1}$ for each $n \geq 1$. Clearly, $R_n < \|x_0 - x_n\|$ for each $n \geq 1$. We may assume that balls $\overline{B}(x_n, R_n)$ are pairwise disjoint. Moreover, $B(x_n, R_n) \subset B(x_0, \|x_0 - x_n\| + R_n)$ and

$$\frac{2R_n}{\|x_0 - x_n\| + R_n} = \frac{\frac{2r_1\|x_0-x_n\|}{2-r_1}}{\|x_0 - x_n\| + \frac{r_1\|x_0-x_n\|}{2-r_1}} = \frac{\frac{2r_1}{2-r_1}}{1+\frac{r_1}{2-r_1}} = \frac{2r_1}{2-r_1+r_1} = r_1 > r$$

for every $n \geq 1$. Therefore

$$p\left(X \setminus \bigcup_{n\geq 1} \overline{B}(x_n, R_n), x_0\right) > r.$$

Define $g\colon X \to \mathbb{R}$ letting

$$g(x) = \begin{cases} -f(x_0), & x \in \{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n), \\ -f(x) + \varepsilon, & x \in X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right). \end{cases}$$

Repeating arguments from the previous part of the proof we can show that $g \in \mathcal{P}_r$ and $x_0 \notin \mathcal{S}_r(f+g)$, which is a contradiction. □

Corollary 2.5. *For every $r \in (0,1)$ we have*

$$\mathfrak{M}_a(\mathcal{M}_r) = C,$$
$$\mathfrak{M}_a(\mathcal{S}_r) = C,$$
$$\mathfrak{M}_a(\mathcal{P}_r) = C.$$

Remark 2.6. One can define in a natural way $\mathcal{M}_1$, $\mathcal{P}_0$ and $\mathcal{S}_0$-continuity. In [6] maximal additive class for these families are described in terms of so called s and p topology defined by Kelar and Zajíček in [4, 8].

3. Maximal Multiplicative Families for Porouscontinuous Functions

In this section we will describe maximal multiplicative classes for $\mathcal{S}_r$ and $\mathcal{M}_r$. It turns out that for this purpose we must define new topologies on a normed space X generated by porosity.

First, recall the definition of maximal multiplicative class for a family of functions.

Definition 3.1. ([2]) Let $\mathcal{F}$ be a family of real functions defined on a normed space $(X, \|\ \|)$. A set $\mathfrak{M}_m(\mathcal{F}) = \{g\colon X \to \mathbb{R} \colon \forall_{f\in\mathcal{F}}\ (f \cdot g \in \mathcal{F})\}$ is called the maximal multiplicative class for $\mathcal{F}$.

Remark 3.2. Let $f\colon X \to \mathbb{R}$, $f(x) = 1$ for $x \in X$, be a constant function. If $f \in \mathcal{F}$, then $\mathfrak{M}_m(\mathcal{F}) \subset \mathcal{F}$.

Remark 3.3. Let $(X, \|\ \|)$ be a normed space and $r \in (0,1)$. If $f\colon X \to \mathbb{R}$ is continuous at $x_0 \in X$ and $g\colon X \to \mathbb{R}$ is $\mathcal{M}_r$-continuous ($\mathcal{S}_r$-continuous) at x_0, then $f \cdot g$ is $\mathcal{M}_r$-continuous ($\mathcal{S}_r$-continuous) at x_0.

Example 3.4. Let $(X, \|\ \|)$ be a normed space. We construct $f\colon X \to \mathbb{R}$ such that f is not continuous and $f \in \mathfrak{M}_m(\mathcal{M}_r) \cap \mathfrak{M}_m(\mathcal{S}_r) \cap \mathfrak{M}_m(\mathcal{P}_r)$ for each $r \in (0,1)$.

Fix $y \in X$, $\|y\| = 1$ and let 0_X denote the zero of X. Then closed balls $\overline{B}\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)$, $n \geq 1$, are pairwise disjoint, $\lim_{n\to\infty} \frac{1}{(3n)!}y = 0_X$ and

$$p\left(X \setminus \bigcup_{n=1}^{\infty} \overline{B}\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right), 0_X\right) = \limsup_{n\to\infty} \frac{2\frac{3n}{(3n+1)!}}{\frac{1}{(3n)!} + \frac{3n}{(3n+1)!}} = \limsup_{n\to\infty} \frac{6n}{6n+1} = 1. \tag{3}$$

Define $f\colon X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in \{0_X\} \cup \bigcup_{n\geq 1} \overline{B}\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right), \\ 1, & x \in X \setminus \left(\{0_X\} \cup \bigcup_{n\geq 1} \overline{B}\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)\right). \end{cases}$$

Fix $r \in (0, 1)$ and $g \in \mathcal{M}_r$ (or $g \in \mathcal{S}_r$, $g \in \mathcal{P}_r$, respectively). If $x \notin \{0_X\} \cup \bigcup_{n\geq 1} S\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)$, then f is continuous at x and $f \cdot g$ is $\mathcal{M}_r$-continuous (or $\mathcal{S}_r$-continuous, $\mathcal{P}_r$-continuous, respectively) at x, because $X \setminus \left(\{0_X\} \cup \bigcup_{n\geq 1} S\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)\right)$ is open. Take any $x_0 \in \bigcup_{n\geq 1} S\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)$. Then $(f \cdot g)(x) = 0$ for $x \in \{0_X\} \cup \bigcup_{n\geq 1} \overline{B}\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)$ and $f \cdot g$ is $\mathcal{M}_r$-continuous (or $\mathcal{S}_r$-continuous, $\mathcal{P}_r$-continuous, respectively) at $x_0 \in \bigcup_{n\geq 1} S\left(\frac{1}{(3n)!}y, \frac{3n}{(3n+1)!}\right)$ by Proposition 1.6. Finally, by (3), $f \cdot g$ is $\mathcal{M}_r$-continuous (or $\mathcal{S}_r$-continuous, $\mathcal{P}_r$-continuous, respectively) at 0_X. Therefore $f \in \mathfrak{M}_m(\mathcal{M}_r) \cap \mathfrak{M}_m(\mathcal{S}_r) \cap \mathfrak{M}_m(\mathcal{P}_r)$.

Theorem 3.5. *Let $(X, \|\,\|)$ be a normed space, $r \in (0, 1)$ and $A \subset X$. The family of sets $U \subset X$ satisfying condition:*

$$\forall_{x\in U} \forall_{E\subset X,\, p(X\setminus E, x)\geq r}\ \left(p\left(X \setminus \left[(E \cap U) \cup A\right], x\right) \geq r\right)$$

forms a topology. We will denote it by $\mathcal{T}_r(A)$. Topology $\mathcal{T}_r(A)$ is stronger than the initial topology generated by the norm.

Proof. Obviously, $\emptyset \in \mathcal{T}_r(A)$. For each $x \in X$ and for each $E \subset X$ satisfying $p(X \setminus E, x) \geq r$ we obtain

$$p(X \setminus \left[(E \cap X) \cup A\right], x) = p(X \setminus (E \cup A), x) \geq p(X \setminus E, x) \geq r.$$

Thus $X \in \mathcal{T}_r(A)$.

Let $U \in \mathcal{T}_r(A)$ and $V \in \mathcal{T}_r(A)$. Fix $x \in U \cap V$. Take $E \subset X$ such that $p(X \setminus E, x) \geq r$. Then $p\left(X \setminus \left[(E \cap U) \cup A\right], x\right) \geq r$. Hence, $p\left(X \setminus \left[(E \cap U \cap V) \cup A\right], x\right) = p\left(X \setminus \left[\left[\left((E \cap U) \cup A\right) \cap V\right] \cup A\right], x\right) \geq r$, because $V \in \mathcal{T}_r(A)$. Thus $U \cap V \in \mathcal{T}_r(A)$.

Let $U_t \in \mathcal{T}_r(A)$ for each $t \in T$. Fix $x \in \bigcup_{t\in T} U_t$. There exists $t_0 \in T$ such that $x \in U_{t_0}$. Take $E \subset X$ such that $p(X \setminus E, x) \geq r$. Then $p\left(X \setminus \left[(E \cap U_{t_0}) \cup A\right], x\right) \geq r$ and $p\left(X \setminus \left[(E \cap \bigcup_{t\in T} U_t) \cup A\right], x\right) \geq p\left(X \setminus \left[(E \cap U_{t_0}) \cup A\right], x\right) \geq r$. Thus $\bigcup_{t\in T} U_t \in \mathcal{T}_r(A)$.

Hence $\mathcal{T}_r(A)$ is a topology in X. The remaining part of the proof is obvious. □

Theorem 3.6. *Let $(X, \|\,\|)$ be a normed space, $r \in (0, 1)$ and $A \subset X$. The family of sets $U \subset X$ satisfying condition:*

$$\forall_{x\in U} \forall_{E\subset X,\, p(X\setminus E, x)> r}\ \left(p\left(X \setminus \left[(E \cap U) \cup A\right], x\right) > r\right)$$

forms a topology. We will denote it by $\tau_r(A)$. Topology $\tau_r(A)$ is stronger than the initial topology generated by the norm.

Proof. The proof is very similar to the proof of the previous theorem and we omit it. □

Lemma 3.7. *Let $(X, \|\,\|)$ be a normed space, $r \in (0, 1)$, $x_0 \in X$ and $f\colon X \to \mathbb{R}$. If f is not continuous at x_0 and $f(x_0) \neq 0$, then there exists $g \in \mathcal{P}_r$ such that $x_0 \notin \mathcal{M}_r(f \cdot g)$.*

Proof. Fix $r_1 \in (r, 1)$ such that $\frac{r_1}{2-r_1} < r$. Assume that $f(x_0) \neq 0$ and f is not continuous at x_0. Then there exist $\varepsilon \in \left(0, \frac{|f(x_0)|}{2}\right)$ and a sequence $(x_n)_{n\geq 1}$ convergent to x_0 such that $|f(x_n) - f(x_0)| \geq \varepsilon$ for each $n \geq 1$. Let $R_n = \frac{r_1\|x_0 - x_n\|}{2-r_1}$ for each $n \geq 1$. Without loss of generality we may assume that balls $\overline{B}(x_n, R_n)$ are pairwise disjoint. Clearly, for every $n \geq 1$ we have $R_n < \|x_0 - x_n\|$ and $B(x_n, R_n) \subset B(x_0, \|x_0 - x_n\| + R_n)$, because if $y \in B(x_n, R_n)$, then

$$\|x_0 - y\| \leq \|x_0 - x_n\| + \|x_n - y\| < \|x_0 - x_n\| + R_n.$$

Moreover,

$$\frac{2R_n}{\|x_0 - x_n\| + R_n} = \frac{\frac{2r_1\|x_0-x_n\|}{2-r_1}}{\|x_0 - x_n\| + \frac{r_1\|x_0-x_n\|}{2-r_1}} = \frac{\frac{2r_1}{2-r_1}}{1 + \frac{r_1}{2-r_1}} = \frac{2r_1}{2 - r_1 + r_1} = r_1 > r.$$

Therefore

$$p\left(X \setminus \bigcup_{n\geq 1} \overline{B}(x_n, R_n), x_0\right) > r. \tag{4}$$

Define $g\colon X \to \mathbb{R}$ letting

$$g(x) = \begin{cases} 1, & x \in \{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n), \\ 0, & x \in X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right). \end{cases}$$

Observe that g is continuous at $x \in X \setminus (\{x_0\} \cup \bigcup_{n\geq 1} S(x_n, R_n))$ and g is $\mathcal{P}_r$-continuous at $x \in \bigcup_{n\geq 1} S(x_n, R_n)$, by Proposition 1.6. By (4) we conclude that g is $\mathcal{P}_r$-continuous at x_0. It follows that $g \in \mathcal{P}_r$.

On the other hand,

$$\begin{aligned} &(f \cdot g)(x_0) = f(x_0), \\ &(f \cdot g)(x) = 0 \quad \text{for} \quad x \in X \setminus \left(\{x_0\} \cup \bigcup_{n\geq 1} \overline{B}(x_n, R_n)\right), \\ &\left|(f \cdot g)(x_n) - (f \cdot g)(x_0)\right| = \left|f(x_n) - f(x_0)\right| \geq \varepsilon \quad \text{for} \quad n \geq 1. \end{aligned} \tag{5}$$

Take any $R_0 > 0$. Let $B(y, R) \subset B(x_0, R_0)$ be any open ball disjoint from $X \setminus \left\{x\colon \left|(f \cdot g)(x) - (f \cdot g)(x_0)\right| < \varepsilon\right\}$. By (5), we obtain

$$B(y, R) \subset \bigcup_{n\geq 1} \left(\overline{B}(x_n, R_n) \setminus \{x_n\}\right).$$

Therefore there exists n_0 such that $B(y, R) \subset \overline{B}(x_{n_0}, R_{n_0}) \setminus \{x_{n_0}\}$. By Lemma 2.3, we get

$$\frac{2R}{R_0} \leq \frac{R_{n_0}}{\|x_0 - x_{n_0}\|} = \frac{r_1\|x_0 - x_{n_0}\|}{(2 - r_1)\|x_0 - x_{n_0}\|} = \frac{r_1}{2 - r_1}.$$

Therefore,

$$p\left(X \setminus \left\{x\colon \left|(f \cdot g)(x) - (f \cdot g)(x_0)\right| < \varepsilon\right\}, x_0\right) \leq \frac{r_1}{2 - r_1} < r.$$

This means $x_0 \notin \mathcal{M}_r(f \cdot g)$. □

Theorem 3.8. *Let $(X, \|\ \|)$ be a normed space, $f\colon X \to \mathbb{R}$ and $r \in (0, 1)$. The following conditions are equivalent:*

(1) *$f \in \mathfrak{M}_m(\mathcal{M}_r)$;*

(2) *for every $x \in X$, if f is not continuous at x, then $f(x) = 0$ and f is $\mathcal{T}_r(N_f)$-continuous at x.*

Proof. Assume that condition (2) is fulfilled. Take $g \in \mathcal{M}_r$ and $x_0 \in X$. If f is continuous at x_0, then obviously $f \cdot g$ is $\mathcal{M}_r$-continuous at x_0. Assume that f is not continuous at x_0. Then, by assumptions, $f(x_0) = 0$ and f is $\mathcal{T}_r(N_f)$-continuous at x_0. Fix $\varepsilon > 0$. Since $x_0 \in \mathcal{M}_r(g)$, there exists $E \subset \{x \in X\colon |g(x) - g(x_0)| < 1\}$ such that $x_0 \in E$, $p(X \setminus E, x_0) \geq r$. By $\mathcal{T}_r(N_f)$-continuity of f at x_0, there is $F \in \mathcal{T}_r(N_f)$ such that $x_0 \in F$ and $|f(x) - f(x_0)| < \frac{\varepsilon}{|g(x_0)|+1}$ for each $x \in F$. Therefore $(f \cdot g)(x_0) = 0$ and $|(f \cdot g)(x)| < \frac{(|g(x_0)|+1)\varepsilon}{|g(x_0)|+1} = \varepsilon$ for each $x \in (E \cap F) \cup N_f$. Moreover, $p\left(X \setminus \left[(E \cap F) \cup N_f\right], x_0\right) \geq r$. Thus $f \cdot g$ is $\mathcal{M}_r$-continuous at x_0. Since x_0 and g were arbitrary, $f \in \mathfrak{M}_m(\mathcal{M}_r)$.

Now assume that $f \in \mathfrak{M}_m(\mathcal{M}_r)$. Take any x_0 and assume that f is not continuous at x_0. By Lemma 3.7, $f(x_0) = 0$. Aiming at a contradiction, suppose that f is not $\mathcal{T}_r(N_f)$-continuous at x_0. Then there exists $\varepsilon > 0$ such that $F = \{x \in X : |f(x) - f(x_0)| < \varepsilon\}$ does not contain any $\mathcal{T}_r(N_f)$-neighborhood of x_0. Hence $x_0 \notin \operatorname{int}_{\mathcal{T}_r(N_f)} F$. In particulary, $\{x_0\} \cup \operatorname{int} F \notin \mathcal{T}_r(N_f)$. Therefore we can find $E \subset X$ such that $p(X \setminus E, x_0) \geq r$ and $p\left(X \setminus \left[(E \cap \operatorname{int} F) \cup N_f\right], x_0\right) < r$. Observe that if $B(y, R) \subset (E \cap F) \cup N_f$, then $B(y, R) \subset F$ and $B(y, R) \subset (E \cap \operatorname{int} F) \cup N_f$, because $N_f \subset F$. Thus

$$p\left(X \setminus \left[(E \cap F) \cup N_f\right], x_0\right) < r.$$

Since $f \in \mathfrak{M}_m(\mathcal{M}_r)$, we have $f \in \mathcal{M}_r$, $x_0 \in \mathcal{M}_r(f)$ and $p(X \setminus F, x_0) \geq r$. Suppose that $x_0 \in \operatorname{int} E$. Then

$$p\left(X \setminus \left[(E \cap F) \cup N_f\right], x_0\right) = p\left(X \setminus (F \cup N_f), x_0\right) = p\left(X \setminus F, x_0\right) \geq r,$$

a contradiction. Thus $x_0 \notin \operatorname{int} E$. Since $p(X \setminus E, x_0) \geq r$, we have $x_0 \in \operatorname{cl}(\operatorname{int} E)$. By Theorem 1.2, there exists a sequence of pairwise disjoint closed balls $\left(\overline{B}(x_n, R_n)\right)_{n \geq 1}$ such that $\lim_{n \to \infty} x_n = x_0$, $\bigcup_{n \geq 1} \overline{B}(x_n, R_n) \subset E \setminus \{x_0\}$ and

$$p(X \setminus E, x_0) = p\left(X \setminus \bigcup_{n \geq 1} \overline{B}(x_n, R_n), x_0\right) \geq r. \tag{6}$$

Put $B = \bigcup_{n=1}^{\infty} \overline{B}(x_n, R_n)$. Then $\operatorname{cl} B = B \cup \{x_0\}$ and

$$p\left(X \setminus \left[(B \cap F) \cup N_f\right], x_0\right) < r.$$

Let $X_n = \overline{B}(x_0, \frac{1}{n}) \setminus \overline{B}(x_0, \frac{1}{n+1})$ for $n \geq 1$. For every n choose a discrete set $A_n \subset X_n \setminus (N_f \cup B)$ such that $X_n \setminus (N_f \cup B) \subset \bigcup_{a \in A_n} B(a, \frac{1}{(n+1)^2})$ and $\|a_1 - a_2\| \geq \frac{1}{(n+1)^2}$ for $a_1, a_2 \in A_n$, $a_1 \neq a_2$ (such a set exists by the Zorn Lemma). Let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is discrete, $A \cap (N_f \cup B) = \emptyset$ and $\operatorname{cl} A \subset A \cup \{x_0\}$. Define $\widetilde{g} : A \to [0, \infty)$ by $\widetilde{g}(a) = \frac{2\varepsilon}{|f(a)|}$ for $a \in A$. Clearly, $|(f \cdot \widetilde{g})(x)| = 2\varepsilon$ for $x \in A$ and $\widetilde{g}$ is continuous, because A is discrete. Since A is a closed subset of $X \setminus (\{x_0\} \cup B)$, by the Tietze Theorem, we can find continuous extension, $\overline{g} : X \setminus (\{x_0\} \cup B) \to [0, \infty)$, of $\widetilde{g}$. Finally, define $g : X \to \mathbb{R}$ letting

$$g(x) = \begin{cases} 1, & x \in \{x_0\} \cup B, \\ \overline{g}(x), & x \in X \setminus (\{x_0\} \cup B). \end{cases}$$

By construction, g is continuous on an open set $X \setminus (\{x_0\} \cup \bigcup_{n \geq 1} S(x_n, R_n))$. By Proposition 1.6, g is $\mathcal{M}_r$-continuous on $\bigcup_{n \geq 1} S(x_n, R_n)$. By (6), we conclude that g is $\mathcal{M}_r$-continuous at x_0. Hence $g \in \mathcal{M}_r$.

Moreover, $(f \cdot g)(x_0) = 0$ and $|(f \cdot g)(x)| > \varepsilon$ for each $x \in A$. We claim $x_0 \notin \mathcal{M}_r(f \cdot g)$, which will be a contradiction. We may assume that $x_0 \in \operatorname{cl}(\operatorname{int}\{x : |(f \cdot g)(x)| < \varepsilon\})$. By Theorem 1.2, we can take a sequence of pairwise disjoint open balls $(B(y_n, s_n))_{n \geq 1} \subset B(x_0, \frac{1}{4}) \setminus \{x_0\}$ disjoint from $X \setminus \{x : |(f \cdot g)(x)| < \varepsilon\}$ such that $\lim_{n \to \infty} y_n = x_0$ and $p(X \setminus \{x \in X : |(f \cdot g)(x)| < \varepsilon\}, x_0) = \lim_{n \to \infty} \frac{2 s_n}{\|x_0 - y_n\| + s_n} > 0$. Hence, $\lim_{n \to \infty} \frac{s_n}{32 \|x_0 - y_n\|^2} = \infty$. Therefore, without loss of generality, we may assume that $s_n > 32 \|x_0 - y_n\|^2$ for every $n \geq 1$. Observe that $B(y_n, s_n) \subset \{x : |(f \cdot g)(x)| < \varepsilon\}$. For every $n \geq 1$ we can find $k_n \geq 4$ such that $\|y_n - x_0\| \in (\frac{1}{k_n + 1}, \frac{1}{k_n}]$. Assume that $B(y_n, s_n) \not\subset B \cup N_f$ and take any $z \in B(y_n, s_n) \setminus (B \cup N_f)$. Then $s_n < \|y_n - x_0\| \leq \frac{1}{k_n}$ and $\|z - x_0\| < \frac{2}{k_n}$. By construction, there exists $a \in A_j$ for some $j \geq \frac{k_n}{2} - 1 \geq \frac{k_n}{2} - \frac{k_n}{4} = \frac{k_n}{4}$ such that $\|z - a\| < \frac{1}{j^2} \leq \frac{16}{(k_n)^2}$. Since $|(f \cdot g)(a)| > \varepsilon$, $a \notin B(y_n, s_n)$. Thus

$$\|z - y_n\| \geq \|y_n - a\| - \|a - z\| > s_n - \frac{16}{{k_n}^2} > s_n - \frac{32}{(k_n + 1)^2} > s_n - 32 \|y_n - x_0\|^2,$$

because $(k+1)^2 < 2k^2$ for $k \geq 4$. Hence $B(y_n, s_n - 32\|y_n - x_0\|^2) \subset B \cup N_f \subset E \cup N_f$. Moreover, $(f \cdot g)(x) = f(x)$ for $x \in B(y_n, s_n - 32\|y_n - x_0\|^2)$ and

$$B(y_n, s_n - 32\|y_n - x_0\|^2) \subset \{x \colon |f(x)| < \varepsilon\} = F.$$

Therefore $B(y_n, s_n - 32\|y_n - x_0\|^2) \subset (E \cap F) \cup N_f$. Since n was chosen arbitrary, $\bigcup_{n=1}^{\infty} B(y_n, s_n - 32\|y_n - x_0\|^2) \subset (E \cap F) \cup N_f$. Therefore,

$$\begin{aligned} p(X \setminus \{x \in X \colon |(f \cdot g)(x)| < \varepsilon\}, x_0) &= \lim_{n\to\infty} \frac{2s_n}{\|x_0 - y_n\| + s_n} \leq \\ &\leq \lim_{n\to\infty} \left(\frac{2(s_n - 32\|y_n - x_0\|^2)}{\|x_0 - y_n\| + s_n - 32\|y_n - x_0\|^2} + \frac{64\|y_n - x_0\|^2}{\|x_0 - y_n\| + s_n} \right) \leq \\ &\leq p(X \setminus [(E \cap F) \cup N_f], x_0) + \lim_{n\to\infty} 64\|x_0 - y_n\| \frac{\|x_0 - y_n\|}{\|x_0 - y_n\| + s_n} = \\ &= p(X \setminus [(E \cap F) \cup N_f], x_0) < r, \end{aligned}$$

a contradiction. This proves that f is $\mathcal{T}_r(N_f)$-continuous at x_0, which completes the proof. □

Theorem 3.9. *Let $(X, \|\ \|)$ be a normed space, $f \colon X \to \mathbb{R}$ and $r \in (0,1)$. The following conditions are equivalent:*

(1) *$f \in \mathfrak{M}_m(\mathcal{S}_r)$;*

(2) *for every $x \in X$, if f is not continuous at x, then $f(x) = 0$ and f is $\tau_r(N_f)$-continuous at x.*

Moreover, if $f \in \mathfrak{M}_m(\mathcal{P}_r)$, then for every $x \in X$, if f is not continuous at x, then $f(x) = 0$ and f is $\tau_r(N_f)$-continuous at x.

Proof. Proof of the implication (2) ⇒ (1) is very similar to the analogous proof of the previous theorem and we omit it.

Now assume that $f \in \mathfrak{M}_m(\mathcal{S}_r)$ (or $f \in \mathfrak{M}_m(\mathcal{P}_r)$, respectively). Take any x_0 and assume that f is not continuous at x_0. By Lemma 3.7, $f(x_0) = 0$. Suppose that f is not $\tau_r(N_f)$-continuous at x_0. Then there exists $\varepsilon > 0$ such that the set $F = \{x \in X \colon |f(x) - f(x_0)| < \varepsilon\}$ does not contain any $\tau_r(N_f)$-neighborhood of x_0. Therefore $x_0 \notin \operatorname{int}_{\tau_r(N_f)} F$. In particular, $\{x_0\} \cup \operatorname{int} F \notin \tau_r(N_f)$. Hence we can find $E \subset X$ for which $p(X \setminus E, x_0) > r$ and $p\left(X \setminus \left[(E \cap \operatorname{int} F) \cup N_f\right], x_0\right) \leq r$. Observe that if $B(y, R) \subset (E \cap F) \cup N_f$, then $B(y, R) \subset F$ and $B(y, R) \subset (E \cap \operatorname{int} F) \cup N_f$. Thus

$$p\left(X \setminus \left[(E \cap F) \cup N_f\right], x_0\right) \leq r.$$

Since $f \in \mathfrak{M}_m(\mathcal{S}_r)$, we have $f \in \mathcal{S}_r$, $x_0 \in \mathcal{S}_r(f)$ and $p(X \setminus F, x_0) > r$. Assume that $x_0 \in \operatorname{int} E$. Then

$$p\left(X \setminus \left[(E \cap F) \cup N_f\right], x_0\right) = p\left(X \setminus (F \cup N_f), x_0\right) = p\left(X \setminus F, x_0\right) > r,$$

a contradiction. Thus $x_0 \notin \operatorname{int} E$. Since $p(X \setminus E, x_0) > r$, we obtain $x_0 \in \operatorname{cl}(\operatorname{int} E)$. By Theorem 1.2, we can find a sequence of pairwise disjoint closed balls $\left(\overline{B}(x_n, R_n)\right)_{n \geq 1}$ such that $\lim_{n\to\infty} x_n = x_0$, $\overline{B}(x_n, R_n) \subset E \setminus \{x_0\}$ for all n and

$$p(X \setminus E, x_0) = p\left(X \setminus \bigcup_{n \geq 1} \overline{B}(x_n, R_n), x_0\right) > r. \tag{7}$$

Put $B = \bigcup_{n=1}^{\infty} \overline{B}(x_n, R_n)$. Then $\operatorname{cl} B = B \cup \{x_0\}$ and

$$p\left(X \setminus \left[(B \cap F) \cup N_f\right], x_0\right) \leq r.$$

Let $X_n = \overline{B}(x_0, \frac{1}{n}) \setminus \overline{B}(x_0, \frac{1}{n+1})$ for $n \geq 1$. For every n choose a discrete set $A_n \subset X_n \setminus (N_f \cup B)$ such that $X_n \setminus (N_f \cup B) \subset \bigcup_{a \in A_n} B(a, \frac{1}{(n+1)^2})$ and $\|a_1 - a_2\| \geq \frac{1}{(n+1)^2}$ for $a_1, a_2 \in A_n$, $a_1 \neq a_2$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is discrete, $A \cap (N_f \cup B) = \emptyset$ and $\operatorname{cl} A \subset A \cup \{x_0\}$. Define $\widetilde{g}\colon A \to [0, \infty)$ by $\widetilde{g}(a) = \frac{2\varepsilon}{|f(a)|}$ for $a \in A$. Clearly, $|(f \cdot \widetilde{g})(x)| = 2\varepsilon$ for $x \in A$ and $\widetilde{g}$ is continuous, because A is discrete. Since A is a closed subset of $X \setminus (\{x_0\} \cup B)$, by the Tietze Theorem, we can find continuous extension, $\overline{g}\colon X \setminus (\{x_0\} \cup B) \to [0, \infty)$, of $\widetilde{g}$. Finally, define $g\colon X \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1, & x \in \{x_0\} \cup B, \\ \overline{g}(x), & x \in X \setminus (\{x_0\} \cup B). \end{cases}$$

Applying similar arguments as in the proof of the previous theorem we can show that $g \in \mathcal{P}_r$ and $p(X \setminus \{x\colon |(f \cdot g)(x)| < \varepsilon\}, x_0) = p(X \setminus [(E \cap F) \cup N_f], x_0) \leq r$, a contradiction. This proves that f is $\tau_r(N_f)$-continuous at x_0, which completes the proof. □

Problem 3.10. *Does $\mathfrak{M}_m(\mathcal{P}_r)$ consist of functions f satisfying condition: for every $x \in X$, if f is not continuous at x, then $f(x) = 0$ and f is $\tau_r(N_f)$-continuous at x?*

Remark 3.11. In [6] maximal multiplicative class for $\mathcal{M}_1$ and $\mathcal{S}_0$ are described.

4. Properties of Topologies

We describe some properties of topologies $\tau_r(A)$ and $\mathcal{T}_r(A)$ for different $r \in (0, 1)$ and for different sets A. By A^d we denote the set of accumulation points of A in the topology generated by $\| \|$.

The following two propositions follow directly from definitions of $\mathcal{T}_r(A)$ and $\tau_r(A)$.

Proposition 4.1. *Let $(X, \| \|)$ be a normed space, $r \in (0, 1)$, $x_0 \in X$, $A \subset X$. If $x_0 \notin A^d$, then for each $U \subset X$ we obtain:*

- $x_0 \in \operatorname{int}_{\mathcal{T}_r(A)} U$ *if and only if* $x_0 \in \operatorname{int} U$;
- $x_0 \in \operatorname{int}_{\tau_r(A)} U$ *if and only if* $x_0 \in \operatorname{int} U$.

Proof. Since $\mathcal{T}_{\| \|} \subset \mathcal{T}_r(A)$, we obtain $\operatorname{int} U \subset \operatorname{int}_{\mathcal{T}_r(A)} U$ for every $U \subset X$. Hence, if $x_0 \in \operatorname{int} U$, then $x_0 \in \operatorname{int}_{\mathcal{T}_r(A)} U$.

Let $x_0 \notin \operatorname{int} U$ for some $U \subset X$. Then we can find a sequence $(x_n)_{n \geq 1}$ convergent to x_0 such that $x_n \notin U$ for each $n \geq 1$. Since $x_0 \notin A^d$, we may assume that $x_n \notin A$ for $n \geq 1$. Denote $R_n = \frac{r\|x_0 - x_n\|}{2-r}$ for each $n \geq 1$. Clearly, for each $n \geq 1$ we have $R_n < \|x_0 - x_n\|$ and $B(x_n, R_n) \subset B(x_0, \|x_0 - x_n\| + R_n)$, because if $y \in B(x_n, R_n)$, then

$$\|x_0 - y\| \leq \|x_0 - x_n\| + \|x_n - y\| < \|x_0 - x_n\| + R_n.$$

Without loss of generality we may assume that balls $\overline{B}(x_n, R_n)$ are pairwise disjoint. Moreover,

$$\frac{2R_n}{\|x_0 - x_n\| + R_n} = \frac{\frac{2r\|x_0 - x_n\|}{2-r}}{\|x_0 - x_n\| + \frac{r\|x_0 - x_n\|}{2-r}} = \frac{\frac{2r}{2-r}}{1 + \frac{r}{2-r}} = \frac{2r}{2 - r + r} = r.$$

Hence

$$p\left(X \setminus \bigcup_{n \geq 1} \overline{B}(x_n, R_n), x_0\right) \geq r.$$

Let $F = \bigcup_{n \geq 1} \overline{B}(x_n, R_n)$.

Assume that $x_0 \in \operatorname{int}_{\mathcal{T}_r(A)} U$ and let $E = \operatorname{int}_{\mathcal{T}_r(A)} U$. Then $x_0 \in E$ and $E \in \mathcal{T}_r(A)$. Since $p(X \setminus F, x_0) \geq r$, we obtain $p(X \setminus [(F \cap E) \cup A], x_0) \geq r$. But

$$p(X \setminus [(F \cap E) \cup A], x_0) = p(X \setminus (F \cap E), x_0) \leq p(X \setminus (F \setminus \{x_n\colon n \geq 1\}), x_0),$$

because $x_0 \notin A^d$.

Take any $R_0 > 0$. Let $B(y, R) \subset B(x_0, R_0)$ be any open ball disjoint from $X \setminus (F \setminus \{x_n : n \geq 1\})$. Then there exists n_0 such that $B(y, R) \subset B(x_{n_0}, R_{n_0}) \setminus \{x_{n_0}\}$. By Lemma 2.3, we get

$$\frac{2R}{R_0} \leq \frac{R_{n_0}}{\|x_0 - x_{n_0}\|} = \frac{r\|x_0 - x_{n_0}\|}{(2-r)\|x_0 - x_{n_0}\|} = \frac{r}{2-r}.$$

Therefore, $p(X \setminus [(F \cap E) \cup A], x_0) \leq \frac{r}{2-r} < r$, a contradiction, because $E \in \mathcal{T}_r(A)$. This completes the proof of the first property. The proof of the second one is analogous and we omit it. □

Proposition 4.2. *Let $(X, \|\ \|)$ be a normed space and $r \in (0, 1)$. If $A_1 \subset A_2 \subset X$, then*

- $\mathcal{T}_r(A_1) \subset \mathcal{T}_r(A_2)$;
- $\tau_r(A_1) \subset \tau_r(A_2)$.

Theorem 4.3. *Let $(X, \|\ \|)$ be a normed space, $A, B \subset X$ and $r \in (0, 1)$. If $B^d = \emptyset$, then*

- $\mathcal{T}_r(A) = \mathcal{T}_r(A \cup B) = \mathcal{T}_r(A \setminus B)$;
- $\tau_r(A) = \tau_r(A \cup B) = \tau_r(A \setminus B)$;
- *$\mathcal{T}_r(B)$ and $\tau_r(B)$ are the initial topology generated by the norm $\|\ \|$.*

Proof. It follows immediately from the fact that B is closed and discrete set. □

Theorem 4.4. *Let $(X, \|\ \|)$ be a normed space, $A_1, A_2 \subset X$ and $r \in (0, 1)$. If $A_1^d \setminus A_2^d \neq \emptyset$, then $\mathcal{T}_r(A_1) \not\subset \mathcal{T}_r(A_2)$ and $\tau_r(A_1) \not\subset \tau_r(A_2)$.*

Proof. Let $x_0 \in A_1^d \setminus A_2^d$. Then there exist a sequence $(x_n)_{n\geq 1}$ of elements of $A_1 \setminus \{x_0\}$ convergent to x_0 and $\delta > 0$ such that $B(x_0, \delta) \cap A_2 \subset \{x_0\}$ and $x_n \in B(x_0, \delta)$ for each $n \geq 1$. Put $U = X \setminus \bigcup_{n=1}^{\infty} \{x_n\}$. Obviously, $U \in \mathcal{T}_r(A_1)$. We shall show that $U \notin \mathcal{T}_r(A_2)$. Let $R_n = \frac{1}{2}\left(\frac{r\|x_0 - x_n\|}{2-r} + r\|x_0 - x_n\|\right)$ for $n \geq 1$. Observe that $\frac{r\|x_0 - x_n\|}{2-r} < R_n < r\|x_0 - x_n\|$ for each $n \geq 1$. Taking, if it is necessary, a subsequence of $(x_n)_{n\geq 1}$ we may assume that balls $B(x_n, R_n)$ are pairwise disjoint. Denote $E = \bigcup_{n=1}^{\infty} B(x_n, R_n)$. Then

$$p(X \setminus E, x_0) = \limsup_{n\to\infty} \frac{2R_n}{\|x_0 - x_n\| + R_n} \geq \limsup_{n\to\infty} \frac{2\frac{r\|x_0 - x_n\|}{2-r}}{\|x_0 - x_n\| + \frac{r\|x_0 - x_n\|}{2-r}} = r,$$

because for every $a > 0$ the function $f(x) = \frac{2x}{a+x}$ is increasing on $[0, \infty)$. Moreover,

$$B(x_0, \delta) \cap [(E \cap U) \cup A_2] \subset \{x_0\} \cup (E \setminus \{x_n : n \geq 1\}) = \{x_0\} \cup \bigcup_{n=1}^{\infty} (B(x_n, R_n) \setminus \{x_n\}).$$

Let $R \in (0, \delta)$. Take any ball $B(y, \gamma)$ included in $B(x_0, R)$ and disjoint with $X \setminus [(E \cap U) \cup A_2]$. Then $B(y, \gamma) \subset \bigcup_{n=1}^{\infty} (B(x_n, R_n) \setminus \{x_n\})$. By Lemma 2.3, $\frac{2\gamma}{R} \leq \frac{R_n}{\|x_0 - x_n\|}$ for each $n \geq 1$. Therefore

$$p(X \setminus [(E \cap U) \cup A_2], x_0) \leq \limsup_{n\to\infty} \frac{R_n}{\|x_0 - x_n\|} = \frac{1}{2}\left(\frac{r}{2-r} + r\right) < r.$$

It follows $x_0 \notin \operatorname{int}_{\mathcal{T}_r(A_2)} U$ and $U \notin \mathcal{T}_r(A_2)$.

In a similar way we can can proof the second statement. □

Corollary 4.5. *Let $(X, \|\ \|)$ be a normed space, $r \in (0, 1)$, $A_1 \subset A_2 \subset X$ and $A_2^d \setminus A_1^d \neq \emptyset$. Then $\mathcal{T}_r(A_1) \subsetneq \mathcal{T}_r(A_2)$ and $\tau_r(A_1) \subsetneq \tau_r(A_2)$.*

Example 4.6. Let $X = \mathbb{R}$ with the standard norm $\|x\| = |x|$, $A_1 = \left\{\frac{1}{n} : n \geq 1\right\}$, $A_2 = \left\{-\frac{1}{n} : n \geq 1\right\}$ and $r \in (0,1)$. Then $A_1^d = A_2^d = \{0\}$, $\mathbb{R}\setminus A_2 \in \big(\mathcal{T}_r(A_2)\cap\tau_r(A_2)\big)\setminus\big(\mathcal{T}_r(A_1)\cup\tau_r(A_1)\big)$ and $\mathbb{R}\setminus A_1 \in \big(\mathcal{T}_r(A_1)\cap\tau_r(A_1)\big)\setminus\big(\mathcal{T}_r(A_2)\cup\tau_r(A_2)\big)$.

Theorem 4.7. *Let $(X, \|\ \|)$ be a normed space, $r \in (0,1)$, $A \subset X$, $x_0 \in X$. The following conditions are equivalent:*

(1) $\{x_0\} \in \mathcal{T}_r(A)$;

(2) $p(X \setminus A, x_0) \geq r$.

Proof. Assume that $p(X \setminus A, x_0) \geq r$. Take any $E \subset X$ such that $p(X \setminus E, x_0) \geq r$. Then

$$p\left(X \setminus [(E \cap \{x_0\}) \cup A], x_0\right) \geq p(X \setminus A, x_0) \geq r.$$

Thus $\{x_0\} \in \mathcal{T}_r(A)$.

Now assume that $\{x_0\} \in \mathcal{T}_r(A)$. By Proposition 4.1, we conclude that $x_0 \in A^d$. If $x_0 \in \operatorname{int}(A \cup \{x_0\})$, then the inequality $p(X \setminus A, x_0) \geq 1 > r$ is obvious. Suppose $x_0 \notin \operatorname{int}(A \cup \{x_0\})$. Put $E = B(x_0, 1)$. Then $p(X \setminus E, x_0) = 2 > r$ and

$$r \leq p\left(X \setminus [(E \cap \{x_0\}) \cup A], x_0\right) = p(X \setminus (\{x_0\} \cup A), x_0) = p(X \setminus A, x_0),$$

which completes the proof. (The last equality follows from the fact that $\operatorname{int} A = \operatorname{int}(\{x_0\} \cup A)$ and $p(X \setminus (\{x_0\} \cup A), x_0) = p(\operatorname{cl}(X \setminus (\{x_0\} \cup A)), x_0) = p(X \setminus \operatorname{int}(\{x_0\} \cup A), x_0) = p(X \setminus \operatorname{int} A, x_0) = p(X \setminus A, x_0)$.) □

In a similar way we can proof the following theorem.

Theorem 4.8. *Let $(X, \|\ \|)$ be a normed space, $r \in (0,1)$, $A \subset X$, $x_0 \in X$. The following conditions are equivalent:*

(1) $\{x_0\} \in \tau_r(A)$;

(2) $p(X \setminus A, x_0) > r$.

Corollary 4.9. *Let $(X, \|\ \|)$ be a normed space, $r \in (0,1)$, $A \subset X$, $U \subset X$. Then*

- $\operatorname{int}_{\mathcal{T}_r(A)} U \supset \operatorname{int} U \cup \{x \in X : p(X \setminus A, x) \geq r\}$;
- $\operatorname{int}_{\tau_r(A)} U \supset \operatorname{int} U \cup \{x \in X : p(X \setminus A, x) > r\}$.

Theorem 4.10. *Let $(X, \|\ \|)$ be a normed space, $0 < r_1 < r_2 < 1$. There exists $A \subset X$ such that $\mathcal{T}_{r_1}(A) \not\subset \mathcal{T}_{r_2}(A)$ and $\tau_{r_1}(A) \not\subset \tau_{r_2}(A)$.*

Proof. Put $r = \frac{r_1+r_2}{2}$ and $x_0 \in X$. Let $(B(x_n, R_n))_{n\geq 1}$ be a sequence of pairwise disjoint open balls such that $x_0 \notin \bigcup_{n=1}^{\infty} B(x_n, R_n)$, $p\left(X \setminus \bigcup_{n=1}^{\infty} B(x_n, R_n), x_0\right) = r$, $\lim_{n\to\infty} x_n = x_0$ and

$$\lim_{n\to\infty} \frac{n\|x_{n+1} - x_0\|}{\|x_n - x_0\|} = 0. \tag{8}$$

Denote $A = \bigcup_{n=1}^{\infty} B(x_n, R_n)$ and $U = \{x_0\} \cup A$. By Theorem 4.7 and Theorem 4.8, $U \in \mathcal{T}_{r_1}(A) \cap \tau_{r_1}(A)$. From (8) it follows that there exists sequence $(B(y_n, \alpha_n))_{n\geq 1}$ of pairwise disjoint open balls with properties: $\lim_{n\to\infty} y_n = x_0$,

$$\bigcup_{n=1}^{\infty} B(y_n, \alpha_n) \cap A = \emptyset,$$

$$p\left(X \setminus \bigcup_{n=1}^{\infty} B(y_n, \alpha_n), x_0\right) = 1.$$

Denote $E = \bigcup_{n=1}^{\infty} B(y_n, \alpha_n)$. Then

$$p\left(X \setminus [(E \cap U) \cup A], x_0\right) = p\left(X \setminus A, x_0\right) = r < r_2.$$

This means that $U \notin \mathcal{T}_{r_2}(A) \cup \tau_{r_2}(A)$. □

Theorem 4.11. *Let $(X, \| \|)$ be a normed space, $0 < r_1 < r_2 < 1$. There exists $A \subset X$ such that $\mathcal{T}_{\| \|} \subsetneq \mathcal{T}_{r_1}(A) = \mathcal{T}_{r_2}(A)$ and $\mathcal{T}_{\| \|} \subsetneq \tau_{r_1}(A) = \tau_{r_2}(A)$.*

Proof. Fix $x_0 \in X$ and let $(B(x_n, R_n))_{n\geq 1}$ be a sequence of pairwise disjoint open balls such that $\lim_{n\to\infty} x_n = x_0$, $x_0 \notin \bigcup_{n=1}^{\infty} B(x_n, R_n)$ and $p\left(X \setminus \bigcup_{n=1}^{\infty} B(x_n, R_n), x_0\right) = 1$. Denote $A = \bigcup_{n=1}^{\infty} B(x_n, R_n)$. Then, from Corollary 4.9, we obtain that

$$\{x_0\} \cup A \in (\mathcal{T}_{r_1}(A) \cap \mathcal{T}_{r_2}(A)) \setminus \mathcal{T}_{\| \|}$$

and

$$\{x_0\} \cup A \in (\tau_{r_1}(A) \cap \tau_{r_2}(A)) \setminus \mathcal{T}_{\| \|}.$$

Now, it is sufficient to show that for each $U \subset X$ and for each $x \in U$ we have

$$x \in \operatorname{int}_{\mathcal{T}_{r_1}(A)} U \Leftrightarrow x \in \operatorname{int}_{\mathcal{T}_{r_2}(A)} U$$

and

$$x \in \operatorname{int}_{\tau_{r_1}(A)} U \Leftrightarrow x \in \operatorname{int}_{\tau_{r_2}(A)} U.$$

Take $U \subset X$ and $x \in U$. We will consider the following two cases.

- $x \notin \{x_0\} \cup \bigcup_{n=1}^{\infty} \overline{B}(x_n, R_n)$. By Proposition 4.1 we get

$$x \in \operatorname{int}_{\mathcal{T}_{r_1}(A)} U \Leftrightarrow x \in \operatorname{int} U \Leftrightarrow x \in \operatorname{int}_{\mathcal{T}_{r_2}(A)} U.$$

- $x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} \overline{B}(x_n, R_n)$. Then $p(X \setminus A, x) = 1$. Applying Theorem 4.7 and Theorem 4.8, we obtain that $\{x\} \in \mathcal{T}_{r_1}(A) \cap \mathcal{T}_{r_2}(A) \cap \tau_{r_1}(A) \cap \tau_{r_2}(A)$.

Finally, $\mathcal{T}_{r_1}(A) = \mathcal{T}_{r_2}(A)$ and $\tau_{r_1}(A) = \tau_{r_2}(A)$. □

Hereafter we will consider the case of $X = \mathbb{R}$. We will need simple technical lemma from [7].

Lemma 4.12. ([7]) *Let $x_0 < a < b < c$, $\alpha = \frac{b-a}{b-x_0}$, $\beta = \frac{c-b}{c-x_0}$ and $\gamma = \frac{c-a}{c-x_0}$. Then $\gamma = \alpha + \beta - \alpha\beta = 1 - (1-\alpha)(1-\beta)$. In particular, $\gamma < \alpha + \beta$.*

Example 4.13. For each $r \in (0, 1)$ there exist $A, U \subset \mathbb{R}$ such that $U \in \mathcal{T}_r(A) \cap \tau_r(A)$ and $U \not\subset \operatorname{int} U \cup \{x \in U : p(\mathbb{R} \setminus A, x) \geq r\}$.

Fix $r \in (0, 1)$. Choose $c \in (0, 1)$ satisfying condition $(1 - cr)\sqrt{1 - cr} < 1 - r$. By induction, we can construct two sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ of positive numbers, both tending to 0, such that

$$0 < \ldots < b_{n+1} < a_n < b_n < \ldots,$$
$$\frac{b_n - a_n}{b_n} = \frac{a_n - b_{n+1}}{a_n} = cr \text{ for each } n \geq 1.$$

Denote $A = \bigcup_{n=1}^{\infty} (b_{n+1}, a_n)$. Then

$$p(\mathbb{R} \setminus A, 0) = p^+(\mathbb{R} \setminus A, 0) = \lim_{n\to\infty} \frac{a_n - b_{n+1}}{a_n} = cr < r.$$

For each $n \geq 1$ we can choose $c_n \in (a_n, b_n)$ such that $\frac{b_n - c_n}{b_n} = \frac{c_n - a_n}{c_n}$. Let $r_n = \frac{b_n - c_n}{b_n} = \frac{c_n - a_n}{c_n}$. Then, by Lemma 4.12, $(1 - r_n)^2 = 1 - rc$, so $r_n = 1 - \sqrt{1 - rc}$ for each $n \geq 1$. In particular, $r_n = r_{n+1}$ for each $n \geq 1$. Let $u_n = \frac{a_{n-1} - c_n}{a_{n-1}}$, $v_n = \frac{c_n - b_{n+1}}{c_n}$ for each $n \geq 2$. By Lemma 4.12, we obtain equality

$$1 - u_n = (1 - cr)(1 - r_n) = 1 - v_n$$

for each $n \geq 2$. Thus $u_n = v_n$ and $u_n = u_{n+1}$ for each $n \geq 2$. Moreover,

$$1 - \frac{a_{n-1} - b_{n+1}}{a_{n-1}} = (1 - cr)^3 = [(1 - cr)(1 - r_n)]^2$$

for each $n \geq 2$. Hence

$$1 - u_n = (1 - cr)(1 - r_n) = (1 - cr)^{\frac{3}{2}} < 1 - r,$$

so $u_n > r$. Denote $U = \mathbb{R} \setminus \bigcup_{n=1}^{\infty}\{c_n\}$. Obviously,

$$U \not\subset \operatorname{int} U \cup \{x \in U\colon p(\mathbb{R} \setminus A, x) \geq r\},$$

because $0 \notin \operatorname{int} U$ and $0 \notin \{x \in U\colon p(\mathbb{R} \setminus A, x) \geq r\}$.

We shall show $U \in \mathcal{T}_r(A) \cap \tau_r(A)$. From Corollary 4.9, we obtain $U \setminus \{0\} \subset \operatorname{int}_{\mathcal{T}_r(A)} U \cap \operatorname{int}_{\tau_r(A)} U$. Take $E \subset \mathbb{R}$ with property $p(\mathbb{R} \setminus E, 0) > r$ (or $p(\mathbb{R} \setminus E, 0) \geq r$, respectively). We will consider the following two cases.

I) $p^-(\mathbb{R} \setminus E, 0) = p(\mathbb{R} \setminus E, 0)$. Then $E \cap (-\infty, 0) \supset [(E \cap U) \cup A] \cap (-\infty, 0)$. Hence

$$p\left(\mathbb{R} \setminus [(E \cap U) \cup A], 0\right) \geq p(\mathbb{R} \setminus E, 0). \tag{9}$$

II) $p^+(\mathbb{R} \setminus E, 0) = p(\mathbb{R} \setminus E, 0)$. Then we can find a sequence of pairwise disjoint intervals $([\alpha_m, \beta_m])_{m\geq1}$ such that $\lim_{m\to\infty} \alpha_m = 0$, $0 < \ldots < \beta_{m+1} < \alpha_m < \beta_m < \ldots$, $[\alpha_m, \beta_m] \cap (\mathbb{R} \setminus E) = \emptyset$ for each $m \geq 1$ and $\lim_{m\to\infty} \frac{\beta_m-\alpha_m}{\beta_m} = p(\mathbb{R} \setminus E, 0) \geq r > cr$. Without loss of generality we may assume that $\frac{\beta_m-\alpha_m}{\beta_m} > cr$ for each $m \geq 1$ and $\alpha_1 < a_1$. Fix $m \in \mathbb{N}$. If $(\alpha_m, \beta_m) \cap \bigcup_{n=1}^{\infty}\{c_n\} = \emptyset$, then

$$(\alpha_m, \beta_m) \cap (\mathbb{R} \setminus [(E \cap U) \cup A]) = \emptyset.$$

So, assume that there exists n_0 such that $c_{n_0} \in (\alpha_m, \beta_m)$. Since $\frac{\beta_m-\alpha_m}{\beta_m} > cr = \frac{b_{n_0}-a_{n_0}}{b_{n_0}}$, we obtain $(\alpha_m, \beta_m) \not\subset (a_{n_0}, b_{n_0})$. Thus $a_{n_0} \in (\alpha_m, \beta_m)$ or $b_{n_0} \in (\alpha_m, \beta_m)$.

Again, we have two possibilities. If $a_{n_0} \in (\alpha_m, \beta_m)$, then $(b_{n_0+1}, c_{n_0}) \cap (\mathbb{R} \setminus [(E \cap U) \cup A]) = \emptyset$ and

$$\frac{c_{n_0} - b_{n_0+1}}{c_{n_0}} = v_n = u_n > r.$$

In the second case, where $b_{n_0} \in (\alpha_m, \beta_m)$, we have $(c_{n_0}, a_{n_0-1}) \cap (\mathbb{R} \setminus [(E \cap U) \cup A]) = \emptyset$ and

$$\frac{a_{n_0-1} - c_{n_0}}{a_{n_0-1}} = u_n > r.$$

For every $m \geq 1$ define an interval (α'_m, β'_m) by

$$(\alpha'_m, \beta'_m) = \begin{cases} (\alpha_m, \beta_m) & \text{if} \quad (\alpha_m, \beta_m) \cap \bigcup_{n=1}^{\infty}\{c_n\} = \emptyset, \\ (b_{n_0+1}, c_{n_0}) & \text{if} \quad a_{n_0}, c_{n_0} \in [\alpha_m, \beta_m], \\ (c_{n_0}, a_{n_0-1}) & \text{if} \quad b_{n_0}, c_{n_0} \in [\alpha_m, \beta_m]. \end{cases}$$

Then $(\alpha'_m, \beta'_m) \cap (\mathbb{R} \setminus [(E \cap U) \cup A]) = \emptyset$ and

$$\frac{\beta'_m - \alpha'_m}{\beta'_m} \geq \min\left\{\frac{\beta_m - \alpha_m}{\beta_m}, u_n\right\}.$$

By definition of $((\alpha_m, \beta_m))_{m\geq1}$, we conclude that

$$p^+\left(\mathbb{R} \setminus [(E \cap U) \cup A], 0\right) \geq \min\left\{p^+(\mathbb{R} \setminus E, 0), u_n\right\}. \tag{10}$$

Combining (9) and (10), we obtain

$$p\left(\mathbb{R} \setminus [(E \cap U) \cup A], 0\right) > r, \text{ if } p(\mathbb{R} \setminus E, 0) > r,$$
$$p\left(\mathbb{R} \setminus [(E \cap U) \cup A], 0\right) \geq r, \text{ if } p(\mathbb{R} \setminus E, 0) \geq r.$$

Hence $0 \in \operatorname{int}_{\mathcal{T}_r(A)} U \cap \operatorname{int}_{\tau_r(A)} U$ and finally $U \in \mathcal{T}_r(A) \cap \tau_r(A)$.

Theorem 4.14. *Let $r_1 \in (0,1)$. For each $r_2 \in \left(r_1, 1-(1-r_1)^{\frac{3}{2}}\right)$ there exists $A \subset \mathbb{R}$ such that $(\mathcal{T}_{r_2}(A) \cap \tau_{r_2}(A)) \not\subset (\mathcal{T}_{r_1}(A) \cup \tau_{r_1}(A))$.*

Proof. Fix $r_2 \in \left(r_1, 1-(1-r_1)^{\frac{3}{2}}\right)$. Take any $R_1 \in (r_1, r_2)$. Then $r_1 < R_1 < r_2 < 1-(1-r_1)^{\frac{3}{2}} < 1-(1-r_1)\sqrt{1-R_1}$. Next, choose $R_2, R_3 > 0$ such that $R_2 \in \left(r_2, 1-(1-r_1)\sqrt{1-R_1}\right)$ and $(1-R_3)^2(1-R_1) = (1-R_2)^2$. Obviously, $R_3 < R_2$. By induction, we can construct sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ all tending to 0 and satisfying conditions:

- $0 < \ldots < c_{n+1} < a_n < b_n < c_n < \ldots$,
- $\frac{c_n - b_n}{c_n} = \frac{a_n - c_{n+1}}{a_n} = R_3$,
- $\frac{b_n - a_n}{b_n} = R_1$

for each $n \geq 1$. Next, for each $n \geq 1$ we choose a point $d_n \in (a_n, b_n)$ such that $\frac{c_n - d_n}{c_n} = R_2$. Therefore by Lemma 4.12, we obtain

$$\left(1 - \frac{d_n - c_{n+1}}{d_n}\right)(1-R_2) = (1-R_3)^2(1-R_1) = (1-R_2)^2.$$

Hence $\frac{d_n - c_{n+1}}{d_n} = R_2$ for each $n \geq 1$. Put $A = \bigcup_{n=1}^{\infty}\left((c_{n+1}, a_n) \cup (b_n, c_n)\right)$, $U = \mathbb{R} \setminus \bigcup_{n=1}^{\infty}\{d_n\}$ and $E = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then

$$p(\mathbb{R} \setminus E, 0) = \limsup_{n\to\infty} \frac{b_n - a_n}{b_n} = R_1 > r_1.$$

Moreover,

$$(E \cap U) \cup A = \bigcup_{n=1}^{\infty}\left((c_{n+1}, a_n) \cup (a_n, d_n) \cup (d_n, b_n) \cup (b_n, c_n)\right).$$

Let $(\alpha, \beta) \subset (0, a_1)$ be any open interval disjoint from $\mathbb{R} \setminus [(E \cap U) \cup A]$. Therefore we can find n_0 such that $(\alpha, \beta) \subset (c_{n_0+1}, a_{n_0}) \cup (a_{n_0}, d_{n_0}) \cup (d_{n_0}, b_{n_0}) \cup (b_{n_0}, c_{n_0})$. Now, we have the following three cases.

1. If $(\alpha, \beta) \subset (c_{n_0+1}, a_{n_0}) \cup (b_{n_0}, c_{n_0})$, then

$$\frac{\beta - \alpha}{\beta} \leq \max\left\{\frac{c_{n_0} - b_{n_0}}{c_{n_0}}, \frac{a_{n_0} - c_{n_0+1}}{a_{n_0}}\right\} = R_3.$$

2. If $(\alpha, \beta) \subset (a_{n_0}, d_{n_0})$, then

$$\frac{\beta - \alpha}{\beta} \leq \frac{d_{n_0} - a_{n_0}}{d_{n_0}}.$$

   Moreover, by Lemma 4.12,

$$\left(1 - \frac{d_{n_0} - a_{n_0}}{d_{n_0}}\right)(1-R_3) = 1 - R_2.$$

   Hence

$$1 - \frac{d_{n_0} - a_{n_0}}{d_{n_0}} = \frac{1-R_2}{1-R_3} = \sqrt{1-R_1}.$$

   Finally, $\frac{d_{n_0} - a_{n_0}}{d_{n_0}} = 1 - \sqrt{1-R_1}$ and $\frac{\beta-\alpha}{\beta} \leq 1 - \sqrt{1-R_1}$.

3. If $(\alpha, \beta) \subset (d_{n_0}, b_{n_0})$, then

$$\frac{\beta - \alpha}{\beta} \leq \frac{b_{n_0} - d_{n_0}}{b_{n_0}}$$

   and $\left(1 - \frac{b_{n_0} - d_{n_0}}{b_{n_0}}\right)(1-R_3) = 1 - R_2$. Therefore $\frac{b_{n_0} - d_{n_0}}{b_{n_0}} = 1 - \sqrt{1-R_1}$ and $\frac{\beta-\alpha}{\beta} \leq 1 - \sqrt{1-R_1}$.

Since (α, β) was arbitrary,

$$p(\mathbb{R} \setminus [(E \cap U) \cup A], 0) \leq \max\left\{R_3, 1 - \sqrt{1 - R_1}\right\}.$$

Obviously, $1 - \sqrt{1 - R_1} < 1 - (1 - r_1)^{\frac{3}{4}} < r_1$ and $R_3 = 1 - \frac{1-R_2}{\sqrt{1-R_1}} < r_1$. Thus $U \notin \mathcal{T}_{r_1}(A) \cup \tau_{r_1}(A)$.

Now, we shall show that $U \in \mathcal{T}_{r_2}(A) \cap \tau_{r_2}(A)$. Obviously, $U \setminus \{0\} \subset \mathrm{int}_{\mathcal{T}_{r_2}(A)} U \cap \mathrm{int}_{\tau_{r_2}(A)} U$. Let $F \subset \mathbb{R}$ be such that $p(\mathbb{R} \setminus F, 0) \geq r_2$ (or $p(\mathbb{R} \setminus F, 0) > r_2$, respectively).

If $p^-(\mathbb{R} \setminus F, 0) = p(\mathbb{R} \setminus F, 0)$, then

$$p(\mathbb{R} \setminus [(F \cap U) \cup A], 0) \geq p^-(\mathbb{R} \setminus F, 0). \tag{11}$$

Therefore, assume that $p^+(\mathbb{R} \setminus F, 0) = p(\mathbb{R} \setminus F, 0)$. Then we can find a sequence of open intervals $\left((\alpha_m, \beta_m)\right)_{m \geq 1}$ such that $(\alpha_m, \beta_m) \subset F$, $0 < \ldots < \beta_{m+1} < \alpha_m < \beta_m < \ldots$ for each $m \geq 1$, $\lim_{n\to\infty} \alpha_m = 0$ and $p(\mathbb{R} \setminus F, 0) = \lim_{m\to\infty} \frac{\beta_m - \alpha_m}{\beta_m} \geq r_2 > R_1$. Without loss of generality, we may assume that $\frac{\beta_m - \alpha_m}{\beta_m} > R_1$ for each $m \geq 1$ and $\alpha_1 < a_1$. Fix $m \in \mathbb{N}$. If $(\alpha_m, \beta_m) \cap \bigcup_{n=1}^{\infty}\{d_n\} = \emptyset$, then

$$(\alpha_m, \beta_m) \cap (\mathbb{R} \setminus [(F \cap U) \cup A]) = \emptyset.$$

So, consider the case, where there exists n_0 such that $d_{n_0} \in (\alpha_m, \beta_m)$. Since $\frac{\beta_m - \alpha_m}{\beta_m} > R_1 = \frac{b_{n_0} - a_{n_0}}{b_{n_0}}$, we obtain $(\alpha_m, \beta_m) \not\subset (a_{n_0}, b_{n_0})$. Thus $a_{n_0} \in (\alpha_m, \beta_m)$ or $b_{n_0} \in (\alpha_m, \beta_m)$.

- If $a_{n_0} \in (\alpha_m, \beta_m)$, then $(c_{n_0+1}, d_{n_0}) \cap (\mathbb{R} \setminus [(F \cap U) \cup A]) = \emptyset$ and

$$\frac{d_{n_0} - c_{n_0+1}}{d_{n_0}} = R_2 > r_2.$$

- If $b_{n_0} \in (\alpha_m, \beta_m)$, then $(d_{n_0}, c_{n_0}) \cap (\mathbb{R} \setminus [(F \cap U) \cup A]) = \emptyset$ and

$$\frac{c_{n_0} - d_{n_0}}{c_{n_0}} = R_2 > r_2.$$

For every $m \geq 1$ define (α'_m, β'_m) by

$$(\alpha'_m, \beta'_m) = \begin{cases} (\alpha_m, \beta_m) & \text{if} \quad (\alpha_m, \beta_m) \cap \bigcup_{n=1}^{\infty}\{d_n\} = \emptyset, \\ (c_{n_0+1}, d_{n_0}) & \text{if} \quad \{a_{n_0}, d_{n_0}\} \subset (\alpha_m, \beta_m), \\ (d_{n_0}, c_{n_0}) & \text{if} \quad \{b_{n_0}, d_{n_0}\} \subset (\alpha_m, \beta_m). \end{cases}$$

Therefore $(\alpha'_m, \beta'_m) \cap (\mathbb{R} \setminus [(E \cap U) \cup A]) = \emptyset$ and

$$\frac{\beta'_m - \alpha'_m}{\beta'_m} \geq \min\left\{\frac{\beta_m - \alpha_m}{\beta_m}, R_2\right\}.$$

By definition of $((\alpha_m, \beta_m))_{m \geq 1}$, we conclude

$$p^+ (\mathbb{R} \setminus [(E \cap U) \cup A], 0) \geq \min\{p^+(\mathbb{R} \setminus E, 0), R_2\}. \tag{12}$$

Combining (11) and (12), we obtain

$$p(\mathbb{R} \setminus [(E \cap U) \cup A], 0) > r_2, \text{ if } p(\mathbb{R} \setminus E, 0) > r_2,$$
$$p(\mathbb{R} \setminus [(E \cap U) \cup A], 0) \geq r_2, \text{ if } p(\mathbb{R} \setminus E, 0) \geq r_2.$$

Hence $0 \in \mathrm{int}_{\mathcal{T}_{r_2(A)}} U \cap \mathrm{int}_{\tau_{r_2(A)}} U$ and finally $U \in \mathcal{T}_{r_2(A)} \cap \tau_{r_2(A)}$. □

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