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The General Induction Functors for the Category of Entwined Hom-Modules

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Abstract. We find a sufficient condition for the category of entwined Hom-modules to be monoidal. Moreover, we introduce morphisms between the underlying monoidal Hom-algebras and monoidal Hom-coalgebras, which give rise to functors between the category of entwined Hom-modules, and we study tensor identities for monodial categories of entwined Hom-modules. Finally, we give necessary and sufficient conditions for the general induction functor from $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ to $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}$ to be separable.

1. Introduction

Entwining modules were introduced in [1], which arise from noncommutative geometry, are modules of an algebra and comodules of a coalgebra such that the action and the coaction satisfy a certain compatibility condition. Unlike Doi-Hopf modules, entwined modules are defined purely using the properties of an algebra and a coalgebra combined into an entwining structure. There is no need for a “background” bialgebra, which is an indispensable part of the Doi-Hopf construction. Entwining modules are more general and easier to deal with, and provide new fields of applications. It is well-known that entwining modules unify modules, comodules, Sweedler’s Hopf modules, Takeuchi’s relative Hopf modules, graded modules, modules graded by G-sets, Long dimodules, Yetter-Drinfeld modules and Doi- Hopf modules [4].

Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov in [16] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity and similar for Hom-coassociativity. They also described the structures of Hom-bialgebras and Hom-Hopf algebras, and extended some important theories from ordinary Hopf algebras to Hom-Hopf algebras in [17] and [18]. Recently, many more properties and structures of Hom-Hopf algebras have been developed, see [5], [6], [7], [8], [9], [10], [12], [14], [20] and references cited therein.

Caenepeel and Goyvaerts studied in [3] Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. In [15], Makhlouf

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and Panaite defined Yetter-Drinfeld modules over Hom-bialgebras and shown that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. Also Liu and Shen [13] studied Yetter-Drinfeld modules over monoidal Hom-bialgebras and called them Hom-Yetter-Drinfeld modules, and shown that the category of Hom-Yetter-Drinfeld modules is a braided monoidal categories. Chen and Zhang [7] defined the category of Hom-Yetter-Drinfeld modules in a slightly different way to [13], and shown that it is a full monoidal subcategory of the left center of left Hom-module category. We have defined in [9] the category of Doi Hom-Hopf modules and we prove there that the category of Hom-Yetter-Drinfeld modules is a subcategory of our category of Doi Hom-Hopf modules.

As a generalization of entwining modules in a Hopf algebra setting, entwined Hom-modules were introduced by Karacuha [11]. It is natural to ask the following question: can we prove a Maschke type theorem for entwined Hom-modules under more general assumptions? This is the motivation of this paper.

In this paper, we discuss the following questions: how do we make the category of entwined Hom-modules into monoidal? We show in Section 3 that it is sufficient that (A, β) and (C, γ) are monoidal Hom-bialgebras with some extra conditions. As an example, we consider the category of Doi Hom-Hopf modules[9], which is well known to be a monoidal category, this category is a special of our theory.

In Section 4, we first give the maps between the underlying Hom-comodule algebras and Hom-module coalgebras, which give rise to functors between the category of entwined Hom-modules. Moreover, we study tensor identities for monodial categories of entwined Hom-modules. As an application, we prove that the category of entwined Hom-modules has enough injective objects.

In Section 5, let $(\Phi, \Psi) : (A, C, \psi) \to (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. The results of [9] can be extended to the general induction functor

$$F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}.$$

In order to avoid technical complications, we will assume that the Hom-entwining map ψ is bijective, and write $\psi^{-1} = \vartheta$.

2. Preliminaries

Throughout this paper we work over a commutative ring k, we recall from [3] and [9] for some informations about Hom-structures which are needed in what follows.

Let C be a category. We introduce a new category $\widetilde{\mathscr{H}}(C)$ as follows: objects are couples (M, μ), with $M \in C$ and $\mu \in Aut_C(M)$. A morphism $f : (M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in C such that $\nu \circ f = f \circ \mu$.

Let $\mathscr{M}_k$ denotes the category of k-modules. $\mathscr{H}(\mathscr{M}_k)$ will be called the Hom-category associated to $\mathscr{M}_k$. If $(M, \mu) \in \mathscr{M}_k$, then $\mu : M \to M$ is obviously a morphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k) = (\mathscr{H}(\mathscr{M}_k), \otimes, (I, I), \widetilde{a}, \widetilde{l}, \widetilde{r})$ is a monoidal category by Proposition 1.1 in [3]: the tensor product of (M, μ) and (N, ν) in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by the formula $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$.

Assume that $(M, \mu), (N, \nu), (P, \pi) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$. The associativity and unit constraints are given by the formulas

$$\widetilde{a}_{M,N,P}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \pi^{-1}(p)),$$

$$\widetilde{l}_M(x \otimes m) = \widetilde{r}_M(m \otimes x) = x\mu(m).$$

An algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ will be called a monoidal Hom-algebra.

Definition 2.1. *A monoidal Hom-algebra is an object $(A, \alpha) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $m_A : A \otimes A \to A$ and an element $1_A \in A$ such that*

$$\alpha(ab) = \alpha(a)\alpha(b); \quad \alpha(1_A) = 1_A,$$

$$\alpha(a)(bc) = (ab)\alpha(c); \quad a1_A = 1_A a = \alpha(a),$$

for all $a, b, c \in A$. Here we use the notation $m_A(a \otimes b) = ab$.

Definition 2.2. *A monoidal Hom-coalgebra is an object $(C,\gamma) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with k-linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\varepsilon : C \to k$ such that*

$$\Delta(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}); \qquad \varepsilon(\gamma(c)) = \varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)}) \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)}), \qquad \varepsilon(c_{(1)})c_{(2)} = \varepsilon(c_{(2)})c_{(1)} = \gamma^{-1}(c)$$

for all $c \in C$.

Definition 2.3. *A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-algebra, $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra and that Δ and ε are morphisms of Hom-algebras, that is,*

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}; \qquad \Delta(1_H) = 1_H \otimes 1_H,$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad \varepsilon(1_H) = 1_H.$$

Definition 2.4. *A monoidal Hom-Hopf algebra is a monoidal Hom-bialgebra (H, α) together with a linear map $S : H \to H$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that*

$$S * I = I * S = \eta\varepsilon, \quad S\alpha = \alpha S.$$

Definition 2.5. *Let (A, α) be a monoidal Hom-algebra. A right (A, α)-Hom-module is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ consists of a k-module and a linear map $\mu : M \to M$ together with a morphism $\psi : M \otimes A \to M, \psi(m \cdot a) = m \cdot a$, in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that*

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab); \quad m \cdot 1_A = \mu(m),$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \to (N, \nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is called right A-linear if it preserves the A-action, that is, $f(m \cdot a) = f(m) \cdot a$. $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ will denote the category of right (A, α)-Hom-modules and A-linear morphisms.

Definition 2.6. *Let (C, γ) be a monoidal Hom-coalgebra. A right (C, γ)-Hom-comodule is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$ notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that*

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); \ m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ)-Hom-comodule are defined in the obvious way. The category of right (C, γ)-Hom-comodules will be denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^C$.

Definition 2.7. *A right-right Hom-entwining structure is a triple (A, C, ψ), where (A, β) is a monoidal Hom-algebra and (C, γ) is a monoidal Hom-coalgebra with a linear map $\psi : C \otimes A \to A \otimes C$ such that $\psi \circ (\gamma \otimes \beta) = (\beta \otimes \gamma) \circ \psi$ satisfying the following conditions:*

$$\begin{aligned}
(ab)_\psi \otimes c^\psi &= a_\psi b_\psi \otimes \gamma((\gamma^{-1}(c)^\psi))^\psi), \\
\psi(c \otimes 1_A) &= 1_A \otimes c, \\
a_\psi \otimes \Delta(c^\psi) &= \beta(\beta^{-1}(a)_{\psi\psi}) \otimes (c_{(1)}{}^\psi \otimes c_{(2)}{}^\psi), \\
\varepsilon(c^\psi)a_\psi &= \varepsilon(c)a.
\end{aligned}$$

Over a Hom-entwining structure (A, C, ψ), a right-right entwined Hom-module (M, μ) is both a right (C, γ)-Hom-comodule and a right (A, β)-Hom-module such that

$$\begin{aligned}\rho_M(m \cdot a) &= \mu(m_{[0]}) \cdot \psi(m_{[1]} \otimes \beta^{-1}(a)) \\ &= m_{[0]} \cdot \beta^{-1}(a)_{\psi} \otimes \gamma(m_{[1]}^{\psi}),\end{aligned}$$

for all $a \in A$ and $m \in M$. $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ will denote the category of right entwined Hom-modules and morphisms between them.

A morphism between right-right entwined Hom-modules is a k-linear map which is a morphism in the categories $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and $\widetilde{\mathscr{C}}(\mathscr{M}_k)^C$ at the same time. $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ will denote the category of right-right entwined Hom-modules and morphisms between them.

3. Making the Category of Entwined Hom-Modules into a Monoidal Category

Now suppose that (A, β) and (C, γ) are both monoidal Hom-bialgebras.

Proposition 3.1. *Let $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$, $(N, \nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$. Then we have $M \otimes N \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ with structures:*

$$(m \otimes n) \cdot a = m \cdot a_{(1)} \otimes n \cdot a_{(2)},$$

$$\rho_{M \otimes N}(m \otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]} n_{[1]}$$

if and only if the following condition holds:

$$a_{(1)\psi} \otimes a_{(2)\psi} \otimes c^{\psi} d^{\psi} = a_{\psi(1)} \otimes a_{\psi(2)} \otimes (cd)^{\psi}, \tag{3. 1}$$

for all $a \in A$ and $c, d \in C$. Furthermore, the category $C = \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ is a monoidal category.

Proof. It is easy to see that $M \otimes N$ is a right (A, β)-module and that $M \otimes N$ is a right (C, γ)-comodule. Now we check that the compatibility condition holds:

$$\begin{aligned}&\rho_{M \otimes N}((m \otimes n) \cdot a) \\ =\ & (m \cdot a_{(1)})_{[0]} \otimes (n \cdot a_{(2)})_{[0]} \otimes (m \cdot a_{(1)})_{[1]} (n \cdot a_{(2)})_{[1]} \\ =\ & m_{[0]} \cdot \beta^{-1}(a_{(1)})_{\psi} \otimes n_{[0]} \cdot \beta^{-1}(a_{(2)})_{\psi} \otimes (\gamma(m_{[1]}^{\psi}) \gamma(n_{[1]}^{\psi})) \\ \overset{(3.1)}{=}\ & m_{[0]} \cdot \beta^{-1}(a)_{\psi(1)} \otimes n_{[0]} \cdot \beta^{-1}(a)_{\psi(2)} \otimes \gamma((m_{[1]} n_{[1]})^{\psi}) \\ =\ & (m_{[0]} \otimes n_{[0]}) \cdot \beta^{-1}(a)_{\psi} \otimes \gamma((m_{[1]} n_{[1]})^{\psi}).\end{aligned}$$

So $M \otimes N \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$.

Conversely, one can easily check that $A \otimes C \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$, let $m = 1 \otimes c$ and $n = 1 \otimes d$ for any $c, d \in C$ and easily get (3.1).

Furthermore, k is an object in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ with structures:

$$x \cdot a = \varepsilon_A(a) x, \quad \rho(x) = x \otimes 1_C,$$

for all $x \in k$ if and only if the following condition holds:

$$\varepsilon_A(a) 1_C = \varepsilon_A(\beta^{-1}(a)_{\psi})(\gamma(1_C^{\psi})), \tag{3. 2}$$

for all $a \in A$. Then it is easy to get that $(C = \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C, \otimes, k, \widetilde{a}, \widetilde{l}, \widetilde{r})$ is a monoidal category, where $\widetilde{a}, \widetilde{l}, \widetilde{r}$ are given by the formulas:

$$\widetilde{a}_{M,N,P}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \pi^{-1}(p)),$$

$$\widetilde{l}_M(x \otimes m) = \widetilde{r}_M(m \otimes x) = x\mu(m),$$

for $(M, \mu), (N, \nu), (P, \pi) \in C$. $\square$

We call $G = (A, C, \psi)$ a *monoidal Hom-entwining structure* if G is a Hom-entwining structure, and A, C are monoidal Hom-bialgebras with the additional compatibility relations (3.1) and (3.2).

If (A, C, ψ) is a monoidal Hom-entwining structure, then (A, β) and (C, γ) can be made into objects of $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$.

Proposition 3.2. *Let (A, C, ψ) be a monoidal Hom-entwining structure. On (A, β) and (C, γ), we consider the following right (A, β)-action and right (C, γ)-coaction:*

$$b \cdot a = ba \text{ and } \rho^r(b) = \psi(1_C \otimes b) = \beta^{-1}(b_\psi) \otimes 1_C^\psi,$$

$$c \cdot a = \varepsilon_A(a_\psi)\gamma(c^\psi) \text{ and } \rho^r(c) = c_{(1)} \otimes c_{(2)}.$$

Then (A, β) and (C, γ) are entwined Hom-modules.

Proof. We will show $(A, \beta) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$, and leave the other statement to the reader. First, (A, β) is a right (C, γ)-comdule, since

$$(id_A \otimes \varepsilon_C)\rho^r(b) = \varepsilon_C(1_C^\psi)\beta^{-1}(b_\psi) = \varepsilon_C(1_C)\beta^{-1}(b) = b,$$

$$(\beta^{-1} \otimes \Delta_C)\rho^r(b) = \beta^{-2}(b_\psi) \otimes \Delta_C(1_C^\psi) = \beta^{-2}(b_{\psi\psi}) \otimes 1_C^\psi \otimes 1_C^\psi = (\rho^r(b) \otimes \gamma^{-1})\rho^r(b),$$

and

$$b_{[0]}\beta^{-1}(a_\psi) \otimes \gamma(b_{[1]}^\psi) = \beta^{-1}(b_\psi)\beta^{-1}(a_\psi) \otimes \gamma(1_C^{\psi\psi}) = \beta^{-1}((ba)_\psi) \otimes \gamma(1_C^\psi) = \rho^r(ba),$$

Thus $(A, \beta) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$. $\square$

Example 3.3. *Let (H, α) be a monoidal Hom-Hopf algebra, (C, γ) a right (H, α)-Hom module bialgebra, and that (H, α) acts on (C, γ) in such a way that (C, γ) is an (H, α)-Hom module algebra and (H, α)-Hom module coalgebra. Now let (A, β) be a monoidal Hom-bialgebra and a right (H, α)-Hom comodule algebra such that the following compatibility relation holds, for all $a \in A$:*

$$a_{(1)[0]} \otimes a_{(2)[0]} \otimes a_{(1)[1]} \otimes (a_{(2)[1]} = a_{[0](1)} \otimes a_{[0](2)} \otimes a_{[1](1)} \otimes a_{[1](2)}.$$

We know that (H, A, C) is a right-right Doi Hom-Hopf datum in [9], and we have a corresponding right-right Hom-entwining structure (A, C, ψ). It is straightforward to check that (A, C, ψ) is monoidal.

4. Tensor Identities

Theorem 4.1. *Given two Hom-entwining structures (A, C, ψ) and (A', C', ψ'), suppose that two maps $\Phi : A \to A'$ and $\Psi : C \to C'$ which are respectively monodial Hom-algebra and monodial Hom-coalgebra maps satisfying*

$$\Phi(a_\psi) \otimes \Psi(c^\psi) = \Phi(a)_{\psi'} \otimes \Psi(c)^{\psi'},$$

then the induction functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}$, defined as follows:

$$F(M) = M \otimes_A A',$$

where (A', β') is a left (A, β)-module via Φ and with structure maps defined by

$$(m \otimes_A a') \cdot b' = \mu(m) \otimes_A a'\beta'^{-1}(b'), \tag{4. 1}$$

$$\rho_{F(M)}(m \otimes_A a') = m_{[0]} \otimes_A (\beta'^{-1}(a'))_{\psi'} \otimes \Psi(\gamma^{-1}(m_{[1]})^{\psi'}), \tag{4. 2}$$

for all $a', b' \in A'$ and $m \in M$.

Proof. Let us show that $M\otimes_A A'$ is an object of ${}_{A'}\widetilde{\mathscr{H}}(\mathscr{M}_k)(H')^{C'}$. It is routine to check that $F(M)$ is a right (A',β')-module. For this, we need to show that $M\otimes_A A'$ is a right (C',γ')-comodule and satisfy the compatible condition, for any $m\in M$ and $a',b'\in A'$, we have

$$\begin{aligned}
\rho_{F(M)}((m\otimes_A a')\cdot b') &= \rho_{F(M)}(\mu(m)\otimes_A a'\beta'^{-1}(b'))\\
&= \mu(m_{[0]})\otimes_A (\beta'^{-1}(a'\beta'^{-1}(b')))_{\psi'}\otimes\Psi(m_{[1]}^{\psi'})\\
&= [m_{[0]}\otimes_A (\beta'^{-1}(a'))_{\psi'}\otimes\Psi(\gamma^{-1}(m_{[1]})^{\psi'})]b'\\
&= \rho_{F(M)}(m\otimes_A a')b',
\end{aligned}$$

i.e., the compatible condition holds. It remains to prove that $M\otimes_A A'$ is a right (C',γ')-comodule. For any $m\in M$ and $a'\in A'$, we have

$$\begin{aligned}
& (\rho_{F(M)}\otimes id_{C'})\rho_{F(M)}(m\otimes_A a')\\
= {} & (\rho_{F(M)}\otimes id'_C)(m_{[0]}\otimes_A (\beta'^{-1}(a'))_{\psi'}\otimes\Psi(\gamma^{-1}(m_{[1]})^{\psi'}))\\
= {} & m_{[0][0]}\otimes_A (\beta'^{-2}(a'))_{\psi'\varphi'}\otimes\Psi(\gamma^{-1}(m_{[0][1]})^{\varphi'})\otimes\Psi(\gamma^{-1}(m_{[1]})^{\psi'})\\
= {} & [m_{[0]}\otimes_A (\beta'^{-1}(a'))_{\psi'\varphi'}]\otimes\Psi(\gamma^{-1}(m_{[1](1)})^{\varphi'})\otimes\Psi(\gamma^{-1}(m_{[1](2)})^{\psi'})\\
= {} & m_{[0]}\otimes_A (\beta'^{-1}(a'))_{\psi'}\otimes\Psi(\gamma^{-1}(m_{[1]})^{\psi'})_{(1)}\otimes\Psi(\gamma^{-1}(m_{[1]})^{\psi'})_{(2)}\\
= {} & (id_{F(M)}\otimes\Delta_{C'})\rho_{F(M)}(m\otimes_A a'),
\end{aligned}$$

and

$$\begin{aligned}
& (id_{F(M)}\otimes\varepsilon)\rho_{F(M)}(m\otimes_A a')\\
= {} & (id_{F(M)}\otimes\varepsilon)(m_{[0]}\otimes_A (\beta'^{-1}(a'))_{\psi'}\otimes\Psi(\gamma^{-1}(m_{[1]})^{\psi'}))\\
= {} & m\otimes_A a',
\end{aligned}$$

as desired. This completes the proof. □

Theorem 4.2. *Under the assumptions of Theorem 4.1, we have a functor* $G:\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')^{C'}_{A'}\to\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)^{C}_{A}$ *which is right adjoint to F. G is defined by*

$$G(M') = M'\square_{C'}C,$$

with structure maps

$$(m'\otimes c)\cdot a = m'\cdot\beta^{-1}(a)_{\psi}\otimes\gamma(c^{\psi}), \tag{4. 3}$$

$$\rho_{G(M')}(m'\otimes c) = \mu'^{-1}(m')\otimes c_{(1)}\otimes\gamma(c_{(2)}), \tag{4. 4}$$

for all $a\in A$.

Proof. We first show that $G(M')$ is an object of $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)^C_A$. It is not hard to check that $G(M')$ is a right (A,β)-module. Now we check that $G(M')$ is a right (C,γ)-comodule and satisfy the compatible condition. For any $m'\in M'$ and $a\in A, c\in C$, we have

$$\begin{aligned}
\rho_{G(M')}((m'\otimes c)\cdot a) &= \rho_{G(M')}(m'\cdot\beta^{-1}(a)_{\psi}\otimes\gamma(c^{\psi}))\\
&= \mu'^{-1}(m')\cdot\beta^{-2}(a_{\psi})\otimes\gamma(c^{\psi})_{(1)}\otimes\gamma(\gamma(c^{\psi})_{(2)})\\
&= (\mu'^{-1}(m')\otimes c_{(1)}\otimes\gamma(c_{(2)}))a\\
&= \rho_{G(M')}(m'\otimes c)a,
\end{aligned}$$

i.e., the compatible condition holds. It remains to prove that $M'\square_{C'}C$ is a right (C,γ)-comodule. For any

$m' \in M'$ and $a \in A$, we have

$$\begin{aligned}
& (\rho_{G(M')} \otimes id_{C'})\rho_{G(M')}(m' \otimes_A c) \\
= \;& (\rho_{G(M')} \otimes id_{C'})(\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)})) \\
= \;& \mu'^{-2}(m') \otimes c_{(1)(1)} \otimes \gamma(c_{(1)(2)}) \otimes \gamma(c_{(2)}) \\
= \;& \mu'^{-2}(m') \otimes \gamma^{-1}(c_{(1)}) \otimes \gamma(c_{(2)(1)}) \otimes \gamma^2(c_{(2)(2)}) \\
= \;& \mu'^{-1}(m') \otimes c_{(1)} \otimes [\gamma(c_{(2)(1)}) \otimes \gamma(c_{(2)(2)})] \\
= \;& (id_{G(M')} \otimes \Delta_C)\rho_{G(M')}(m' \otimes c),
\end{aligned}$$

and

$$\begin{aligned}
& (id_{G(M')} \otimes \varepsilon)\rho_{G(M')}(m' \otimes c) \\
= \;& (id_{G(M')} \otimes \varepsilon)(\mu'^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)})) \\
= \;& \mu'^{-1}(m') \otimes c_{(1)}\varepsilon(c_{(2)}) \otimes 1_C = m' \otimes c,
\end{aligned}$$

as required.

$G(M') \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ and the functorial properties can be checked in a straightforward way. Finally, we show that G is a right adjoint to F. Take $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_{A'}^C$, define $\eta_M : M \to GF(M) = (M \otimes_A A')\Box_{C'} C$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes_A 1_{A'} \otimes m_{[1]}.$$

It is easy to see that $\eta_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$. Take $(M', \mu') \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}$, define $\delta_{M'} : FG(M') \to M'$, where

$$\delta_{M'}((m' \otimes c) \otimes_A a') = \varepsilon_C(c)\mu'(m') \cdot a',$$

It is easy to check that $\delta_{M'}$ is (A, β)-linear and therefore $\delta_{M'} \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}$. We can also verify η and δ defined above are all natural transformations and satisfy

$$G(\delta_{M'}) \circ \eta_{G(M')} = I, \;\; \delta_{F(M)} \circ F(\eta_M) = I,$$

for all $M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ and $M' \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}$. And this completes the proof. □

A morphism (Φ, Ψ) between two monoidal Hom-entwining structures is called *monoidal* if Φ and Ψ are monoidal Hom-bialgebra maps. We now consider the particular situation where $A = A'$ and $\Phi = I_A$. The following result is a generalization of [4].

Theorem 4.3. *Let $(I_A, \Psi) : (A, C, \psi) \to (A, C', \psi')$ be a monoidal morphism of monoidal Hom entwining structures. Then*

$$G(C') = C. \tag{4. 5}$$

Let $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ be flat as a k-module, and take $(N, \nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_A^{C'}$. If (C, γ) is a monoidal Hom-Hopf algebra, then

$$M \otimes G(N) \cong G(F(M) \otimes N) \;\; in \;\; \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C. \tag{4. 6}$$

If (C, γ) has a twisted antipode $\overline{S}$, then

$$G(N) \otimes M \cong G(N \otimes F(M)) \;\; in \;\; \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C. \tag{4. 7}$$

Proof. We know that $\varepsilon_{C'} \otimes id_C : C'\Box_C C \to C$ is an isomorphism; the inverse map is $(\Psi \otimes id_C)\Delta_C : C \to C'\Box_C C$. It is clear that $\varepsilon_{C'} \otimes id_C$ is (A, β)-linear and (C, γ)-colinear. And this prove (4.5).

Now we define the map

$$\Gamma : M \otimes G(N) = M \otimes (N\Box_{C'} C) \to G(F(M) \otimes N) = (F(M) \otimes N)\Box_{C'} C,$$

which is given by

$$\Gamma(m \otimes (n_i \otimes c_i)) = (m_{[0]} \otimes n_i) \otimes m_{[1]}c_i.$$

Recall that $F(M) = M$ as an (A, β)-module, with (C', γ')-coaction given by

$$\rho_{F(M)}(m) = m_{[0]} \otimes \Psi(m_{[1]}).$$

(1) Γ is well-defined, we have to show that

$$\Gamma(m_i \otimes (n_i \otimes c_i)) \in (F(M) \otimes N)\Box'_C C.$$

This may be seen as follows: for any $m \in M$ and $n_i \Box_{C'} c \in N \Box_{C'} C$, we have

$$\begin{aligned}
& (\rho_{F(M)\otimes N} \otimes id_C)((m_{[0]} \otimes n_i) \otimes m_{[1]}c_i) \\
= \ & (m_{[0][0]} \otimes n_{i[0]}) \otimes \Psi(m_{[0][1]})n_{i[1]} \otimes m_{[1]}c_i \\
= \ & (\mu(m_{[0]}) \otimes \nu(n_i)) \otimes \Psi(m_{[0][1]})\Psi(c_{i(1)}) \otimes \gamma^{-1}(m_{[1]}c_{i(2)}) \\
= \ & (m_{[0]} \otimes n_i) \otimes [\phi(m_{[0][1]})\Psi(c_{i(1)}) \otimes m_{[1]}c_{i(2)}] \\
= \ & (id_{F(M)\otimes N} \otimes \rho_{C'})((m_{[0]} \otimes n_i) \otimes m_{[1]}c_i).
\end{aligned}$$

(2) Γ is (A, β)-linear. Indeed, for any $a \in A$, $m \in M$ and $n_i \Box_{C'} c \in N \Box_{C'} C$, we have

$$\begin{aligned}
& \Gamma((m \otimes (n_i \otimes c_i)) \cdot a) \\
= \ & \Gamma(m \cdot a_{(1)} \otimes (n_i \cdot \beta^{-1}(a)_{(2)\psi} \otimes \gamma(c_i^{\psi}))) \\
= \ & (m_{[0]} \cdot \beta^{-1}(a_{(1)\psi}) \otimes n_i \cdot \beta^{-1}(a)_{(2)\psi}) \otimes \gamma(m_{[1]}^{\psi})\gamma(c_i^{\psi}) \\
= \ & (m_{[0]} \cdot \beta^{-1}(a_{\psi(1)}) \otimes n_i \cdot \beta^{-1}(a)_{\psi(2)}) \otimes \gamma((m_{[1]}c_i)^{\psi}) \\
= \ & (m_{[0]} \otimes n_i) \cdot \beta^{-1}(a_{\psi}) \otimes \gamma((m_{[1]}c_i)^{\psi}) \\
= \ & \Gamma(m \otimes (n_i \otimes c_i)) \cdot a.
\end{aligned}$$

(3) Γ is (C, γ)-colinear. Indeed, for any $m \in M$ and $n_i \Box_{C'} c \in N \Box_{C'} C$, we have

$$\begin{aligned}
& \rho \circ \Gamma(m \otimes (n_i \otimes c_i)) \\
= \ & \rho((m_{[0]} \otimes n_i) \otimes m_{[1]}c_i) \\
= \ & (\mu^{-1}(m_{[0]}) \otimes \nu^{-1}(n_i)) \otimes m_{[1](1)}c_{i(1)} \otimes \gamma(m_{[1](2)}c_{i(2)}) \\
= \ & (m_{[0]} \otimes \nu^{-1}(n_i)) \otimes m_{[0][1]}c_{i(1)} \otimes m_{[1]}\gamma(c_{i(2)}) \\
= \ & (\Gamma \otimes id_C)(m_{[0]} \otimes (\nu^{-1}(n_i) \otimes c_{i(1)})) \otimes m_{[1]}\gamma(c_{i(2)}) \\
= \ & (\Gamma \otimes id_C) \circ \rho(m \otimes (n_i \otimes c_i)).
\end{aligned}$$

Assume (C, γ) has an antipode and define

$$\begin{aligned}
& \Theta : (F(M) \otimes N)\Box_{C'} C \to M \otimes (N \Box_{C'} C), \\
& \Theta((m_i \otimes n_i) \otimes c_i) = \mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]})\gamma^{-2}(c_i)).
\end{aligned}$$

We have to show that Ψ is well-defined. (M, μ) is flat, so $M \otimes (N \Box_{C'} C)$ is the equalizer of the maps

$$id_M \otimes id_N \otimes \rho_C : \quad M \otimes N \otimes C \to M \otimes N \otimes C' \otimes C,$$

and

$$id_M \otimes \rho_N \otimes id_C : \ M \otimes N \otimes C \to M \otimes N \otimes C' \otimes C.$$

Now take $(m_i \otimes n_i) \otimes c_i \in (F(M) \otimes N)\Box_{C'} C$, then

$$(m_{i[0]} \otimes n_{i[0]}) \otimes \phi(m_{i[1]})n_{i[1]} \otimes c_i = (\mu^{-1}(m_i) \otimes \nu^{-1}(n_i)) \otimes \Psi(c_{i(1)}) \otimes \gamma(c_{i(2)}). \tag{4. 8}$$

Therefore, we get

$$\begin{aligned}
& id_M \otimes id_N \otimes \rho_C(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]})\gamma^{-2}(c_i))) \\
= \ & \mu^2(m_{i[0]}) \otimes (n_i \otimes \Psi(S(m_{i[1](2)})\gamma^{-2}(c_{i(1)})) \otimes S(m_{i[1](1)})\gamma^{-2}(c_{i(2)})) \\
= \ & m_{i[0]} \otimes \nu^{-1}(n_i) \otimes \Psi(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})) \otimes S(\gamma^2(m_{i[1](1)}))c_{i(2)},
\end{aligned}$$

and

$$\begin{aligned}
& id_M \otimes \rho_N \otimes id_C(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]})\gamma^{-2}(c_i))) \\
= \ & \mu^2(m_{i[0]}) \otimes (n_{i[0]} \otimes n_{i[1]} \otimes S(m_{i[1]})\gamma^{-2}(c_i)) \\
= \ & m_{i[0]} \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i).
\end{aligned}$$

Applying $(id_M \otimes \Psi \otimes id_C) \circ (id_M \otimes (\Delta_C \circ S_C)) \circ \rho_M$ to the first factor of (4.8), we obtain

$$\begin{aligned}
& m_{i[0][0]} \otimes \Psi(S(m_{i[0][1](2)})) \otimes S(m_{i[0][1](1)}) \otimes n_{i[0]} \otimes \Psi(m_{i[1]})n_{i[1]} \otimes c_i \\
= \ & \mu^{-1}(m_{i[0]}) \otimes \Psi(S(\gamma^{-1}(m_{i[1](2)}))) \otimes S(\gamma^{-1}(m_{i[1](1)})) \otimes \nu^{-1}(n_i) \otimes \phi(c_{i(1)}) \otimes \gamma(c_{i(2)}).
\end{aligned}$$

Applying $id_M \otimes \gamma^2 \otimes id_C \otimes id_N \otimes \gamma^{-1} \otimes \gamma^{-1}$ to the above identity, we have

$$\begin{aligned}
& m_{i[0][0]} \otimes \Psi(S(\gamma^2(m_{i[0][1](2)}))) \otimes S(m_{i[0][1](1)}) \otimes n_{i[0]} \otimes \gamma^{-1}(\phi(m_{i[1]})n_{i[1]}) \otimes \gamma^{-1}(c_i) \\
= \ & \mu^{-1}(m_{i[0]}) \otimes \Psi(S(\gamma(m_{i[1](2)}))) \otimes S(\gamma^{-1}(m_{i[1](1)})) \otimes \nu^{-1}(n_i) \otimes \phi(\gamma^{-1}(c_{i(1)})) \otimes c_{i(2)}.
\end{aligned}$$

Multiplying the second and the fifth factor, and also the third and sixth factor, we have

$$\begin{aligned}
& \mu(m_{i[0]}) \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i) \\
= \ & \mu(m_{i[0]}) \otimes \nu^{-1}(n_i) \otimes \Psi(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})) \otimes S(\gamma^2(m_{i[1](1)}))c_{i(2)},
\end{aligned}$$

and applying $\mu^{-1} \otimes id_N \otimes id_C \otimes id_C$ to the above identity, we obtain

$$\begin{aligned}
& m_{i[0]} \otimes n_{i[0]} \otimes \gamma(n_{i[1]}) \otimes S(\gamma(m_{i[1]}))\gamma^{-1}(c_i) \\
= \ & m_{i[0]} \otimes \nu^{-1}(n_i) \otimes \Psi(S(\gamma(m_{i[1](2)}))\gamma^{-1}(c_{i(1)})) \otimes S(\gamma^2(m_{i[1](1)}))c_{i(2)},
\end{aligned}$$

or

$$id_M \otimes \rho_N \otimes id_C \circ (\Theta((m_i \otimes n_i) \otimes c_i)) = id_M \otimes id_N \otimes \rho_C \circ (\Theta((m_i \otimes n_i) \otimes c_i)).$$

Let us point out that Γ and Θ are each other's inverses. In fact,

$$\begin{aligned}
& \Gamma \circ \Theta((m_i \otimes n_i) \otimes c_i) \\
= \ & \Gamma(\mu^2(m_{i[0]}) \otimes (n_i \otimes S(m_{i[1]}\gamma^{-2}(c_i)))) \\
= \ & (\mu^2(m_{i[0][0]}) \otimes n_i) \otimes \gamma^2(m_{i[0][1]})S(m_{i[1]})\gamma^{-2}(c_i)) \\
= \ & (\mu^2(m_{i[0][0]}) \otimes n_i) \otimes [\gamma(m_{i[0][1]})S(m_{i[1]})]\gamma^{-1}(c_i)) \\
= \ & (\mu(m_{i[0]}) \otimes n_i) \otimes [\gamma(m_{i[1](1)})S(\gamma(m_{i[1](2)}))]\gamma^{-1}(c_i)) \\
= \ & (m_i \otimes n_i) \otimes c_i,
\end{aligned}$$

and

$$\begin{aligned}
& \Theta \circ \Gamma(m \otimes (n_i \otimes c_i)) \\
= \ & \Theta((m_{[0]} \otimes n_i) \otimes m_{[1]}c_i) \\
= \ & \mu^2(m_{[0][0]}) \otimes (n_i \otimes [S(\gamma^{-1}(m_{[0][1]}))\gamma^{-2}(m_{[1]})]\gamma^{-1}(c_i)) \\
= \ & \mu(m_{[0]}) \otimes (n_i \otimes [S(\gamma^{-1}(m_{[1](1)}))\gamma^{-1}(m_{[1](2)})]\gamma^{-1}(c_i)) \\
= \ & m \otimes (n_i \otimes c_i).
\end{aligned}$$

The proof of (4.7) is similar and left to the reader. □

Corollary 4.4. *Let (A, C, ψ) be a monoidal Hom-entwining structure, Λ: $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C \to \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ the functor forgetting the (C, γ)-coaction. For any flat entwined Hom-module (M, μ), we have an isomorphism*

$$M \otimes C \cong \Lambda(M) \otimes C$$

in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$. If k is a field, then $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ has enough injective objects, and any injective object in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ is a direct summand of an object of the form $I \otimes C$, where I is an injective (A, β)-module.

We have already proved that the category of Doi Hom-Hopf modules may be viewed as the category of entwined Hom-modules corresponding to a monoidal Hom-entwining structure. Then we have the following corollary.

Corollary 4.5. *Let (H, A, C) be a monoidal Doi Hom-Hopf Datum. If k is a field, then $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ has enough injective objects, and any injective object in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(H)_A^C$ is a direct summand of an object of the form $I \otimes C$, where I is an injective (A, β)-module.*

We continue with the dual version of Theorem 4.3.

Theorem 4.6. *Let $(\Phi, I_C) : (A, C, \psi) \to (A', C, \psi)$ be a monoidal morphism of monoidal Hom-entwining structures. Then*

$$F(A) = A'. \tag{4. 9}$$

Let $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C$ be flat as a k-module, and take $(N, \nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_{A'}^C$. If (A', β') is a monoidal Hom-Hopf algebra, then

$$F(M) \otimes N \cong F(M \otimes G(N)) \ \text{ in } \ \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C. \tag{4. 10}$$

If (A', β') has a twisted antipode $\overline{S}$, then

$$N \otimes F(M) \cong F(G(N) \otimes M) \ \text{ in } \ \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C. \tag{4. 11}$$

Proof. We only prove (4.10) and similar for (4.9) and (4.11). Assume that (A', β') is a monoidal Hom-Hopf algebra and define

$$\Gamma : F(M \otimes G(N)) = M \otimes G(N) \otimes_A A' \to F(M) \otimes N = (M \otimes_A A') \otimes N$$

by

$$\Gamma((m \otimes n) \otimes a') = (m \otimes a'_{(1)}) \otimes n \cdot a'_{(2)},$$

for all $a' \in A', m \in M$ and $n \in N$. Γ is well-defined since

$$\begin{aligned}
\Gamma((m \otimes n) \otimes \Phi(a)a') &= (m \otimes \Phi(a_{(1)})a'_{(1)}) \otimes n \cdot \Phi(a_{(2)})a'_{(2)} \\
&= (m \cdot a_{(1)} \otimes a'_{(1)}) \otimes n \cdot \Phi(a_{(2)})a'_{(2)} \\
&= \Gamma((m \cdot a_{(1)} \otimes n \cdot \Phi(a_{(2)})) \otimes a') \\
&= \Gamma((m \otimes n) \cdot a \otimes a').
\end{aligned}$$

It is easy to check that Γ is (A', β')-linear. Now we shall verify that Γ is (C, γ)-colinear based on (3.1). For any $a' \in A', m \in M$ and $n \in N$, we have

$$\begin{aligned}
\rho(\Gamma((m \otimes n) \otimes a')) &= \rho((m \otimes a'_{(1)}) \otimes n \cdot a'_{(2)}) \\
&= (m_{[0]} \otimes \beta'^{-1}(a'_{(1)\psi})) \otimes (n_{[0]} \cdot \beta'^{-1}(a'_{(2)\psi})) \otimes \gamma(m_{[1]})_\psi \gamma(n_{[1]})_\psi \\
&\overset{(3.1)}{=} (m_{[0]} \otimes \beta'^{-1}(a'_{\psi(1)})) \otimes (n_{[0]} \cdot \beta'^{-1}(a'_{\psi(2)})) \otimes \gamma(m_{[1]}n_{[1]})^\psi \\
&= (\Gamma \otimes id_c)(((m_{[0]} \otimes n_{[0]}) \otimes \beta'^{-1}(a')_\psi) \otimes \gamma(m_{[1]}n_{[1]})^\psi) \\
&= (\Gamma \otimes id_c)\rho((m \otimes n) \otimes a').
\end{aligned}$$

The inverse of Γ is given by

$$\Pi((m\otimes a')\otimes n)=(m\otimes \nu^{-2}(n)S^{-1}(a'_{(2)}))\otimes \beta'^2(a'_{(1)})$$

for all $a'\in A', m\in M$ and $n\in N$. One can check that Π is well-defined similar to Γ. Finally, we have

$$\begin{aligned}\Pi(\Gamma((m\otimes n)\otimes a')) &= \Pi((m\otimes a'_{(1)})\otimes n\cdot a'_{(2)})\\ &= (m\otimes \nu^{-2}(n\cdot a'_{(2)})S(a'_{(1)(2)}))\otimes \beta'^2(a'_{(1)(1)})\\ &= (m\otimes \nu^{-1}(n)\cdot[\beta'^{-1}(a'_{(2)(2)}S^{-1}(\beta'^{-1}(a'_{(2)(1)}))])\otimes \beta'(a'_{(1)})\\ &= (m\otimes n)\otimes a',\end{aligned}$$

and

$$\begin{aligned}\Gamma(\Pi((m\otimes a')\otimes n)) &= \Gamma((m\otimes \nu^{-2}(n)S^{-1}(a'_{(2)}))\otimes \beta'^2(a'_{(1)}))\\ &= (m\otimes \beta'^2(a'_{(1)(1)}))\otimes \nu^{-2}(n)\cdot S^{-1}(a'_{(2)})\beta'^2(a'_{(1)(2)})\\ &= (\beta'(a'_{(1)})\otimes m)\otimes \nu^{-1}(n)\cdot[S^{-1}(\beta'(a'_{(2)(2)})\beta'(a'_{(2)(1)})]\\ &= (m\otimes a')\otimes n,\end{aligned}$$

as needed. The proof is completed. □

5. The General Induction Functor

Let $(\Phi,\Psi):(A,C,\psi)\to(A',C',\psi')$ be a morphism of (right-right) Hom-entwining structures. The results of [9] can be extended to the general induction functor

$$F:\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)_A^C\to\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')_{A'}^{C'}$$

and its right adjoint G (see Theorem 4.2). In order to avoid technical complications, we will assume that the Hom-entwining map ψ is bijective, and write $\psi^{-1}=\vartheta$.

Proposition 5.1. *Let $(\Phi,\Psi):(A,C,\psi)\to(A',C',\psi')$ be a morphism of (right-right) Hom-entwining structures. With ψ invertible, and $\vartheta: A\otimes C\to C\otimes A$ its inverse. Let V_2 consist of all left and right (A,β)-linear maps $\lambda: GF(C\otimes A)\to A$ satisfying*

$$\lambda((\gamma^{-1}(c_i)\otimes a'_i)\otimes d_{i(1)})\otimes\gamma(d_{i(2)})=\sum\lambda((c_{i(2)}\otimes a'_i)\otimes\gamma^{-1}(d_i))_\psi\otimes\gamma^2(c_{i(1)})^\psi \quad (5.\ 1)$$

for all $(c_i\otimes a'_i)\otimes d_i\in GF(C\otimes A)$. We have a k-linear isomorphism

$$f_1: V_1={}^C_A Hom^C_A(GF(C\otimes A),C\otimes A)\to V_2,\quad f_1(\overline{v})=(\varepsilon\otimes I_A)\circ\overline{v}.$$

Proof. $\lambda=f_1(\overline{v})$ is left and right (A,β)-linear since $\overline{v}$ and $\varepsilon\otimes I_A$ are left and right (A,β)-linear. Take $\sum_i(c_i\otimes a'_i)\otimes d_i\in GF(C\otimes A)$, and we write

$$\overline{v}(\sum_i(c_i\otimes a'_i)\otimes d_i)=\sum_j c_j\otimes a_j.$$

Using the left (C,γ)-colinearity of $\overline{v}$, we have

$$\gamma^2(c_{i(1)})\otimes\overline{v}(\sum_i(c_{i(2)}\otimes\beta'^{-1}(a'_i))\otimes\gamma^{-1}(d_i))=\sum_j\gamma(c_{j(1)})\otimes(c_{j(2)}\otimes\beta^{-1}(a_j)),$$

and applying ε_C to the second factor

$$\gamma^2(c_{i(1)}) \otimes \overline{\lambda}(\sum_i (c_{i(2)} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(d_i)) = \sum_j c_j \otimes \beta^{-1}(a_j),$$

$\overline{v}$ is also right (C,γ)-colinear, hence

$$\overline{v}(\sum_i (\gamma^{-1}(c_i) \otimes \beta'^{-1}(a'_i)) \otimes d_{i(1)}) \otimes \gamma(d_{i(2)}) = \sum_j [c_{j(1)} \otimes \beta^{-1}(a_{j\psi})] \otimes \gamma(c^{\psi}_{j(2)}).$$

Applying ε_C to the first factor, we obtain

$$\overline{\lambda}(\sum_i (\gamma^{-1}(c_i) \otimes \beta'^{-1}(a'_i) \otimes d_{i(1)}) \otimes \gamma(d_{i(2)}) = \sum_j \beta^{-1}(a_{j\psi}) \otimes c^{\psi}_j,$$

and we have shown that $\overline{\lambda}$ satisfies (5.1), and f_1 is well-defined. The inverse of f_1 is given by

$$g_1(\sum_i (c_i \otimes a'_i) \otimes d_i) = \sum_i \gamma^2(c_{i(1)}) \otimes \lambda(\sum_i (c_{i(2)} \otimes \beta^{-1}(a_i)) \otimes \gamma^{-1}(d_i)).$$

It is obvious that $\overline{v} = g_1(\lambda)$ is left (C,γ)-colinear and right (A,β)-linear. $\overline{v}$ is right (C,γ)-colinear since

$$\begin{aligned}
& \overline{v}(\sum_i (\gamma^{-1}(c_i) \otimes \beta'^{-1}(a'_i) \otimes d_{i(1)}) \otimes \gamma(d_{i(2)}) \\
= \ & \sum_i \gamma(c_{i(1)}) \otimes \overline{\lambda}(\sum_i (\gamma^{-1}(c_{i(2)}) \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(d_{i(1)})) \otimes \gamma(d_{i(2)}) \\
= \ & \sum_i \gamma(c_{i(1)}) \otimes \overline{\lambda}(\sum_i (c_{i(2)(2)} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-2}(d_i))_{\psi} \otimes \gamma^2(c^{\psi}_{i(2)(1)}) \\
= \ & \rho(\sum_i \gamma^2(c_{i(1)}) \otimes \overline{\lambda}((c_{i(2)} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(d_i))) \\
= \ & \rho(\overline{v}((c_i \otimes a'_i) \otimes d_i)),
\end{aligned}$$

and $\overline{v}$ is left (A,β)-linear since

$$\begin{aligned}
& \overline{v}(a(\sum_i (c_i \otimes a'_i) \otimes d_i)) \\
= \ & \overline{v}(\sum_i (\gamma(c^{\vartheta}_i) \otimes \Phi(\beta^{-2}(a_{\vartheta}))\beta'^{-1}(a'_i)) \otimes \gamma(d_i)) \\
= \ & \gamma^3(c^{\vartheta}_{i(1)}) \otimes \lambda((\gamma(c^{\vartheta}_{i(2)}) \otimes \Phi(\beta^{-3}(a_{\vartheta}))\beta'^{-2}(a'_i)) \otimes d_i) \\
= \ & \gamma^3(c^{\vartheta}_{i(1)}) \otimes \lambda((\gamma(c^{\psi}_{i(2)}) \otimes \Phi(\beta^{-3}(a_{\vartheta\psi}))\beta'^{-2}(a'_i)) \otimes d_i) \\
= \ & \gamma^3(c^{\vartheta}_{i(1)}) \otimes \lambda(\beta^{-2}(a_{\vartheta})(c_{i(2)} \otimes \beta'^{-3}(a'_i)) \otimes d_i) \\
= \ & a(\sum_i \gamma^2(c_{i(1)}) \otimes \lambda(c_{i(2)} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(d_i)) \\
= \ & a\overline{v}(\sum_i (c_i \otimes a'_i) \otimes d_i).
\end{aligned}$$

We have it to the reader to show that $g_1 = f_1^{-1}$.

Theorem 5.2. *Let $(\Phi, \Psi) : (A, C, \psi) \to (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. With ψ invertible, and $\vartheta : A \otimes C \to C \otimes A$ its inverse. Define the A-action on $C \otimes A'$ by*

$$a \cdot (c \otimes b') = \sum \gamma^{-1}(c^{\vartheta}) \otimes \beta^{-1}(a)_{\vartheta} b', \quad \text{where } a \in A,\ c \in C,\ b' \in B'.$$

If (C,γ) is left (C',γ')-coflat, then V_1 and V_2 are isomorphic as k-modules.

Proof. In view of the previous results, it suffices to show that $f \circ f_1 : V \to V_2$ is surjective. Starting from $\lambda \in V_2$, we have to construct a natural transformation v, that is, for all $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)^C_A$, we have to construct a morphism

$$v_M : GF(M) = (M \otimes_A A')\Box_{C'} C \to M.$$

First we remark that the map

$$\phi : M \otimes_A A' \to M \otimes_A (C \otimes A'),\quad \phi(m \otimes_A a') = \mu(m_{[0]}) \otimes_A (m_{[1]} \otimes \beta'^{-1}(a'))$$

is well-defined. Indeed,

$$\begin{aligned}
\phi(ma \otimes_A a') &= \mu((ma)_{[0]}) \otimes_A ((ma)_{[1]} \otimes \beta'^{-1}(a'))\\
&= \sum \mu(m_{[0]}) \cdot \beta(\beta^{-1}(a)_{\psi}) \otimes_A (\gamma(m_{[1]}{}^{\psi}) \otimes \beta'^{-1}(a'))\\
&= \sum \mu(m_{[0]}) \otimes_A \beta(\beta^{-1}(a)_{\psi}) \cdot (\gamma(m_{[1]}{}^{\psi}) \otimes \beta'^{-1}(a'))\\
&= \sum \mu(m_{[0]}) \otimes_A (\gamma^{-1}(\gamma(m_{[1]}{}^{\psi})^{\vartheta}) \otimes \beta^{-1}(a)_{\psi\,\vartheta}\beta'^{-1}(a'))\\
&= \mu(m_{[0]}) \otimes_A (m_{[1]} \otimes \beta'^{-1}(aa')) = \phi(m \otimes_A aa').
\end{aligned}$$

From the fact that (C, γ) is left (C', γ')-coflat, so we have

$$(M \otimes_A (C \otimes A'))\Box_{C'} C \cong M \otimes_A ((C \otimes A')\Box_{C'} C),$$

and we consider the map

$$v_M = (I_M \otimes_A \lambda) \circ \widetilde{a} \circ (\phi \Box_{C'} I_C) : GF(M) \to M \otimes_A A \cong M$$

given by

$$v_M(\sum (m_i \otimes a'_i) \otimes c_i) = \sum \mu^2(m_{i[0]}) \cdot \lambda((m_{i[1]} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(c_i)).$$

Let us first show that v is right A-linear.

$$\begin{aligned}
&v_M((\sum (m_i \otimes a'_i) \otimes c_i) \cdot a)\\
= \;& v_M(\sum (\mu(m_i) \otimes a'_i\beta'^{-1}(\alpha(\beta^{-1}(a)_{\psi}))) \otimes \gamma(c_i{}^{\psi}))\\
= \;& \sum \mu^3(m_{i[0]}) \cdot \lambda((\gamma(m_{i[1]}) \otimes \beta'^{-1}(a'_i)\beta'^{-2}(\alpha(\beta^{-1}(a)_{\psi}))) \otimes c_i{}^{\psi})\\
= \;& \sum \mu^3(m_{i[0]}) \cdot \lambda((\gamma(m_{i[1]}) \otimes \beta'^{-1}(a'_i)\beta'^{-1}(\alpha\beta^{-1}(\beta^{-1}(a)_{\psi}))) \otimes \gamma(\gamma^{-1}(c_i{}^{\psi})))\\
= \;& \sum \mu^3(m_{i[0]}) \cdot \lambda((\gamma(m_{i[1]}) \otimes \beta'^{-1}(a'_i)\beta'^{-1}(\alpha(\beta^{-2}(a)_{\psi}))) \otimes \gamma(\gamma^{-1}(c_i)^{\psi}))\\
= \;& \sum \mu^3(m_{i[0]}) \cdot \lambda(((m_{i[1]} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(c_i)) \cdot \beta^{-1}(a))\\
= \;& \sum \mu^3(m_{i[0]}) \cdot (\lambda((m_{i[1]} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(c_i))\beta^{-1}(a))\\
= \;& \sum (\mu^2(m_{i[0]}) \cdot \lambda((m_{i[1]} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(c_i))) \cdot a\\
= \;& v_M(\sum (m_i \otimes a'_i) \otimes c_i) \cdot a.
\end{aligned}$$

v is right C-colinear since

$$\begin{aligned}
&\rho^r(v_M(\sum(m_i \otimes a'_i) \otimes c_i)) \\
= \quad & \rho^r(\sum \mu^2(m_{i[0]}) \cdot \lambda((m_{i[1]} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(c_i))) \\
= \quad & \sum \mu^2(m_{i[0][0]}) \cdot (\beta^{-1}(\lambda((m_{i[1]} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(c_i))))_\psi \otimes \gamma(\gamma^2(m_{i[0][1]})^\psi) \\
= \quad & \sum \mu(m_{i[0]}) \cdot (\lambda((m_{i[1](2)} \otimes \beta^{-2}(a'_i)) \otimes \gamma^{-2}(c_i)))_\psi \otimes \gamma(\gamma^2(m_{i[1](1)})^\psi) \\
\overset{(5.1)}{=} \quad & \sum \mu(m_{i[0]}) \cdot \lambda((\gamma^{-1}(m_{i[1]}) \otimes \beta^{-2}(a'_i)) \otimes \gamma^{-1}(c_{i[1]})) \otimes \gamma(c_{i[2]}) \\
= \quad & \sum v_M((\mu^{-1}(m_i) \otimes \beta^{-1}(a'_i)) \otimes c_{i[1]}) \otimes \gamma(c_{i[2]}) \\
= \quad & \sum v_M(\sum(m_i \otimes a'_i) \otimes c_i)_{[0]} \otimes v_M(\sum(m_i \otimes a'_i) \otimes c_i)_{[1]}.
\end{aligned}$$

Let us show that v is natural. Let $g : (M, \mu) \to (N, \nu)$ be a morphism in $\widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)^C_{A'}$ and take $x = \sum(m_i \otimes a'_i) \otimes c_i \in (M \otimes_A A')\Box_{C'} C$. Then

$$\begin{aligned}
v_N(GF(g))(x) &= \sum v_N((g(m_i) \otimes a'_i) \otimes c_i) \\
&= \sum \mu^2(g(m_{i[0]})) \cdot \lambda((m_{i[1]} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(c_i)) \\
&= \sum g(\mu^2(m_{i[0]}) \cdot \lambda((m_{i[1]} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(c_i))) \\
&= \sum g(v_M(x)).
\end{aligned}$$

Finally, we have to show that $f_1(f(v)) = \lambda$. Indeed, we have

$$\begin{aligned}
&(\widetilde{l_A} \circ (\varepsilon_C \otimes I_A))(v_{C\otimes A}(\sum((c_i \otimes 1_A) \otimes a'_i) \otimes d_i)) \\
= \quad & (\widetilde{l_A} \circ (\varepsilon_C \otimes I_A))(\sum(\gamma^2(c_{i(1)}) \otimes 1_A) \cdot \lambda((c_{i(2)} \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(d_i))) \\
= \quad & \sum 1_A \lambda((\gamma^{-1}(c_i) \otimes \beta'^{-1}(a'_i)) \otimes \gamma^{-1}(d_i))) \\
= \quad & \sum \lambda((c_i \otimes a'_i) \otimes d_i)),
\end{aligned}$$

as needed.

Corollary 5.3. *Let $(\Phi, \Psi) : (A, C, \psi) \to (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. with ψ invertible, and $\vartheta : A \otimes C \to C \otimes A$ its inverse. If (C, γ) is left (C', γ')-coflat, then induction functor $F : \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi)^C_A \to \widetilde{\mathscr{H}}(\mathscr{M}_k)(\psi')^{C'}_{A'}$ is separable if and only if there exists $\lambda \in V_2$ such that*

$$\lambda((\gamma^{-1}(c_{(1)}) \otimes 1_{A'}) \otimes c_{(2)}) = \varepsilon(c)1_A \tag{5. 2}$$

for all $c \in c$ and $a \in A$. F is full and faithful if and only if $\eta_{C\otimes A}$ is an isomorphism.

Proof. If F is separable, then there exists $v \in V$ such that $v \circ \eta$ is the identity natural transformation, in particular

$$v_{C\otimes A} \circ \eta_{C\otimes A} = I_{C\otimes A}.$$

Write $\overline{v} = f(v)$ and $\lambda = f_1(\overline{v})$, and apply both sides to $c \otimes 1_A$:

$$\overline{v}((\gamma^{-1}(c_{(1)}) \otimes \Phi((1_A)_\psi)) \otimes c^\psi_{(2)}) = c \otimes 1_A,$$

and (5.2) follows after we apply ε to the first factor. Conversely, if $\lambda \in V_2$ satisfies (5.2), and v is the natural transformation corresponding to λ, then

$$\begin{aligned} v_M(\eta_M(m)) &= v_M((\mu^{-1}(m_{[0]}) \otimes 1'_A) \otimes m_{[1]}) \\ &= \mu(m_{[0][0]}) \otimes \lambda((\gamma^{-1}(m_{[0][1]}) \otimes 1'_A) \otimes \gamma^{-1}(m_{[1]})) \\ &= m_{[0]} \otimes \lambda((\gamma^{-1}(m_{[1](1)}) \otimes 1'_A) \otimes m_{[1](2)}) \\ &= m_{[0]})\varepsilon(m_{[1]})1_A = m. \end{aligned}$$

The second statement is proved in the same way.

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