Filomat 33:10 (2019), 3023–3034
https://doi.org/10.2298/FIL1910023G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Difference of Operators with Different Basis Functions

Vijay Gupta[a], Ana Maria Acu[b]

[a] *Department of Mathematics*
Netaji Subhas University of Technology
(formerly Netaji Subhas Institute of Technology)
Sector 3 Dwarka, New Delhi-110078, INDIA
[b] *Department of Mathematics and Informatics*
Lucian Blaga University of Sibiu
Str. Dr. I. Ratiu, No. 5-7, Sibiu, 550012, Romania

Abstract. In the recent years several researchers have studied problems concerning the difference of two linear positive operators, but all the available literature on this topic is for operators having same basis functions. In the present paper, we deal with the general quantitative estimate for the difference of operators having different basis functions. In the end we provide some examples. The estimates for the differences of two operators can be obtained also using classical result of Shisha and Mond. Using numerical examples we will show that for particular cases our result improves the classical one.

1. Introduction

Recently Acu et al. [1] and Aral et al. [2] established some interesting results for the difference of operators in order to generalize the problem posed by A. Lupaş [12] on polynomial differences. Recently Gupta in [8] and [9] established some results on the difference of operators having same basis functions. Some of the results on this topic are compiled in the recent book by Gupta et al [10]. Let $I \subseteq \mathbb{R}$ be an interval and $C(I) = \{f : I \to \mathbb{R}$ continous $\}$, $C(I)$ containing the polynomials and $C_B(I)$ the space of all $f \in C(I)$ such that

$$\|f\| = \sup\{|f(x)| : x \in I\} < \infty.$$

Also, let $F : C(I) \to \mathbb{R}$ be a positive linear functional such that $F(e_0) = 1$. Denote

$$b^F := F(e_1), \mu_r^F = F(e_1 - b^F e_0)^r, r \in \mathbb{N}.$$

An inequality for the functional F was establish in [1] as follows:

Lemma 1.1. *[1] Let $f \in C(I)$ with $f'' \in C_B(I)$. Then*

$$|F(f) - f(b^F)| \leq \frac{\mu_2^F}{2}\|f''\|.$$

2010 *Mathematics Subject Classification*. Primary 41A25 ; Secondary 41A30
Keywords. Difference of operators, Szász operators, Baskakov operators, modulus of continuity.
Received: 18 October 2018; Accepted: 09 January 2019
Communicated by Dragan S. Djordjević
The work of the second author was financed from Lucian Blaga University of Sibiu research grants LBUS-IRG-2018-04.
Email addresses: vijaygupta2001@hotmail.com (Vijay Gupta), anamaria.acu@ulbsibiu.ro (Ana Maria Acu)

Let

$$\omega_k(f;h) := \sup\left\{\left|\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x+j\delta)\right| : |\delta| \le h, x, x+j\delta \in I\right\}$$

be the classical k^{th} order modulus of smoothness given for a compact interval I and $h \in \mathbb{R}_+$.

In order to obtain one of the main results we will mention the following lemma proved by Gonska and Kovacheva in [4]:

Lemma 1.2. *[4] If $f \in C^q[0,1]$, then for all $0 < h \le \frac{1}{2}$ there are functions $\nu \in C^{q+2}[0,1]$, such that*

i) $\|f^{(q)} - \nu^{(q)}\| \le \frac{3}{4}\omega_2(f^{(q)};h)$,

ii) $\|\nu^{(q+1)}\| \le \frac{5}{h}\omega_1(f^{(q)};h)$,

iii) $\|\nu^{(q+2)}\| \le \frac{3}{2h^2}\omega_2(f^{(q)};h)$.

In [1], Acu and Raşa gave an inequality for the functional F in terms of second modulus of continuity.

Proposition 1.3. *[1] Let $f \in C[0,1]$, $\lambda \ge 2\sqrt{\mu_2^F} > 0$. Then*

$$|F(f) - f(b^F)| \le \frac{3}{2}\omega_2\left(f, \frac{\sqrt{\mu_2^F}}{\lambda}\right)(1+\lambda^2).$$

2. Difference of operators

Let K be a set of non-negative integers and $v_{n,k}, u_{n,k} \ge 0$, $v_{n,k}, u_{n,k} \in C(I)$, $k \in K$, such that $\sum_{k\in K} u_{n,k} = \sum_{k\in K} v_{n,k} = e_0$. Consider the positive linear functionals $F_{n,k}, G_{n,k} : C(I) \to \mathbb{R}$ such that $F_{n,k}(e_0) = G_{n,k}(e_0) = 1$ and denote $D(I)$ the set of all $f \in C(I)$ for which

$$\sum_{k\in K} u_{n,k}F_{n,k}(f) \in C(I) \text{ and } \sum_{k\in K} v_{n,k}G_{n,k}(f) \in C(I).$$

Let $U_n, V_n : D(I) \to C(I)$ be two positive linear operators defined as

$$U_n(f;x) := \sum_{k\in K} u_{n,k}(x)F_{n,k}(f), \quad V_n(f;x) := \sum_{k\in K} v_{n,k}(x)G_{n,k}(f).$$

Theorem 2.1. *If $f \in D(I)$ with $f'' \in C_B(I)$, then*

$$|(U_n - V_n)(f;x)| \le \alpha(x)\|f''\| + 2\omega_1(f;\delta_1(x)) + 2\omega_1(f;\delta_2(x)), \tag{1}$$

where

$$\alpha(x) = \frac{1}{2}\sum_{k\in K}\left(u_{n,k}(x)\mu_2^{F_{n,k}} + v_{n,k}(x)\mu_2^{G_{n,k}}\right),$$

$$\delta_1^2(x) = \sum_{k\in K} u_{n,k}(x)\left(b^{F_{n,k}} - x\right)^2, \ \ \delta_2^2(x) = \sum_{k\in K} v_{n,k}(x)\left(b^{G_{n,k}} - x\right)^2.$$

Proof. We can write

$$\begin{aligned}
|(U_n - V_n)(f;x)| &\leq \sum_{k\in K} |u_{n,k}(x)F_{n,k}(f) - v_{n,k}(x)G_{n,k}(f)| \\
&\leq \sum_{k\in K} u_{n,k}(x)|F_{n,k}(f) - f(b^{F_{n,k}})| + \sum_{k\in K} v_{n,k}(x)|G_{n,k}(f) - f(b^{G_{n,k}})| \\
&+ \sum_{k\in K} u_{n,k}(x)|f(b^{F_{n,k}}) - f(x)| + \sum_{k\in K} v_{n,k}(x)|f(b^{G_{n,k}}) - f(x)| \\
&\leq \frac{1}{2}\sum_{k\in K} u_{n,k}(x)\mu_2^{F_{n,k}}\|f''\| + \frac{1}{2}\sum_{k\in K} v_{n,k}(x)\mu_2^{G_{n,k}}\|f''\| \\
&+\sum_{k\in K} u_{n,k}(x)\left(1+\frac{(b^{F_{n,k}}-x)^2}{\delta_1^2(x)}\right)\omega_1(f,\delta_1(x))+\sum_{k\in K} v_{n,k}(x)\left(1+\frac{(b^{G_{n,k}}-x)^2}{\delta_2^2(x)}\right)\omega_1(f,\delta_2(x)) \\
&= \alpha(x)\|f''\| + 2\omega_1(f;\delta_1(x)) + 2\omega_1(f;\delta_2(x)).
\end{aligned}$$

□

Remark 2.2. *If we use result of Shisha and Mond [19], we can write*

$$\begin{aligned}
|(U_n - V_n)(f;x)| &\leq |U_n(f;x) - f(x)| + |V_n(f;x) - f(x)| \\
&\leq 2\omega_1(f,\nu_1(x)) + 2\omega_1(f,\nu_2(x)),
\end{aligned}$$

where

$$\begin{aligned}
\nu_1^2(x) &= U_n\left((e_1-x)^2;x\right) = \sum_{k\in K} u_{n,k}(x)F_{n,k}\left((e_1-x)^2;x\right), \\
\nu_2^2(x) &= V_n\left((e_1-x)^2;x\right) = \sum_{k\in K} v_{n,k}(x)G_{n,k}\left((e_1-x)^2;x\right).
\end{aligned}$$

Since $F_{n,k}^2(e_1) \leq F_{n,k}(e_1^2)$ and $G_{n,k}^2(e_1) \leq G_{n,k}(e_1^2)$, it follows $\delta_i(x) \leq \nu_i(x), i = 1,2$.

Remark 2.3. *If $F_{n,k}^2(e_1) = F_{n,k}(e_1^2)$ and $G_{n,k}^2(e_1) = G_{n,k}(e_1^2)$, then the relation (1) becomes*

$$|(U_n - V_n)(f;x)| \leq 2\omega_1(f,\nu_1(x)) + 2\omega_1(f,\nu_2(x)),$$

where ν_1, ν_2 are defined in Remark 2.2.

Theorem 2.4. *Let $I = [0,1]$, $f \in C[0,1]$, $0 < h \leq \frac{1}{2}$, $x \in [0,1]$. Then*

$$|(U_n - V_n)(f;x)| \leq \frac{3}{2}\left(1 + \frac{\alpha(x)}{h^2}\right)\omega_2(f,h) + (\delta_1(x)+\delta_2(x))\frac{10}{h}\omega_1(f,h).$$

Proof. From relation (1) for $\nu \in C^2[0,1]$, we get

$$|(U_n - V_n)(\nu;x)| \leq \alpha(x)\|\nu''\| + 2(\delta_1(x)+\delta_2(x))\|\nu'\|.$$

We can write

$$\begin{aligned}
|(U_n - V_n)(f;x)| &\leq |U_n(f;x) - U_n(\nu;x)| + |U_n(\nu;x) - V_n(\nu;x)| + |V_n(\nu;x) - V_n(f;x)| \\
&\leq 2\|f-\nu\| + \alpha(x)\|\nu''\| + 2(\delta_1(x)+\delta_2(x))\|\nu'\|.
\end{aligned}$$

Using Lemma 1.2, there exists $\nu \in C^2[0,1]$ such that

$$\|f-\nu\| \leq \frac{3}{4}\omega_2(f,h);\;\; \|\nu'\| \leq \frac{5}{h}\omega_1(f,h),\;\; \|\nu''\| \leq \frac{3}{2h^2}\omega_2(f,h).$$

Therefore,

$$|(U_n - V_n)(\nu;x)| \leq \frac{3}{2}\omega_2(f,h) + \frac{3}{2h^2}\alpha(x)\omega_2(f,h) + \frac{10}{h}\left(\delta_1(x)+\delta_2(x)\right)\omega_1(f,h),$$

and the proof is complete.

□

3. Applications

3.1. *Difference of two genuine-Durrmeyer type operators*

Păltănea [17] and further Păltănea and Gonska ([5, 6]) propossed a new class of Bernstein-Durrmeyer type operators $U_n^{\rho} : C[0,1] \to \prod_n$, defined as

$$\begin{aligned} U_n^{\rho}(f;x) &:= \sum_{k=0}^{n} F_{n,k}^{\rho}(f)p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1}\left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}f(t)dt\right)p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \end{aligned}$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. These operators constitute a link between genuine Bernstein-Durrmeyer operators U_n ($\rho = 1$) and the classical Bernstein operators B_n ($\rho \to \infty$).

Stancu [20] introduced a sequence of positive linear operators $P_n^{<\alpha>} : C[0,1] \to C[0,1]$, depending on a parameter $\alpha \geq 0$ as follows

$$P_n^{<\alpha>}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}^{<\alpha>}(x),\; x \in [0,1], \tag{2}$$

where

$$p_{n,k}^{<\alpha>}(x) = \binom{n}{k}\frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}} \text{ and } x^{[n,\alpha]} := x(x-\alpha)\cdots(x-\overline{n-1}\alpha).$$

Note that for $\alpha = 0$ are obtained Bernstein operators and for $\alpha = \dfrac{1}{n}$ these operators reduce to the special case introduced by L. Lupaş and A. Lupaş [13] as follows

$$P_n^{<\frac{1}{n}>}(f;x) = \frac{2n!}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)(nx)_k(n-nx)_{n-k}. \tag{3}$$

The moments of the operators (3) are given below

$$P_n^{<\frac{1}{n}>}(e_0;x) = 1,\; P_n^{<\frac{1}{n}>}(e_1;x) = x,\; P_n^{<\frac{1}{n}>}(e_2;x) = x^2 + \frac{2x(1-x)}{n+1}.$$

In 2017, Neer and Agrawal [16] introduced a genuine-Durrmeyer type modification of the operators (3) as follows:

$$\tilde{U}_n^{\rho}(f;x) = \sum_{k=0}^{n} F_{n,k}^{\rho}(f)p_{n,k}^{<\frac{1}{n}>}(x),\; \rho > 0,\; f \in C[0,1]. \tag{4}$$

Proposition 3.1. *The following inequalities hold*

i) $$\left|\left(U_n^\rho-\tilde{U}_n^\rho\right)(f;x)\right|\le\frac{x(1-x)}{n\rho+1}\|f''\|+2\omega_1\left(f,\sqrt{\frac{x(1-x)}{n}}\right)+2\omega_1\left(f,\sqrt{\frac{2x(1-x)}{n+1}}\right),\ f''\in C[0,1];$$

ii) $$\left|\left(U_n^\rho-\tilde{U}_n^\rho\right)(f;x)\right|\le 3\omega_2\left(f,\sqrt{\frac{x(1-x)}{n\rho+1}}\right)+10(1+\sqrt{2})\sqrt{\frac{n\rho+1}{n}}\omega_1\left(f,\sqrt{\frac{x(1-x)}{n\rho+1}}\right),f\in C[0,1].$$

Proof. Denote

$$F_{n,k}^\rho(f)=G_{n,k}(f)=\begin{cases}\displaystyle\int_0^1\frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}f(t)dt, & 1\le k\le n-1,\\ f(0), & k=0,\\ f(1), & k=n.\end{cases}$$

We have

$$\begin{aligned}
&b^{F_{n,k}}=b^{G_{n,k}}=\frac{k}{n};\\
&\mu_2^{F_{n,k}}=\mu_2^{G_{n,k}}=F_{n,k}\left(e_1-b^{F_{n,k}}\right)^2=\frac{k(n-k)}{n^2(n\rho+1)};\\
&\alpha(x)=\frac{1}{2}\sum_{k=0}^{n}\left(p_{n,k}(x)+p_{n,k}^{<\frac{1}{n}>}(x)\right)\mu_2^{F_{n,k}}\\
&\quad=\frac{1}{2(n\rho+1)}\left\{B_n(e_1;x)-B_n(e_2;x)+P_n^{<\frac{1}{n}>}(e_1;x)-P_n^{<\frac{1}{n}>}(e_2;x)\right\}\\
&\quad=\frac{x(1-x)}{2(n\rho+1)}\left\{2-\frac{1}{n}-\frac{2}{n+1}\right\}\le\frac{x(1-x)}{n\rho+1},
\end{aligned}$$

where B_n is the classical Bernstein operators. Also,

$$\begin{aligned}
&\delta_1^2(x)=\sum_{k=0}^{n}p_{n,k}(x)\left(b^{F_{n,k}}-x\right)^2=B_n(e_2;x)-2xB_n(e_1;x)+x^2=\frac{x(1-x)}{n};\\
&\delta_2^2(x)=\sum_{k=0}^{n}p_{n,k}^{<\frac{1}{n}>}(x)\left(b^{G_{n,k}}-x\right)^2=P_n^{<\frac{1}{n}>}(e_2;x)-2xP_n^{<\frac{1}{n}>}(e_1;x)+x^2=\frac{2x(1-x)}{n+1}.
\end{aligned}$$

Using Theorem 2.1 and Theorem 2.4 the proof is complete. □

Example 3.2. *Let* $f(x)=\dfrac{x}{x^2+1}$, $x\in[0,1]$ *and* $\rho=2$. *From Proposition 3.1 we obtain the following estimate for the differences of the operators* U_n^ρ *and* $\tilde{U}_n^\rho$

$$|(U_n^\rho-\tilde{U}_n^\rho)(f;x)|\le E_n(f), \tag{5}$$

where

$$E_n(f)=\frac{1}{4(n\rho+1)}\|f''\|+\left(\frac{1}{\sqrt{n}}+\frac{\sqrt{2}}{\sqrt{n+1}}\right)\|f'\|,\ f\in C^2[0,1]$$

Using the result of Shisha and Mond (see Remark 2.2) and the central moments of the operators U_n^ρ *and* $\tilde{U}_n^\rho$, *namely*

$$U_n^\rho((e_1-x)^2;x)=\frac{(\rho+1)x(1-x)}{n\rho+1},\ \tilde{U}_n^\rho((e_1-x)^2;x)=\frac{(2n\rho+n+1)x(1-x)}{(n+1)(n\rho+1)},$$

Estimate for the differences of $U_n^\rho f$ and $\tilde{U}_n^\rho f$

n	$E_n(f)$	$E_n^{(SM)}(f)$
10	0.759975714300	0.847835967200
10^2	0.242531830000	0.279263643100
10^3	0.076503841440	0.088687680300
10^4	0.024159641940	0.028057503200
10^5	0.007636212520	0.008872941165
10^6	0.002414394970	0.002805882368
10^7	0.000763459548	0.000887298292
10^8	0.000241423174	0.000280588368

we get the next result

$$|(U_n^\rho - \tilde{U}_n^\rho)(f;x)| \leq E_n^{(SM)}(f), \tag{6}$$

where

$$E_n^{(SM)}(f) = \frac{1}{\sqrt{n\rho+1}}\left(\sqrt{\rho+1} + \sqrt{\frac{2n\rho+n+1}{n+1}}\right)\|f'\|,\ f \in C^1[0,1].$$

Table 3.2 contains the errors for the differences of the operators $U_n^\rho f$ and $\tilde{U}_n^\rho f$. Note that for this particular case the estimate (5) is better than the estimate given by Shisha and Mond's result (6).

3.2. Difference of Durrmeyer operators and Lupaş-Durrmeyer operators

The classical Durrmeyer operators were introduced by Durrmeyer [3] and, independently, by Lupaş [14] and are defined as

$$M_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\, f(t)\, dt,\ \ x \in [0,1]. \tag{7}$$

The moments of the operator (7) are given by:

$$M_n(e_0;x) = 1,\ M_n(e_1;x) = x + \frac{1-2x}{n+2},\ M_n(e_2;x) = \frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)}.$$

The Durrmeyer type modification of the operator (3) was introduced by Gupta et al. [11] as follows

$$D_n^{<\frac{1}{n}>}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n}>}\int_0^1 p_{n,k}(t)f(t)dt,\ f \in C[0,1]. \tag{8}$$

The moments of the operators (8) are given below (see [11])

$$D_n^{<\frac{1}{n}>}(e_0;x) = 1,\ D_n^{<\frac{1}{n}>}(e_1;x) = \frac{nx+1}{n+2},\ D_n^{<\frac{1}{n}>}(e_2;x) = \frac{n^3x^2+5n^2x-n^2x^2+3nx+2n+2}{(n+1)(n+2)(n+3)}.$$

Proposition 3.3. *The following inequalities hold*

i) $$\left|\left(M_n - D_n^{<\frac{1}{n}>}\right)(f;x)\right| \leq \frac{n(2n+1)(n-1)x(1-x)+2(n+1)^2}{2(n+1)(n+3)(n+2)^2}\|f''\|$$
$$+ 2\omega_1\left(f, \sqrt{\frac{nx(1-x)+(2x-1)^2}{(n+2)^2}}\right) + 2\omega_1\left(f, \sqrt{\frac{2n^2x(1-x)+(2x-1)^2(n+1)}{(n+2)^2(n+1)}}\right),\ f'' \in C[0,1];$$

ii) $\left|\left(M_n - D_n^{<\frac{1}{n}>}\right)(f;x)\right| \leq 3\omega_2\left(f, \sqrt{\frac{n(2n+1)(n-1)x(1-x)+2(n+1)^2}{2(n+1)(n+3)(n+2)^2}}\right)$
$+ 20\sqrt{3}\omega_1\left(f, \sqrt{\frac{n(2n+1)(n-1)x(1-x)+2(n+1)^2}{2(n+1)(n+3)(n+2)^2}}\right), f \in C[0,1].$

Proof. The Durrmeyer operators and the Lupaş-Durrmeyer operators can be written as follows:

$$M_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)F_{n,k}(f), \ \ D_n^{<\frac{1}{n}>}(f;x) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n}>}(x)G_{n,k}(f),$$

where

$$F_{n,k}(f) = G_{n,k}(f) = (n+1)\int_0^1 p_{n,k}(t)f(t)dt.$$

We have

$$b^{F_{n,k}} = b^{G_{n,k}} = \frac{k+1}{n+2};$$
$$\mu_2^{F_{n,k}} = \mu_2^{G_{n,k}} = F_{n,k}\left(e_1 - b^{F_{n,k}}\right)^2 = \frac{(k+1)(n-k+1)}{(n+2)^2(n+3)};$$
$$\begin{aligned}\alpha(x) &= \frac{1}{2}\sum_{k=0}^{n}\left(p_{n,k}(x) + p_{n,k}^{<\frac{1}{n}>}(x)\right)\mu_2^{F_{n,k}} \\ &= \frac{1}{2(n+2)^2(n+3)}\left\{-n^2B_n(e_2;x) + n^2B_n(e_1;x) - n^2P_n^{<\frac{1}{n}>}(e_2;x) + n^2P_n^{<\frac{1}{n}>}(e_1;x) + 2(n+1)\right\} \\ &= \frac{1}{2(n+2)^2(n+3)}\left\{2n^2x(1-x) - nx(1-x) - \frac{2n^2x(1-x)}{n+1} + 2(n+1)\right\} \\ &= \frac{n(2n+1)(n-1)x(1-x)+2(n+1)^2}{2(n+1)(n+3)(n+2)^2},\end{aligned}$$

where B_n is the classical Bernstein operators. Also,

$$\delta_1^2(x) = \sum_{k=0}^{n} p_{n,k}(x)\left(b^{F_{n,k}} - x\right)^2 = \frac{nx(1-x)+(2x-1)^2}{(n+2)^2};$$
$$\delta_2^2(x) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n}>}(x)\left(b^{G_{n,k}} - x\right)^2 = \frac{2n^2x(1-x)+(2x-1)^2(n+1)}{(n+2)^2(n+1)}.$$

Using Theorem 2.1 and Theorem 2.4 the proof is complete. □

Example 3.4. *Let $f(x) = sin(2\pi x)$, $x \in [0,1]$. From Proposition 3.3 we obtain the following estimate for the differences of the operators M_n and $D_n^{<\frac{1}{n}>}$*

$$|(M_n - D_n^{<\frac{1}{n}>})(f;x)| \leq E_n(f), \tag{9}$$

where

$$E_n(f) = \frac{n(2n+1)(n-1)+8(n+1)^2}{8(n+2)^2(n+3)(n+1)}\|f''\| + \frac{1}{n+2}\left(\sqrt{n+4} + \sqrt{\frac{2(n^2+2n+2)}{n+1}}\right)\|f'\|, \ f \in C^2[0,1]$$

Estimate for the differences of M_n and $D_n^{<\frac{1}{n}>}$

n	$E_n(f)$	$E_n^{(SM)}(f)$
10	5.11006052700	5.12324449500
10^2	1.59818810900	1.93302197800
10^3	0.48908880900	0.62371818010
10^4	0.15266270140	0.19764065390
10^5	0.04806670559	0.06251225990
10^6	0.01517880741	0.01976851709
10^7	0.00479783008	0.00625136679
10^8	0.00151699380	0.00197685616

Using the result of Shisha and Mond (see Remark 2.2) and the central moments of the operators M_n and $D_n^{<\frac{1}{n}>}$, namely

$$M_n((e_1-x)^2;x)=\frac{2x(1-x)(n-3)+2}{(n+2)(n+3)},\ D_n^{<\frac{1}{n}>}((e_1-x)^2;x)=\frac{x(1-x)(3n^2-5n-6)+2(n+1)}{(n+1)(n+2)(n+3)},$$

we get the next result

$$|(M_n-D_n^{<\frac{1}{n}>})(f;x)|\le E_n^{(SM)}(f), \tag{10}$$

where

$$E_n^{(SM)}(f)=2\left(\sqrt{\frac{n+1}{2(n+2)(n+3)}}+\sqrt{\frac{3n^2+3n+2}{4(n+1)(n+2)(n+3)}}\right)\|f'\|,\ f\in C^1[0,1].$$

Table 3.4 contains the errors for the differences of the operators M_n and $D_n^{<\frac{1}{n}>}$. Note that for this particular case the estimate (9) is better than the estimates given by Shisha and Mond's result (10).

3.3. Baskakov operators and Szász operators

This section deals with the quantitative estimates for the differences of Szász operators and the Baskakov operators.

The Baskakov operators are defined as

$$V_n(f;x)=\sum_{k=0}^{\infty}v_{n,k}(x)F_{n,k}(f), \tag{11}$$

where

$$v_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},\ F_{n,k}(f)=f\left(\frac{k}{n}\right).$$

Remark 3.5. *We have*

$$b^{F_{n,k}}=F_{n,k}(e_1)=\frac{k}{n},$$

$$\mu_2^{F_{n,k}}:=F_{n,k}(e_1-b^{F_{n,k}}e_0)^2=0$$

Some of the moments of Baskakov operators are given below

$$V_n(e_0,x)=1,\ V_n(e_1,x)=x,\ V_n(e_2,x)=\frac{x^2(n+1)+x}{n},$$
$$V_n((e_1-x)^2;x)=\frac{x(x+1)}{n}.$$

The Szász operators are defined as

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) G_{n,k}(f), \tag{12}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!},\ G_{n,k}(f) = f\left(\frac{k}{n}\right).$$

Remark 3.6. *We have*

$$b^{G_{n,k}} = G_{n,k}(e_1) = \frac{k}{n}$$

$$\mu_2^{G_{n,k}} := G_{n,k}(e_1 - b^{G_{n,k}}e_0)^2 = 0$$

Some of the moments of Szász operators defined by (12) are given as:

$$S_n(e_0;x) = 1,\ S_n(e_1;x) = x,\ S_n(e_2;x) = x^2 + \frac{x}{n}$$

$$S_n((e_1-x)^2;x) = \frac{x}{n}.$$

Using Remark 2.3 the following quantitative estimate for the difference of Baskakov operators and Szász operators is obtained.

Proposition 3.7. *Let $f \in C_B[0,\infty)$ and $x \in [0,\infty)$. Then for $n \in \mathbb{N}$, we have*

$$|(V_n - S_n)(f,x)| \le 2\omega_1\left(f, \sqrt{\frac{x(1+x)}{n}}\right) + 2\omega_1\left(f, \sqrt{\frac{x}{n}}\right).$$

3.4. Baskakov and the integral modification of the Szász-Baskakov operators

The integral modification of the Szász-Baskakov type operators considered in [18] and later improved by Gupta in [7] are defined as

$$M_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) H_{n,k}(f), \tag{13}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ is the Szász basis function defined in (12) and

$$H_{n,k}(f) = (n-1)\int_0^{\infty} v_{n,k}(t) f(t)dt,$$

with $v_{n,k}(t)$ the Baskakov basis functions defined by

$$v_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

Remark 3.8. *We have*

$$b^{H_{n,k}} = H_{n,k}(e_1) = \frac{k+1}{n-2}.$$

Also, we have

$$\begin{aligned}\mu_2^{H_{n,k}} &:= H_{n,k}(e_1 - b^{H_{n,k}}e_0)^2 \\ &= H_{n,k}(e_2,x) + \left(\frac{k+1}{n-2}\right)^2 - 2H_{n,k}(e_1,x)\left(\frac{k+1}{n-2}\right) \\ &= \frac{(k+2)(k+1)}{(n-2)(n-3)} - \left(\frac{k+1}{n-2}\right)^2 = \frac{k^2+nk+n-1}{(n-2)^2(n-3)}.\end{aligned}$$

Following proposition provides the estimate for the difference of Baskakov and integral modification of Szász-Baskakov operators, which is application of Theorem 2.1:

Proposition 3.9. *Let $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, $n > 3$, we have*

$$|(M_n - V_n)(f;x)| \leq \frac{n^2x^2 + nx + n^2x + n - 1}{2(n-2)^2(n-3)}\|f''\| + 2\omega_1\left(f; \sqrt{\frac{x(1+x)}{n}}\right) + 2\omega_1\left(f; \frac{\sqrt{4x^2 + (4+n)x + 1}}{n-2}\right).$$

Proof. By Remark 3.5 and Remark 3.6, we have

$$\begin{aligned}\alpha(x) &= \frac{1}{2}\left[\sum_{k=0}^{\infty} v_{n,k}(x)\mu_2^{F_{n,k}} + \sum_{k=0}^{\infty} s_{n,k}(x)\mu_2^{H_{n,k}}\right] \\ &= \frac{1}{2}\sum_{k=0}^{\infty} s_{n,k}(x)\frac{k^2 + nk + n - 1}{(n-2)^2(n-3)} \\ &= \frac{n^2x^2 + nx + n^2x + n - 1}{2(n-2)^2(n-3)}.\end{aligned}$$

Further by using Remark 3.5, we get

$$\begin{aligned}\delta_1^2(x) &= \sum_{k=0}^{\infty} v_{n,k}(x)(b^{F_{n,k}} - x)^2 \\ &= \sum_{k=0}^{\infty} v_{n,k}(x)\left(\frac{k}{n} - x\right)^2 = \frac{x(1+x)}{n}.\end{aligned}$$

Finally by Remark 3.6 and Remark 3.8, we get

$$\begin{aligned}\delta_2^2(x) &= \sum_{k=0}^{\infty} s_{n,k}(x)(b^{H_{n,k}} - x)^2 \\ &= \sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k+1}{n-2} - x\right)^2 = \frac{4x^2 + (4+n)x + 1}{(n-2)^2}.\end{aligned}$$

□

3.5. Lupaş and Szász-Durrmeyer operators

Lupaş [12] proposed the following operators

$$L_n(f;x) = \sum_{k=0}^{\infty} l_{n,k}(x)L_{n,k}(f), \tag{14}$$

where $l_{n,k}(x) = \frac{2^{-nx}(nx)_k}{k!2^k}$ and

$$L_{n,k}(f) = f\left(\frac{k}{n}\right).$$

Remark 3.10. *We have*

$$b^{L_{n,k}} = L_{n,k}(e_1) = \frac{k}{n}$$

$$\mu_2^{L_{n,k}} := L_{n,k}(e_1 - b^{L_{n,k}}e_0)^2 = 0.$$

Some of the moments of Lupaş operators are given as:

$$L_n(e_0,x)=1,\ L_n(e_1,x)=x,\ L_n(e_2,x)=x^2+\frac{2x}{n}$$
$$L_n((e_1-x)^2;x)=\frac{2x}{n}.$$

The Szász-Durrmeyer operators [15] are defined as

$$D_n(f;x)=\sum_{k=0}^{\infty}s_{n,k}(x)Q_{n,k}(f), \tag{15}$$

where $s_{n,k}(x)$ is the Szász basis function defined in (12) and

$$Q_{n,k}(f)=n\int_0^{\infty}s_{n,k}(t)f(t)dt.$$

Remark 3.11. *By simple computation, we have*

$$Q_{n,k}(e_r)=\frac{(k+r)!}{k!n^r}$$

implying

$$b^{Q_{n,k}}=Q_{n,k}(e_1)=\frac{k+1}{n}.$$

Next, we have

$$\mu_2^{Q_{n,k}}:=Q_{n,k}(e_1-b^{Q_{n,k}}e_0)^2=\frac{k+1}{n^2}$$

Following result provides the estimate for the difference of Lupaş and Szász-Durrmeyer operators, which is application of Theorem 2.1:

Proposition 3.12. *Let $f^{(s)}\in C_B[0,\infty), s\in\{0,1,2\}$ and $x\in[0,\infty)$, then for $n\in\mathbb{N}$, we have*

$$|(D_n-L_n)(f;x)|\le\frac{nx+1}{2n^2}\|f''\|+2\omega_1\left(f;\sqrt{\frac{2x}{n}}\right)+2\omega_1\left(f;\frac{\sqrt{nx+1}}{n}\right).$$

Proof. By Remark 3.10 and Remark 3.11, we have

$$\begin{aligned}\alpha(x)&=\frac{1}{2}\left[\sum_{k=0}^{\infty}l_{n,k}(x)\mu_2^{L_{n,k}}+\sum_{k=0}^{\infty}s_{n,k}(x)\mu_2^{Q_{n,k}}\right]\\&=\frac{1}{2}\sum_{k=0}^{\infty}s_{n,k}(x)\frac{k+1}{n^2}=\frac{nx+1}{2n^2}.\end{aligned}$$

Further by using Remark 3.10, we get

$$\begin{aligned}\delta_1^2(x)&=\sum_{k=0}^{\infty}l_{n,k}(x)(b^{L_{n,k}}-x)^2\\&=\sum_{k=0}^{\infty}l_{n,k}(x)\left(\frac{k}{n}-x\right)^2=\frac{2x}{n}.\end{aligned}$$

Finally by Remark 3.6 and Lemma 3.11, we get

$$\begin{aligned}\delta_2^2 &= \sum_{k=0}^{\infty} s_{n,k}(x)(b^{Q_{n,k}} - x)^2 \\ &= \sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k+1}{n} - x\right)^2 = \frac{nx+1}{n^2}.\end{aligned}$$

□

References

[1] A. M. Acu and I. Rasa, New estimates for the differences of positive linear operators, Numer. Algorithms 73(2016), 775-789.
[2] A. Aral, D. Inoan and I. Rasa, On differences of linear positive operators, Anal. Math. Phys. 9(3) (2019), 1227–1239.
[3] J. L. Durrmeyer, Une formule d'inversion de la transforme de Laplace: Applications a la theorie des moments, These de 3e cycle, Paris, (1967).
[4] H. Gonska, R. Kovacheva, The second order modulus revised: remarks, applications, problems, Confer. Sem. Mat. Univ. Bari, 257 (1994), 1-32.
[5] H. Gonska, R. Păltănea, Quantitative convergence theorems for a class of Bernstein-Durrmeyer operators preserving linear functions, Ukrainian Math. J. 62, 2010, 913-922.
[6] H. Gonska, R. Păltănea, Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions, Czechoslovak Math. J. 60(135), 2010, 783-799.
[7] V. Gupta, A note on modified Szász operators, Bull. Inst. Math. Acad. Sinica 21(3)(1993), 275-278.
[8] V. Gupta, Differences of operators of Lupas type, Constructive Mathematical Analysis 1(1) (2018), 9-14.
[9] V. Gupta, On difference of operators with applications to Szász type operators, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemáticas, 113 (3) (2019), 2059–2071.
[10] V. Gupta, T. M. Rassias, P. N. Agrawal and A. M. Acu, Estimates for the Differences of Positive Linear Operators. In: Recent Advances in Constructive Approximation Theory. Springer Optimization and Its Applications, vol 138, (2018), Springer, Cham.
[11] V. Gupta and T. M. Rassias, Lupaş-Durrmeyer operators based on Polya distribution, Banach J. Math. Anal. 8 (2) (2014) 146-155.
[12] A. Lupaş, The approximation by means of some linear positive operators. In: Approximation Theory (M.W. Muller others, eds), pp. 201-227. Akademie-Verlag, Berlin (1995).
[13] L. Lupaş, A. Lupaş, *Polynomials of binomial type and approximation operators*, Studia Univ. Babes-Bolyai, Mathematica 32(1987), 61-69.
[14] A. Lupaş, Die Folge der Beta operatoren, Dissertation, Universität Stuttgart, 1972.
[15] S. M. Mazhar, V. Totik, Approximation by modified Szász operators, Acta Sci. Math. (Szeged) 49 (1985), 257–269.
[16] T. Neer, P.N. Agrawal, A genuine family of Bernstein-Durrmeyer type operators based on Pólya basis functions, Filomat 31:9 (2017), 2611-2623.
[17] R. Păltănea, *A class of Durrmeyer type operators preserving linear functions*, Ann. Tiberiu Popoviciu Sem. Funct. Equat. Approxim. Convex. (Cluj-Napoca) 5, 2007, 109-117.
[18] G. Prasad, P. N. Agrawal and H. S. Kasana, Approximation of functions on $[0, \infty]$ by a new sequence of modified Szász operators, Math. Forum 6(2)(1983), 1-11.
[19] O. Shisha, B. Mond, The degree of convergence of linear positive operators, Proc. Nat. Acad. Sci. USA, 60(1968), 1196-1200.
[20] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roum. Math. Pures et Appl., **13**, 1968, 1173-1194.

Further Readings:
V. Gupta, General estimates for the difference of operators, Computational and Mathematical Methods 1(2) (2019) e1018 https://doi.org/10.1002/cmm4.1018