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Applications of Third Order Differential Subordination and Superordination Involving Generalized Struve Function

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Abstract. In the present paper, by making use of the linear operator associated with generalized Struve functions suitable classes of admissible functions are investigated and the dual properties of the third-order differential subordinations are presented. As a consequence, various sandwich-type results are established for a class of univalent analytic functions involving generalized Struve functions. Relevant connections of the new results presented here with those that were considered in earlier works are pointed out.

1. Introduction and Preliminaries

The special functions have great importance in geometric function theory especially after the proof of famous Bieberbach conjecture which is solved by de-Branges [12]. Since then, there are extensive literature dealing with various geometric aspects of analytic univalent function involving special functions. In the present paper, we are dealing with one of such function which is introduced and studied by Struve [39] (also see [1], [44]), is the series solution of inhomogeneous second order Bessel differential equation. Struve function and its generalizations are applied to various areas of applied mathematics and physics. For example see the recent works [2, 7–11] and the references therein.

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Denote $\mathcal{H}[\kappa, n]$ $(n \in \mathbb{N} := \{1, 2, 3, \cdots\})$, the subclass of $\mathcal{H}$ consists of functions of the form $f(z) = \kappa + a_n z^n + a_{n+1} z^{n+1} + \cdots,\quad (z \in \mathbb{D})$ and $\mathcal{A}(\subset \mathcal{H})$ be the class of functions analytic in $\mathbb{D}$ and has the Taylor-Maclaurin series representation: $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. We consider a new linear operator $S_{a,c} : \mathcal{A} \longrightarrow \mathcal{A}$, which is defined by the Hadamard product or convolution as follows:

$$S_{a,c} f(z) = U_{a,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n}{(3/2)_n (a)_n} a_{n+1} z^{n+1}, \qquad (z \in \mathbb{D}), \tag{1}$$

where $*$ denote the convolution or Hadamard product [32] and $U_{a,c} := zU_{p,b,c}(z)$ is the normialized form of *generalized Struve function of order p* (cf. [28], [34] and [46]) having following series representation:

$$U_{p,b,c}(z) = 2^p \sqrt{\pi}\,\Gamma\left(p + \frac{b+2}{2}\right) z^{\frac{-p-1}{2}} \sum_{n=0}^{\infty} \frac{(-c)^n \left(\sqrt{z}/2\right)^{2n+p+1}}{\Gamma(n + 3/2)\Gamma(p + n + (b+2)/2)} = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(3/2)_n (a)_n} z^n, \quad (z \in \mathbb{C}), \tag{2}$$

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where $p, b, c \in \mathbb{C}$, $a = p + (b+2)/2 \neq 0, -1, -2, \cdots$, and $(\lambda)_n$ is the Pochhammer symbol (or *shifted factorial*) defined in terms of the gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & ;\ (n=0), \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & ;\ (n \in \mathbb{N} := \{1, 2, \cdots\}). \end{cases}$$

In [20], Habibullah et al. derived the conditions on parameters p, b and c such that $zU_{p,b,c}(z)$ is univalent in $\mathbb{D}$. We observed that, for suitable choices of the parameters b and c in (2) we obtain following new operators:

(i) The operator $\mathfrak{S} : \mathcal{A} \longrightarrow \mathcal{A}$ familiar with Struve function, defined by

$$\mathfrak{S}f(z) = zU_{p,1,1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1} z^{n+1}}{(p+3/2)_n (2n+1)!}. \tag{3}$$

(ii) The operator $\mathfrak{I} : \mathcal{A} \longrightarrow \mathcal{A}$ related with modified Struve function, defined by

$$\mathfrak{I}f(z) = zU_{p,1,-1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{a_{n+1} z^{n+1}}{(p+3/2)_n (2n+1)!}. \tag{4}$$

It is easy to verify from (1) that

$$z(S_{a+1,c}f(z))' = aS_{a,c}f(z) - (a-1)S_{a+1,c}f(z), \tag{5}$$

where $a = p + (b+1)/2 \neq 0, -1, -2 \cdots$.

The theory of first and second order differential subordination and superordination have been used by numerous authors to solve various problems in geometric function theory. For detail treatment we refer the monographs [13, 23, 35], also see [5, 6, 14–16, 18, 19, 24–26, 33, 37, 38, 45] and the references therein. It is challenging to consider the dual problems for higher order cases. Though, the concept of third order differential subordination have originally found in the work of Ponnusamy and Juneja [29], the recent work due to Antonino and Miller [4] revive the attention among the researcher in this directions. In 2014, Tang et al. [43] introduced the concept of third order differential superordination, as a generalization of the second order case. In the recent years, few works have been carried out on results related to the third order differential subordination and superordination in the different context. For example see [17, 21, 22, 27, 31, 36, 40–43]. In the present investigation our aim is to determine third order differential subordination and superordination of generalized Struve function by using the technique developed in [4] and [43]. Thus, to achieve our aim we recall some definitions and preliminary results from the theory of differential subordination and superordination.

Suppose that f and g are in $\mathcal{H}$. We say that f is *subordinate* to g, (or g is *superordinate* to f), write as $f \prec g$ in $\mathbb{D}$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a function $\omega \in \mathcal{H}$, satisfying the conditions of the Schwarz lemma (i.e. $\omega(0) = 0$ and $|\omega(z)| < 1$) such that $f(z) = g(\omega(z))$ $(z \in \mathbb{D})$. It follows that $f(z) \prec g(z)$ $(z \in \mathbb{D}) \Longrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. In particular, if g is *univalent* in $\mathbb{D}$, then the reverse implication also holds (cf.[23]).

Definition 1.1. *[[4], Definition 1, p.440]. Let $\psi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in $\mathbb{D}$. If the function $p(z)$ is analytic in $\mathbb{D}$ and satisfies the following third-order differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{6}$$

then $p(z)$ is called a solution *of the differential subordination.*

A univalent function $q(z)$ is called a *dominant* of the solutions of the differential subordination,or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (6). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (6) is said to be the *best dominant*.

Definition 1.2. *[43] Let $\psi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in $\mathbb{D}$. If the function $p(z)$ and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in $\mathbb{D}$ and satisfies the following third-order differential superordination*

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \tag{7}$$

then $p(z)$ is called a solution *of the differential superordination. An analytic function $q(z)$ is called a* subordinant *of the solutions of the differential superordination, or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (7).*

A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinant $q(z)$ of (7) is said to be the *best subordinant*. We note that both the best dominant and best subordinant are unique up to rotation of $\mathbb{D}$.

Definition 1.3. *[[4], Definition 2, p.441]. Let $\mathcal{Q}$ denote the set of functions q that are analytic and univalent on the set $\overline{\mathbb{D}} \setminus E(q)$, where $E(q) = \{\xi : \xi \in \partial\mathbb{D} : \lim_{z\to\xi} q(z) = \infty\}$ and are such that* $\min |q'(\xi)| = \rho > 0$ *for $\xi \in \partial\mathbb{D} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = \kappa$ be denoted by $\mathcal{Q}(\kappa), \mathcal{Q}(0) = \mathcal{Q}_0$ and $\mathcal{Q}(1) = \mathcal{Q}_1$.*

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions was given by Antonino and Miller.

Definition 1.4. *[[4], Definition 3, p.449]. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$, and $n \in \mathbb{N}\setminus\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ achieving the admissibility condition: $\psi(r, s, t, u; z) \notin \Omega$ whenever*

$$r = q(\zeta), \quad s = k\zeta q'(\zeta), \quad \Re\left(\frac{t}{s} + 1\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right), \quad \textit{and} \quad \Re\left(\frac{u}{s}\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq n$.

The next lemma is the foundation result in the theory of third-order differential subordination.

Lemma 1.5. *[[4], Theorem 1, p.449]. Let $q \in \mathcal{H}[\kappa, n]$ with $n \geq 2$, $q \in \mathcal{Q}(\kappa)$ achieving the following conditions: $\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0$, $\left|\frac{zp'(z)}{q'(\zeta)}\right| \leq k$, where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D}\setminus E(q)$ and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and $\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \subset \Omega$, then $p(z) \prec q(z), \qquad (z \in \mathbb{D})$.*

Definition 1.6. *[43] Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[\kappa, n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t, u; z) \in \Omega$ whenever*

$$r = q(z), \ \ s = \frac{zq'(z)}{m}, \ \ \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right), \ \ \textit{and} \ \ \Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D}$ and $m \geq n \geq 2$.

Lemma 1.7. *[43] Let $q \in \mathcal{H}[\kappa, n]$ with $\psi \in \Psi'_n[\Omega, q]$. If $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is univalent in $\mathbb{D}$ and $p \in \mathcal{Q}(\kappa)$ satisfying the following conditions: $\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0$, $\left|\frac{zp'(z)}{q'(z)}\right| \leq m$, where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq n \geq 2$, then $\Omega \subset \{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in \mathbb{D}\}$ implies that $q(z) \prec p(z), \qquad (z \in \mathbb{D})$.*

In this investigation, by considering suitable classes of admissible functions, we obtained some interesting inclusion results on third order differential subordination and superordination involving $S_{a,c}$. More precisely, we have shown that the sandwich-type relations of the form

$$q_1(z) \prec \Xi(z) \prec q_2(z), \qquad (z \in \mathbb{D}),$$

holds, where q_1, q_2 are univalent in $\mathbb{D}$ with suitable normalization, and $\Xi(z)$ is one of the variant of $S_{a,c}f(z)$. The proof of the main results are much akin to that of results found in [36] and [43] and hence omitted.

2. Results based on differential subordination

In this section the following class of admissible functions is introduced which are required to prove the main third-order differential subordination theorems involving the operator $S_{a,c}$ defined by (1).

Definition 2.1. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_S[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$ whenever*

$$\alpha = q(\zeta),\ \beta = \frac{k\zeta q'(\zeta) + (a-1)q(\zeta)}{a},\ \Re\left(\frac{a(a-1)\gamma - (a-2)(a-1)\alpha}{a\beta - (a-1)\alpha} - (2a-3)\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{a(a-1)((1-a)\alpha + (3a\beta + (1-3a)\gamma + (a-2)\delta)}{\alpha + a(\beta - \alpha)}\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq 2$.

Theorem 2.2. *Let $\phi \in \Phi_S[\Omega, q]$. If the function $f \in \mathcal{A}, q \in \mathcal{Q}_0$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \quad , \left|\frac{S_{a,c}f(z)}{q'(\zeta)}\right| \leq k, \tag{8}$$

and

$$\left\{\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z) : z \in \mathbb{D}\right\} \subset \Omega, \tag{9}$$

then $S_{a+1,c}f(z) \prec q(z), \quad (z \in \mathbb{D})$.

Proof. Define the analytic function $p(z)$ in $\mathbb{D}$ by

$$p(z) = S_{a+1,c}f(z). \tag{10}$$

From equation (10) and (5), we have $S_{a,c}f(z) = \frac{zp'(z)+(a-1)p(z)}{a}$. Similar argument yields,

$$S_{a-1,c}f(z) = \frac{z^2p''(z) + 2z(a-1)p'(z) + (a-2)(a-1)p(z)}{a(a-1)}$$

and

$$S_{a-2,c}f(z) = \frac{z^3p'''(z) + 3(a-1)z^2p''(z) + 3(a-1)(a-2)zp'(z) + (a-1)(a-2)(a-3)p(z)}{a(a-1)(a-2)}. \tag{11}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r, s, t, u) = r, \qquad \beta(r, s, t, u) = \frac{s + (a-1)r}{a}, \qquad \gamma(r, s, t, u) = \frac{t + 2(a-1)s + (a-2)(a-1)r}{a(a-1)}$$

and

$$\delta(r, s, t, u) = \frac{u + 3(a-1)t + 3(a-1)(a-2)s + (a-1)(a-2)(a-3)r}{a(a-1)(a-2)}.$$

Let

$$\psi(r, s, t, u) = \phi(\alpha, \beta, \gamma, \delta; z) = \phi\Bigg(r, \frac{s + (a-1)r}{a}, \frac{t + 2(a-1)s + (a-2)(a-1)r}{a(a-1)}, \frac{u + 3(a-1)t + 3(a-1)(a-2)s + (a-1)(a-2)(a-3)r}{a(a-1)(a-2)}; z\Bigg). \tag{12}$$

The proof will make use of Lemma 1.5. Using equations (10) to (11), and from (12), we have

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) = \phi\left(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z\right). \tag{13}$$

Hence, (9) becomes $\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Omega$. Note that

$$\frac{t}{s} + 1 = \frac{a(a-1)\gamma - (a-2)(a-1)\alpha}{a\beta - (a-1)\alpha} - (2a-3) \text{ and } \frac{u}{s} = \frac{a(a-1)((1-k)\alpha + 3a\beta + (1-3a)\gamma + (a-2)\delta)}{\alpha + a(\beta - \alpha)}.$$

Thus, the admissibility condition for $\phi \in \Phi_S[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 1.4 with $n = 2$. Therefore, by using (8) and Lemma 1.5, we have $S_{a+1,c}f(z) \prec q(z)$. This completes the proof. □

The next result is an extension of theorem 2.2 to the case where the behavior of $q(z)$ on $\partial\mathbb{D}$ is not known.

Corollary 2.3. *Let $\Omega \subset \mathbb{C}$ and the function q be univalent in $\mathbb{D}$ with $q(0) = 0$. Let $\phi \in \Phi_S[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}, q_\rho$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q_\rho''(\zeta)}{q_\rho'(\zeta)}\right) \geq 0, \qquad \left|\frac{S_{a,c}f(z)}{q_\rho'(\zeta)}\right| \leq k \quad (z \in \mathbb{D}, \zeta \in \partial\mathbb{D} \setminus E(q_\rho)), \tag{14}$$

and $\phi\left(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z\right) \in \Omega$, then $S_{a+1,c}f(z) \prec q(z), \quad (z \in \mathbb{D})$.

Proof. From Theorem 2.2, then $S_{a+1,c}f(z) \prec q_\rho(z)$. The result asserted by Corollary 2.3 is now deduced from the following subordination property $q_\rho(z) \prec q(z), \quad (z \in \mathbb{D})$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_S[h(\mathbb{D}), q]$ is written as $\Phi_S[h, q]$. The following result follows immediately as a consequence of Theorem 2.2.

Theorem 2.4. *Let $\phi \in \Phi_S[h, q]$. If the function $f \in \mathcal{A}, q \in \mathcal{Q}_0$ satisfy the conditions (8), and*

$$\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z) \prec h(z), \tag{15}$$

then $S_{a+1,c}f(z) \prec q(z), \quad (z \in \mathbb{D})$.

The next result is an immediate consequence of Corollary 2.3.

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$ and the function q be univalent in $\mathbb{D}$ with $q(0) = 0$. Let $\phi \in \Phi_S[h, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}, q_\rho$ satisfy the conditions (14), and*

$$\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z) \prec h(z), \ (z \in \mathbb{D})$$

then $S_{a+1,c}f(z) \prec q(z), \quad (z \in \mathbb{D})$.

The following result yields the best dominant of the differential subordination (15).

Theorem 2.6. *Let the function h be univalent in $\mathbb{D}$. Also let $\phi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ and ψ be given by (12). Suppose that the differential equation:*

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{16}$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies condition (8). If the function $f \in \mathcal{A}$ satisfies condition (15) and if $\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z)$ is analytic in $\mathbb{D}$, then $S_{a+1,c}f(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. From Theorem 2.2, we have q is a dominant of (15). Since q satisfies (16), it is also a solution of (15) and therefore q will be dominated by all dominants. Hence q is the best dominant. This completes the proof. □

In view of Definition 2.1, and in the special case $q(z) = Mz,\ M > 0$, the class of admissible functions $\Phi_S[\Omega, q]$, denoted by $\Phi_S[\Omega, M]$, is expressed as follows.

Definition 2.7. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_S[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ such that*

$$\phi\Big(Me^{i\theta}, \frac{(k+a-1)Me^{i\theta}}{a}, \frac{L + [(2k+a-2)(a-1)]Me^{i\theta}}{a(a-1)}, \frac{N + 3(a-1)L + [(a-1)(a-2)(3k+a-3)]Me^{i\theta}}{a(a-1)(a-2)}; z\Big) \notin \Omega, \quad (17)$$

where $z \in \mathbb{D}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$, $\Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

Corollary 2.8. *Let $\phi \in \Phi_S[\Omega, M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|S_{a,c}f(z)\right| \leq kM, \quad (k \geq 2;\ M > 0) \text{ and } \phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z) \in \Omega,$$

then $\left|S_{a+1,c}f(z)\right| < M, \quad (z \in \mathbb{D})$.

In this special case $\Omega = q(\mathbb{D}) = \{w : |w| < M\}$, the class $\Phi_S[\Omega, M]$ is simply denoted by $\Phi_S[M]$. Corollary 2.8 can now be written in the following form:

Corollary 2.9. *Let $\phi \in \Phi_S[M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|S_{a,c}f(z)\right| \leq kM, \quad (k \geq 2;\ M > 0), \text{ and } \left|\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z\right| < M,$$

then $\left|S_{a+1,c}f(z)\right| < M, \quad (z \in \mathbb{D})$.

Corollary 2.10. *Let $\Re(a) \geq \frac{1-k}{2}$, $\quad k \geq 2$ and $M > 0$. If $f \in \mathcal{A}$ satisfies $|S_{a,c}f(z)| \leq M$, then $|S_{a+1,c}f(z)| < M$, $(z \in \mathbb{D})$.*

Proof. This follows from Corollary 2.9 by taking $\phi(\alpha, \beta, \gamma, \delta; z) = \beta = \frac{k+a-1}{a}Me^{i\theta}$. □

Remark 2.11. *For $f(z) = \frac{z}{1-z}$ in Corollary 2.10, we have $|U_{a,c}(z)| < M \Longrightarrow |U_{a+1,c}(z)| < M, \quad (z \in \mathbb{D})$, which is a generalization of result given by Prajapat [30].*

Corollary 2.12. *Let $0 \neq a \in \mathbb{C}, k \geq 2$ and $M > 0$. If $f \in \mathcal{A}$ satisfies $|S_{a,c}f(z)| \leq kM$ and $|S_{a,c}f(z) - S_{a+1,c}f(z)| < \frac{M}{|a|}$, then $|S_{a+1,c}f(z)| < M, \quad (z \in \mathbb{D})$.*

Proof. Let $\phi(\alpha, \beta, \gamma, \delta; z) = \beta - \alpha$ and $\Omega = h(\mathbb{D})$, where $h(z) = \frac{Mz}{a},\ M > 0$. In order to use Corollary 2.8, we need to show that $\phi \in \Phi_S[\Omega, M]$, that is, the admissibility condition (17) is satisfied. This follows since

$$|\phi(\alpha, \beta, \gamma, \delta; z)| = \left|\frac{(k-1)Me^{i\theta}}{a}\right| \geq \frac{M}{|a|},$$

whenever $z \in \mathbb{D}, \theta \in \mathbb{R}$ and $k \geq 2$. The required result now follows from Corollary 2.8.

Theorem 2.6 shows that the result is sharp. The differential equation $zq'(z) = Mz$ has a univalent solution $q(z) = Mz$. It follows from Theorem 2.6 that $q(z) = Mz$ is the best dominant. □

Example 2.13. *For $p = \pm 1/2$, $b = 1$ and $c = -1$, we have $U_{2,-1}(z) = zU_{1/2,1,-1}(z) = 2(\cosh\sqrt{z} - 1)$ and $U_{1,-1}(z) = zU_{-1/2,1,-1}(z) = \sqrt{z}\sinh\sqrt{z}$, where $U_{p,b,c}$ is given by (2). Furthermore, taking $f(z) = \frac{z}{1-z}$ in Corollary 2.12, we have*

$$\left|\sqrt{z}\sinh\sqrt{z} - 2(\cosh\sqrt{z} - 1)\right| < M \text{ implies that } \left|\cosh\sqrt{z} - 1\right| < \frac{M}{2}, \quad (z \in \mathbb{D}).$$

Definition 2.14. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{S,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$ whenever*

$$\alpha = q(\zeta), \ \ \beta = \frac{k\zeta q'(\zeta) + aq(\zeta)}{a}, \ \ \Re\left(\frac{(a-1)(\gamma-\alpha)}{\beta-\alpha} + (1-2a)\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{(a-1)(a-2)(\delta-\alpha) - 3a(a-1)(\gamma-2\alpha+\beta)}{\beta-\alpha} + 6a^2\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq 2$.

The proof the following theorem run parallel to that of Theorem 2.2 and we choose to omit the details.

Theorem 2.15. *Let $\phi \in \Phi_{S,1}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \left|\frac{S_{a,c}f(z)}{zq'(\zeta)}\right| \leq k, \tag{18}$$

and $\left\{\phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right) : z \in \mathbb{D}\right\} \subset \Omega$, then $\frac{S_{a+1,c}f(z)}{z} \prec q(z), \quad (z \in \mathbb{D})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_{S,1}[h(\mathbb{D}), q]$ is written as $\Phi_{S,1}[h, q]$. Proceeding similarly as in the previous theorem, the following result is an immediate consequence of Theorem 2.15.

Theorem 2.16. *Let $\phi \in \Phi_{S,1}[h, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the conditions (18) and*

$$\phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right) \prec h(z), \quad \text{then} \quad \frac{S_{a+1,c}f(z)}{z} \prec q(z), \quad (z \in \mathbb{D}).$$

In the particular case $q(z) = 1 + Mz, \ M > 0$, and in view of Definition 2.14 the class of admissible functions $\Phi_{S,1}[\Omega, q]$, denoted by $\Phi_{S,1}[\Omega, M]$, is expressed as follows.

Definition 2.17. *Let Ω be a set in $\mathbb{C}, a \in \mathbb{C} \setminus \{0, 1, 2\}$ and $M > 0$. The class of admissible functions $\Phi_{S,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ such that*

$$\begin{aligned}\phi\Big(1 + Me^{i\theta}, \frac{a + [k+a]Me^{i\theta}}{a}, \frac{L + a(a-1) + [a(2k+a-1)]Me^{i\theta}}{a(a-1)}, \\ \frac{N + 3aL + a(a-1)(a-2) + [(a-1)a(3k+a-2)]Me^{i\theta}}{a(a-1)(a-2)}; z\Big) \notin \Omega,\end{aligned}$$

whenever $z \in \mathbb{D}, \Re(Le^{-i\theta}) \geq (k-1)kM, \ \Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

Corollary 2.18. *Let $\phi \in \Phi_{S,1}[\Omega, M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|\frac{S_{a,c}f(z)}{z}\right| \leq kM, \quad (k \geq 2; M > 0) \ \text{and} \ \phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right) \in \Omega,$$

then $\left|\frac{S_{a+1,c}f(z)}{z} - 1\right| < M, \quad (z \in \mathbb{D})$.

In this special case $\Omega = q(\mathbb{D}) = \{w : |w-1| < M\}$, the class $\Phi_{S,1}[\Omega, M]$ is simply denoted by $\Phi_{S,1}[M]$. Corollary 2.18 can now be written in the following form:

Corollary 2.19. *Let $\phi \in \Phi_{S,1}[M]$. If the function $f \in \mathcal{A}$ satisfies*

$$\left|\frac{S_{a,c}f(z)}{z}\right| \leq kM, \quad (k \geqslant 2; M > 0) \text{ and } \left|\phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right) - 1\right| < M,$$

then $\left|\frac{S_{a+1,c}f(z)}{z} - 1\right| < M, \quad (z \in \mathbb{D}).$

Corollary 2.20. *Let $\Re(a) \geq \frac{-k}{2}$, $0 \neq a \in \mathbb{C}, k \geq 2$ and $M > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{S_{a,c}f(z)}{z}\right| \leq kM \text{ and } \left|\frac{S_{a,c}f(z)}{z} - 1\right| < M, \text{ then } \left|\frac{S_{a+1,c}f(z)}{z} - 1\right| < M, \quad (z \in \mathbb{D}).$$

Proof. This follows from Corollary 2.18 by taking $\phi(\alpha, \beta, \gamma, \delta; z) = \beta - 1$. □

Remark 2.21. *For $f(z) = \frac{z}{1-z}$ in Corollary 2.20, we have*

$$\left|\frac{U_{a,c}(z)}{z} - 1\right| < M \text{ implies that } \left|\frac{U_{a+1,c}(z)}{z} - 1\right| < M, \quad (z \in \mathbb{D}),$$

which is given by Andras and Baricz [3].

Example 2.22. *For $p = \pm 1/2$, $b = 1$ and $c = 1$, we have $U_{2,1}(z) = zU_{1/2,1,1}(z) = 2(1 - \cos\sqrt{z})$ and $U_{1,1}(z) = zU_{-1/2,1,1}(z) = \sqrt{z}\sin\sqrt{z}$, where $U_{p,b,c}$ is given by (2). Therefore, from Remark 2.21, we get*

$$\left|\frac{\sqrt{z}\sin\sqrt{z}}{z} - 1\right| < M \text{ implies that } \left|\frac{1 - \cos\sqrt{z}}{z} - \frac{1}{2}\right| < \frac{M}{2}, \quad (z \in \mathbb{D}).$$

Corollary 2.23. *Let $a \in \mathbb{C} \setminus \{0, 1, -2\}, k \geq 2$ and $M > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{S_{a,c}f(z)}{z}\right| \leq kM \text{ and } \left|\frac{S_{a-1,c}f(z)}{z} - \frac{S_{a,c}f(z)}{z}\right| < \frac{2(a+2)M}{a(a-1)}, \text{ then } \left|\frac{S_{a+1,c}f(z)}{z} - 1\right| < M, \quad (z \in \mathbb{D}).$$

Proof. This follows from Corollary 2.18 by taking $\phi(\alpha, \beta, \gamma, \delta; z) = \gamma - \beta$. □

Definition 2.24. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{S,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$ whenever*

$$\alpha = q(\zeta),\ \beta = \frac{1}{(a-1)}\left(\frac{k\zeta q'(\zeta)}{q(\zeta)} + aq(\zeta) - 1\right),\ \Re\left(\frac{[(a-2)\gamma - (a-1)\beta + 1](a-1)\beta\alpha}{(a-1)\beta\alpha - a\alpha^2 + \alpha} + (a-1)\beta + 1\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\begin{aligned}\Re\Bigg[\Big(&\delta\gamma\beta\alpha(a-1)(a-2)(a-3) - (a-2)^2(a-1)\beta\gamma^2\alpha - \beta\gamma\alpha(a-1)(a-2) - \\ &\gamma\beta^2\alpha(a-1)^2(a-2) + \beta^3\alpha(a-1)^3 - 2\beta\alpha(a-1) + \beta^2\alpha(a-1)^2 - \alpha\beta\gamma a(a-1)(a-2) + \\ &\beta^2\alpha a(a-1)^2 - a(a-1)\beta\alpha\Big) \times \Big((a-1)\beta\alpha - a\alpha^2 + \alpha\Big)^{-1} + 3\gamma\beta(a-1)(a-2) - 4\beta(a-1)(a\alpha - 1) - \\ &2\beta^2(a-1)^2 - \beta a(a-1) - 3a\alpha\beta(a-1) + 2a^2\alpha - a + 4a^2\alpha^2 + a\alpha\Bigg] \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),\end{aligned}$$

where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq 2$.

Following the same method as in the proof of Theorem 2.2, (also see [27, 36]) we arrive the following results:

Theorem 2.25. *Let $\phi \in \Phi_{S,2}[\Omega, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \left|\frac{S_{a-1,c}f(z)}{S_{a,c}f(z)q'(\zeta)}\right| \leq k, \tag{19}$$

and $\left\{\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}, \frac{S_{a-1,c}f(z)}{S_{a,c}f(z)}, \frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)}, \frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)}; z\right) : z \in \mathbb{D}\right\} \subset \Omega$, *then* $\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)} \prec q(z), \quad (z \in \mathbb{D})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_{S,1}[h(\mathbb{D}), q]$ is written as $\Phi_{S,2}[h, q]$. The following result is an immediate consequence of Theorem 2.25 and hence we omit the proof.

Theorem 2.26. *Let $\phi \in \Phi_{S,2}[h, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathcal{Q}_1$ satisfy the conditions* (19) *and*

$$\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}, \frac{S_{a-1,c}f(z)}{S_{a,c}f(z)}, \frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)}, \frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)}; z\right) \prec h(z), \quad \textit{then} \quad \frac{S_{a,c}f(z)}{S_{a+1,c}f(z)} \prec q(z), \quad (z \in \mathbb{D}).$$

3. Results based on differential superordination

In this section, the third-order differential superordination theorems for the operator $S_{a,c}$ defined in (1) is investigated. For the purpose, we considered the following class of admissible functions.

Definition 3.1. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_S[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(\alpha, \beta, \gamma, \delta; \zeta) \in \Omega$ whenever*

$$\alpha = q(z), \beta = \frac{zq'(z) + m(a-1)q(z)}{ma}, \quad \Re\left(\frac{a(a-1)\gamma - (a-2)(a-1)\alpha}{a\beta - (a-1)\alpha} - (2a-3)\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{a(a-1)((1-a)\alpha + (3a\beta + (1-3a)\gamma + (a-2)\delta)}{\alpha + a(\beta - \alpha)}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq 2$.

Theorem 3.2. *Let $\phi \in \Phi'_S[\Omega, q]$. If the function $f \in \mathcal{A}, S_{a+1,c}f(z) \in \mathcal{Q}_0$ and if $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfying the following conditions:*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \left|\frac{S_{a,c}f(z)}{q'(z)}\right| \leq m, \tag{20}$$

and $\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z)$ is univalent in $\mathbb{D}$, then

$$\Omega \subset \{\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z)\}, \tag{21}$$

implies that $q(z) \prec S_{a+1,c}f(z) \qquad (z \in \mathbb{D})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω. In this case, the class $\Phi'_S[h(\mathbb{D}), q]$ is written as $\Phi'_S[h, q]$. The following result is an immediate consequence of Theorem 3.2.

Theorem 3.3. *Let $\phi \in \Phi'_S[h, q]$ and h be analytic in $\mathbb{D}$. If the function $f \in \mathcal{A}$, $S_{a+1,c}f(z) \in \mathcal{Q}_0$ and if $q \in \mathcal{H}_0$ with $q'(z) \neq 0$, satisfying the conditions (20) and $\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z)$ is univalent in $\mathbb{D}$, then*

$$h(z) \prec \phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z) \tag{22}$$

implies that $q(z) \prec S_{a+1,c}f(z), \quad (z \in \mathbb{D})$.

Theorem 3.2 and 3.3 can only be used to obtain subordinants of the third-order differential superordination of the forms (21) or (22). The next result shows the existence of the best subordinant of (22) for a suitable ϕ.

Theorem 3.4. *Let the function h be univalent in $\mathbb{D}$ and let $\phi : \mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ and ψ be given by (12). Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{23}$$

has a solution $q(z) \in \mathcal{Q}_0$. If the function $f \in \mathcal{A}$, $S_{a+1,c}f(z) \in \mathcal{Q}_0$ and if $q \in \mathcal{H}_0$ with $q'(z) \neq 0$, which satisfying the conditions (20) and $\phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z)); z)$ is analytic in $\mathbb{D}$, then $h(z) \prec \phi(S_{a+1,c}f(z), S_{a,c}f(z), S_{a-1,c}f(z), S_{a-2,c}f(z); z)$ implies that $q(z) \prec S_{a+1,c}f(z)$, $\quad (z \in \mathbb{D})$ and $q(z)$ is the best dominant.

Definition 3.5. *Let Ω be a set in $\mathbb{C}, q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_{S,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(\alpha, \beta, \gamma, \delta; z) \in \Omega$, whenever*

$$\alpha = q(z), \beta = \frac{zq'(z) + amq(z)}{am}, \quad \Re\left(\frac{(a-1)(\gamma - \alpha)}{\beta - \alpha} + (1-2a)\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right), \text{ and}$$

$$\Re\left(\frac{(a-1)(a-2)(\delta - \alpha) - 3a(a-1)(\gamma - 2\alpha + \beta)}{\beta - \alpha} + 6a^2\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{D}, \zeta \in \partial\mathbb{D}$ and $m \geq 2$.

Theorem 3.6. *Let $\phi \in \Phi'_{S,1}[\Omega, q]$. If the function $f \in \mathcal{A}$, $\frac{S_{a+1,c}f(z)}{z} \in \mathcal{Q}_1$ and if $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfying the conditions:*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \quad \left|\frac{S_{a,c}f(z)}{zq'(z)}\right| \leq m, \tag{24}$$

and $\phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right)$ is univalent in $\mathbb{D}$, then

$$\Omega \subset \left\{\phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right) : z \in \mathbb{D}\right\}, \tag{25}$$

implies that $q(z) \prec \frac{S_{a+1,c}f(z)}{z}$, $\quad (z \in \mathbb{D})$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω. In this case, the class $\Phi'_{S,1}[h(\mathbb{D}), q]$ is written as $\Phi'_{S,1}[h, q]$. The following result is an immediate consequence of Theorem 3.6.

Theorem 3.7. *Let $\phi \in \Phi'_{S,1}[h, q]$. If the function $f \in \mathcal{A}$, $\frac{S_{a+1,c}f(z)}{z} \in \mathcal{Q}_1$ and if $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the conditions (24) and $\phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right)$ is univalent in $\mathbb{D}$, then*

$$h(z) \prec \phi\left(\frac{S_{a+1,c}f(z)}{z}, \frac{S_{a,c}f(z)}{z}, \frac{S_{a-1,c}f(z)}{z}, \frac{S_{a-2,c}f(z)}{z}; z\right) \quad \text{implies that} \quad q(z) \prec \frac{S_{a+1,c}f(z)}{z}, \qquad (z \in \mathbb{D}).$$

Definition 3.8. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_{S,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ that satisfy the admissibility conditions: $\phi(\alpha, \beta, \gamma, \delta; z) \in \Omega$, whenever*

$$\alpha = q(z), \beta = \frac{1}{a-1}\left(\frac{zq'(z)}{mq(z)} + aq(z) - 1\right), \ \Re\left(\frac{[(a-2)\gamma - (a-1)\beta + 1](a-1)\beta\alpha}{(a-1)\beta\alpha - a\alpha^2 + \alpha} + (a-1)\beta + 1\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\Big[\Big(\delta\gamma\beta\alpha(a-1)(a-2)(a-3)-(a-2)^2(a-1)\beta\gamma^2\alpha-\beta\gamma\alpha(a-1)(a-2)-\gamma\beta^2\alpha(a-1)^2(a-2)+$$
$$\beta^3\alpha(a-1)^3-2\beta\alpha(a-1)+\beta^2\alpha(a-1)^2-\alpha\beta\gamma a(a-1)(a-2)+\beta^2\alpha a(a-1)^2-a(a-1)\beta\alpha\Big)\times$$
$$\Big((a-1)\beta\alpha-a\alpha^2+\alpha\Big)^{-1}+3\gamma\beta(a-1)(a-2)-4\beta(a-1)(a\alpha-1)-2\beta^2(a-1)^2-\beta a(a-1)-$$
$$3a\alpha\beta(a-1)+2a^2\alpha-a+4a^2\alpha^2+a\alpha\Big]\leq\frac{1}{m^2}\Re\Big(\frac{z^2q'''(z)}{q'(z)}\Big),$$

where $z\in\mathbb{D}, \zeta\in\partial\mathbb{D}$ and $m\geq 2$.

Theorem 3.9. *Let $\phi\in\Phi'_{S,2}[\Omega,q]$. If the function $f\in\mathcal{A}$, $\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}\in\mathcal{Q}_1$, and if $q\in\mathcal{H}_1$ with $q'(z)\neq 0$ satisfying the following conditions:*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right)\geq 0,\ \left|\frac{S_{a-1,c}f(z)}{S_{a,c}q'(z)}\right|\leq m, \tag{26}$$

and $\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)},\frac{S_{a-1,c}f(z)}{S_{a,c}f(z)},\frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)},\frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)};z\right)$ is univalent in $\mathbb{D}$, then

$$\Omega\subset\left\{\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)},\frac{S_{a-1,c}f(z)}{S_{a,c}f(z)},\frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)},\frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)};z\right):z\in\mathbb{D}\right\}, \tag{27}$$

implies that $q(z)\prec\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}$, $(z\in\mathbb{D})$.

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{D})$, for some conformal mapping h(z) of $\mathbb{D}$ on to Ω. In this case the class $\Phi'_{S,2}[h(\mathbb{D}),q]$ is written as $\Phi'_S[h,q]$. The following result is a consequence of Theorem 3.9.

Theorem 3.10. *Let $\phi\in\Phi'_{S,2}[\Omega,q]$. If the function $f\in\mathcal{A}$, $\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}\in\mathcal{Q}_1$, and if $q\in\mathcal{H}_1$ with $q'(z)\neq 0$ satisfying the conditions (26), and $\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)},\frac{S_{a-1,c}f(z)}{S_{a,c}f(z)},\frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)},\frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)};z\right)$ is univalent in $\mathbb{D}$, then*

$$h(z)\prec\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)},\frac{S_{a-1,c}f(z)}{S_{a,c}f(z)},\frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)},\frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)};z\right)\quad \textit{implies that}\quad q(z)\prec\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)},\qquad (z\in\mathbb{D}).$$

4. Sandwich type results

Combining Theorem 2.4 and 3.3, we obtain the following sandwich-type result.

Corollary 4.1. *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be univalent function in $\mathbb{D}$, $q_2\in\mathcal{Q}_0$ with $q_1(0)=q_2(0)=0$ and $\phi\in\Phi_S[h_2,q_2]\cap\Phi'_S[h_1,q_1]$. If the function $f\in\mathcal{A}$, $S_{a+1,c}\in\mathcal{Q}_0\cap\mathcal{H}_0$, and $\phi(S_{a+1,c}f(z),S_{a,c}f(z),S_{a-1,c}f(z),S_{a-2,c}f(z);z)$ is univalent in $\mathbb{D}$, and if the conditions (8) and (20) are satisfied, then*

$$h_1(z)\prec\phi(S_{a+1,c}f(z),S_{a,c}f(z),S_{a-1,c}f(z),S_{a-2,c}f(z);z)\prec h_2(z)\quad \textit{implies that}\quad q_1(z)\prec S_{a+1,c}f(z)\prec q_1(z),\quad (z\in\mathbb{D}).$$

Combining Theorems 2.16 and 3.7, we obtain the following sandwich-type result.

Corollary 4.2. *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be univalent function in $\mathbb{D}$, $q_2\in\mathcal{Q}_1$ with $q_1(0)=q_2(0)=1$ and $\phi\in\Phi_{S,1}[h_2,q_2]\cap\Phi'_{S,1}[h_1,q_1]$. If the function $f\in\mathcal{A}$, $\frac{S_{a+1,c}f(z)}{z}\in\mathcal{Q}_1\cap\mathcal{H}_1$, and $\phi\left(\frac{S_{a+1,c}f(z)}{z},\frac{S_{a,c}f(z)}{z},\frac{S_{a-1,c}f(z)}{z},\frac{S_{a-2,c}f(z)}{z};z\right)$ is univalent in $\mathbb{D}$, and if the conditions (18) and (24) are satisfied, then*

$$h_1(z)\prec\phi\left(\frac{S_{a+1,c}f(z)}{z},\frac{S_{a,c}f(z)}{z},\frac{S_{a-1,c}f(z)}{z},\frac{S_{a-2,c}f(z)}{z};z\right)\prec h_2(z)\quad \textit{implies that}\quad q_1(z)\prec\frac{S_{a+1,c}f(z)}{z}\prec q_1(z),\ (z\in\mathbb{D}).$$

Combining Theorem 2.26 and 3.10, we obtain the following sandwich-type result.

Corollary 4.3. *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be univalent function in $\mathbb{D}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{S,1}[h_2, q_2] \cap \Phi'_{S,1}[h_1, q_1]$. If the function $f \in \mathcal{A}$, $\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)} \in \mathcal{Q}_1 \cap \mathcal{H}_1$, and $\phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}, \frac{S_{a-1,c}f(z)}{S_{a,c}f(z)}, \frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)}, \frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)}; z\right)$ is univalent in $\mathbb{D}$, and if the conditions (19) and (26) are satisfied, then*

$$h_1(z) \prec \phi\left(\frac{S_{a,c}f(z)}{S_{a+1,c}f(z)}, \frac{S_{a-1,c}f(z)}{S_{a,c}f(z)}, \frac{S_{a-2,c}f(z)}{S_{a-1,c}f(z)}, \frac{S_{a-3,c}f(z)}{S_{a-2,c}f(z)}; z\right) \prec h_2(z) \quad \textit{implies that} \quad q_1(z) \prec \frac{S_{a,c}f(z)}{S_{a+1,c}f(z)} \prec q_1(z), \; (z \in \mathbb{D}).$$

Remark 4.4. *For special cases all of above results, we can obtain the corresponding results for family associated with the operators $\mathfrak{S}$, $\mathfrak{I}$, which are defined by (3), and (4), respectively.*

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