Filomat 33:10 (2019), 3231–3242
https://doi.org/10.2298/FIL1910231E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Some Types of Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

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Abstract. The main purpose of the present paper is to study the geometry of screen transversal lightlike submanifolds and radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of Golden semi-Riemannian manifolds. We investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on these manifolds to be metric connection. We also obtain characterizations of screen transversal anti-invariant lightlike submanifolds of Golden semi-Riemannian manifolds. Finally, we give two examples.

1. Introduction

Lightlike submanifolds are one of the most interesting topics in differential geometry. It is well known that a submanifold of a Riemannian manifold is always a Riemannian one. Contrary to that case, in semi-Riemannian manifolds, the induced metric by the semi-Riemannian metric on the ambient manifold is not necessarily non-degenerate. Since the induced metric is degenerate on lightlike submanifolds, the tools which are used to investigate the geometry of submanifolds in Riemannian case are not favorable in semi-Riemannian case and so the classical theory can not be used to define an induced object on a lightlike submanifold. The main difficulties arise from the fact that the intersection of the normal bundle and the tangent bundle of a lightlike submanifold is nonzero. In 1996, K. Duggal-A. Bejancu [14] put forward the general theory of lightlike submanifolds of semi-Riemannian manifolds in their book. In order to resolve the difficulties that arise during studying lightlike submanifolds, they introduced a non-degenerate distribution called screen distribution to construct a lightlike transversal vector bundle which does not intersect to its lightlike tangent bundle. It is well-known that a suitable choice of screen distribution gives rises to many substantial results in lightlike geometry. Many authors have studied the geometry of lightlike submanifolds[2–4, 16–18, 33, 34, 37] in different manifolds. For further read we refer [14, 15] and the references therein.

In recent years, one of the most studied type of manifolds are Riemannian manifolds with metallic structures. Metallic structures on Riemannian manifolds allow many geometric results to be given on a submanifold.

As a generalization of the Golden mean, which contains the Silver mean, the Bronze mean, the Copper mean and the Nickel mean etc., Metallic means family was introduced by V. W. de Spinadel [12] in 2002.

2010 *Mathematics Subject Classification*. Primary 53C15 53C25; Secondary 53C35
Keywords. Golden Semi-Riemannian Manifold,screen transversal anti-invaryant lightlike manifold
Received: 29 November 2018; Accepted: 08 February 2019
Communicated by Mića Stanković
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The positive solution of the equation given by

$$x^2 - px - q = 0,$$

for some positive integer p and q, is called a (p, q)-metallic number [9, 11] which has the form

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

For $p = q = 1$ and $p = 2$, $q = 1$, it is well-known that we have the Golden mean $\phi = \frac{1+\sqrt{5}}{2}$ and Silver mean $\sigma_{2,1} = 1 + \sqrt{2}$, respectively. The metallic mean family plays an important role to establish a relationship between mathematics and architecture. For example Golden mean and Silver mean can be seen in the sacred art of Egypt, Turkey, India, China and other ancient civilizations [13].

S. I. Goldberg, K. Yano and N. C. Petridis in ([23] and [24]) introduced polynomial structures on manifolds. As some particular cases of polynomial structures, C. E. Hretcanu and M. Crasmareanu defined Golden structure [6–8, 25] and some generalizations of this, called Metallic structure [21]. Being inspired by the Metallic mean, the notion of Metallic manifold $\tilde{N}$ was defined in [21] by a $(1, 1)$-tensor field $\tilde{J}$ on $\tilde{N}$, which satisfies $\tilde{J}^2 = p\tilde{J} + qI$, where I is the identity operator on the Lie algebra $\chi(\tilde{N})$ of vector fields on $\tilde{N}$ and p, q are fixed positive integer numbers. Moreover, if $(\tilde{N}, g)$ is a Riemannian manifold endowed with a Metallic structure $\tilde{J}$ such that the Riemannian metric $\tilde{g}$ is $\tilde{J}$-compatible, i.e., $\tilde{g}(\tilde{J}V, W) = \tilde{g}(V, \tilde{J}W)$, for any $V, W \in \chi(\tilde{N})$, then $(\tilde{g}, \tilde{J})$ is called Metallic Riemannian structure and$(\tilde{N}, \tilde{g}, \tilde{J})$ is a Metallic Riemannian manifold. Metallic structure on the ambient Riemannian manifold provides important geometrical results on the submanifolds, since it is an important tool while investigating the geometry of submanifolds. Invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds of a Metallic Riemannian manifold are studied in [5, 26–28] and the authors obtained important characterizations for submanifolds of Metallic Riemannian manifolds.

One of the most important subclass of Metallic Riemannian manifolds is the Golden Riemannian manifolds. Many authors have studied Golden Riemannian manifolds and their submanifolds in recent years (see [6–8, 19, 20, 32, 36]). N. Poyraz Önen and E. Yaşar [34] initiated the study of lightlike geometry in Golden semi-Riemannian manifolds, by investigating lightlike hypersurfaces of Golden semi-Riemannian manifolds. B. E. Acet introduced lightlike hypersurfaces in Metallic semi-Riemannian manifolds [1].

Let $\tilde{N}$ be a manifold and $\tilde{P}$ be a $(1, 1)$-tensor field on $\tilde{N}$. If $\tilde{P}$ satisfies $\tilde{P}^2 - \tilde{P} - I = 0$, then $\tilde{P}$ is called a Golden structure on $\tilde{N}$ and $(\tilde{N}, \tilde{P})$ is a Golden manifold[7]. This structure was inspired by the Golden proportion, which was described by Kepler (1571-1630). The number ϕ, which is the real positive root of equation $x^2 - x - 1 = 0$ (*thus*, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618...$) is the Golden proportion. We remark that for Golden structure $\tilde{P} \neq \phi I$. If $\tilde{P} = \phi I$, $\phi = \frac{1+\sqrt{5}}{2}$, then its minimal polynomial is $X - \phi$. However, $X^2 - X - I$ is the minimal polynomial of the Golden structure $\tilde{P}$.

Considering the above information, in this paper, we introduce lightlike submanifolds of Golden semi-Riemannian manifolds and study their geometry. The paper is organized as follows: In section 2, we give basic information needed for this paper. In section 3, section 4 and section 5, we introduce Golden semi-Riemannian manifold along with its subclasses (screen transversal, radical screen transversal and screen transversal anti-invariant lightlike submanifolds) and obtain some characterizations. We investigate the geometry of distributions and find necessary and sufficient conditions for the induced connection to be metric a connection. Furthermore, we give two examples.

2. Preliminaries

A submanifold $\dot{N}^m$ immersed in a semi-Riemannian manifold $(\tilde{N}^{m+k}, \tilde{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\tilde{g}$ whose radical distribution which is a semi-Riemannian complementary distribution of $RadT\dot{N}$ is of rank r, where $1 \leq r \leq m$: $RadT\dot{N} = T\dot{N} \cap T\dot{N}^{\perp}$, where

$$T\dot{N}^{\perp} = \cup_{x \in \dot{N}} \left\{ u \in T_x\tilde{N} \mid \tilde{g}(u, v) = 0, \forall v \in T_x\dot{N} \right\}. \tag{1}$$

Let $S(T\dot{N})$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadT\dot{N}$ in $T\dot{N}$, i.e., $T\dot{N} = RadT\dot{N} \perp S(T\dot{N})$.

We consider a screen transversal vector bundle $S(T\dot{N}^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadT\dot{N}$ in $T\dot{N}^{\perp}$. Let $tr(T\dot{N})$ be complementary (but not orthogonal) vector bundle to $T\dot{N}$ in $T\tilde{N}^{\perp}|_{\dot{N}}$. Then

$$
\begin{aligned}
tr(T\dot{N}) &= ltr(T\dot{N}) \perp S(T\dot{N}^{\perp}),\\
T\tilde{N}|_{\dot{N}} &= S(T\dot{N}) \perp [RadT\dot{N} \oplus ltr(T\dot{N})] \perp S(T\dot{N}^{\perp}).
\end{aligned}
$$

Although $S(T\dot{N})$ is not unique, it is canonically isomorphic to the factor vector bundle $T\dot{N}/RadT\dot{N}$ [14]. The following result is important for this paper.

Proposition 2.1. *[14]. The lightlike second fundamental forms of a lightlike submanifold $\dot{N}$ do not depend on $S(T\dot{N})$, $S(T\dot{N}^{\perp})$ and $ltr(T\dot{N})$.*

We say that a submanifold $(\dot{N}, g, S(T\dot{N}), S(T\dot{N}^{\perp}))$ of $\tilde{N}$ is
Case1: r-lightlike if $r < \min\{m, k\}$;
Case2: Co-isotropic if $r = k < m$; $S(T\dot{N}^{\perp}) = \{0\}$;
Case3: Isotropic if $r = m = k; S(T\dot{N}) = \{0\}$;
Case4: Totally lightlike if $r = k = m; S(T\dot{N}) = \{0\} = S(T\dot{N}^{\perp})$.

The Gauss and Weingarten equations are:

$$
\begin{aligned}
\tilde{\nabla}_W U &= \nabla_W U + h(W,U), \forall W, U \in \Gamma(T\dot{N}), &(2)\\
\tilde{\nabla}_W V &= -A_V W + \nabla^t_W V, \forall W \in \Gamma(T\dot{N}), V \in \Gamma(tr(T\dot{N})), &(3)
\end{aligned}
$$

where $\{\nabla_W U, A_V W\}$ and $\left\{h(W,U), \nabla^t_W V\right\}$ belong to $\Gamma(T\dot{N})$ and $\Gamma(tr(T\dot{N}))$, respectively. ∇ and ∇^t are linear connections on $\dot{N}$ and the vector bundle $tr(T\dot{N})$, respectively. Moreover, we have

$$
\begin{aligned}
\tilde{\nabla}_W U &= \nabla_W U + h^{\ell}(W,U) + h^s(W,U), \forall W, U \in \Gamma(T\dot{N}), &(4)\\
\tilde{\nabla}_W N &= -A_N W + \nabla^{\ell}_W N + D^s(W,N), N \in \Gamma(ltr(T\dot{N})), &(5)\\
\tilde{\nabla}_W Z &= -A_Z W + \nabla^s_W Z + D^{\ell}(W,Z), Z \in \Gamma(S(T\dot{N}^{\perp})). &(6)
\end{aligned}
$$

Denote by $\tilde{P}$ the projection of $T\dot{N}$ on $S(T\dot{N})$. Then by using (2), (4)-(6) and the metric connection $\tilde{\nabla}$ of $\tilde{g}$, we obtain

$$
\begin{aligned}
\tilde{g}(h^s(W,U), Z) + \tilde{g}\left(U, D^{\ell}(W,Z)\right) &= g(A_Z W, U), &(7)\\
\tilde{g}(D^s(W,N), Z) &= \tilde{g}(N, A_Z W). &(8)
\end{aligned}
$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$
\begin{aligned}
\nabla_W \tilde{P}U &= \nabla^*_W \tilde{P}U + h^*(W, \tilde{P}U), &(9)\\
\nabla_W \xi &= -A^*_{\xi} W + \nabla^{*t}_W \xi, &(10)
\end{aligned}
$$

for $W, U \in \Gamma(T\dot{N})$ and $\xi \in \Gamma(RadT\dot{N})$. By using the above equations, we obtain

$$
\begin{aligned}
g(h^{\ell}(W, \tilde{P}U), \xi) &= g(A^*_{\xi} W, \tilde{P}U), &(11)\\
g(h^s(W, \tilde{P}U), N) &= g(A_N W, \tilde{P}U), &(12)\\
g(h^{\ell}(W, \xi), \xi) &= 0, \quad A^*_{\xi}\xi = 0. &(13)
\end{aligned}
$$

In general, the induced connection ∇ on $\dot{N}$ is not a metric connection. Since $\tilde{\nabla}$ is a metric connection, by using (4) we get

$$
(\nabla_W g)(U, V) = \tilde{g}\left(h^{\ell}(W,U), V\right) + \tilde{g}\left(h^{\ell}(W,V), U\right). \tag{14}
$$

However, ∇^* is a metric connection on $S(T\dot{N})$.

Let (N_1, g_1) and (N_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, and $q_2 > 0$, respectively. Let $\pi : N_1 \times N_2 \to N_1$ and $\sigma : N_1 \times N_2 \to N_2$ be the projections which are given by $\pi(w, u) = w$ and $\sigma(w, u) = u$, for any $(w, u) \in N_1 \times N_2$, respectively.

We denote by $\tilde{N} = (N_1 \times N_2, \tilde{g})$, the product manifold where

$$\tilde{g}(W, U) = g_1(\pi_* W, \pi_* U) + g_2(\sigma_* W, \sigma_* U)$$

for any $W, U \in \Gamma(T\tilde{N})$. Then we have

$$\pi_*^2 = \pi_*, \quad \pi_*\sigma_* = \sigma_*\pi_* = 0,$$

$$\sigma_*^2 = \sigma_* \quad , \pi_* + \sigma_* = I,$$

where I is identity transformation. Thus $(\tilde{N}, \tilde{g})$ is an $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that

$$F^2 \quad = \quad \pi_*^2 - \pi_*\sigma_* - \sigma_*\pi_* + \sigma_*^2 = I.$$

Also, we have

$$\tilde{g}\,(FW, U) = \tilde{g}\,(W, FU)\,,$$

for any $W, U \in \Gamma(T\tilde{N})$. If we denote the levi-civita connection on $\tilde{N}$ by $\tilde{\nabla}$, then it can be seen that $(\tilde{\nabla}_W F)Y = 0$, for any $W, U \in \Gamma(T\tilde{N})$, that is, F is parallel with respect to $\tilde{\nabla}$ [24].

Let $(\tilde{N}, \tilde{g})$ be a semi-Riemannian manifold. Then $\tilde{N}$ is called Golden semi-Riemannian manifold if there exists a $(1, 1)$-tensor field $\tilde{P}$ on $\tilde{N}$ such that

$$\tilde{P}^2 = \tilde{P} + I, \tag{15}$$

where I is the identity map on $\tilde{N}$. Also, we have

$$\tilde{g}(\tilde{P}W, U) = \tilde{g}(W, \tilde{P}U). \tag{16}$$

The semi-Riemannian metric (16) is called $\tilde{P}$-compatible and $(\tilde{N}, \tilde{g}, \tilde{P})$ is named a Golden semi-Riemannian manifold. Also we have [37]

$$\tilde{\nabla}_W \tilde{P} U = \tilde{P} \tilde{\nabla}_W U. \tag{17}$$

If $\tilde{P}$ is a Golden structure, then (16) is equivalent to

$$\tilde{g}(\tilde{P}W, \tilde{P}U) = \tilde{g}(\tilde{P}W, U) + \tilde{g}(W, U) \tag{18}$$

for any $W, U \in \Gamma(T\tilde{N})$.

If F is an almost product structure on $\tilde{N}$, then

$$\tilde{P} = \frac{1}{\sqrt{2}}(I + \sqrt{5}F) \tag{19}$$

is a Golden structure on $\tilde{N}$, where I is the identity transformation. Conversely, if $\tilde{P}$ is a Golden structure on $\tilde{N}$, then

$$F = \frac{1}{\sqrt{5}}(2\tilde{P} - I)$$

is an almost product structure on $\tilde{N}$[34].

3. Screen Transversal Lightlike Submanifolds Of Golden semi-Riemannian Manifolds

In this section, we investigate screen transversal, radical screen transversal and screen transversal anti-invariant lightlike submanifolds of Golden semi-Riemannian manifolds.

Lemma 3.1. *Let $\dot{N}$ be a lightlike submanifold of Golden semi-Riemannian manifold. Also, let*

$$\tilde{P}RadT\dot{N} \subseteq S(T\dot{N}^{\perp}).$$

Then, $\tilde{P}ltrT\dot{N}$ is also a subvector bundle of the screen transversal vector bundle.

Proof. Let us accept the reversal of hypothesis. Namely, $ltrT\dot{N}$ is invariant with respect to $\tilde{P}$, i.e. $\tilde{P}ltrT\dot{N} = ltrT\dot{N}$. From the definition of lightlike submanifold, we have

$$\tilde{g}(N, \xi) = 1,$$

for $\xi \in \Gamma(RadT\dot{N})$ and $N \in \Gamma(ltrT\dot{N})$. Also from (18), we find that

$$\tilde{g}(N, \xi) = \tilde{g}(\tilde{P}N, \tilde{P}\xi) = 1.$$

However, if $\tilde{P}N \in \Gamma(ltrT\dot{N})$, then by hypothesis, we get $\tilde{g}(\tilde{P}N, \tilde{P}\xi) = 0$. Hence, we find a contradiction which implies that $\tilde{P}N$ does not belong to $ltrT\dot{N}$. Now, if we accept that $\tilde{P}N \in \Gamma(S(T\dot{N}))$, then, similarly, we obtain that

$$1 = \tilde{g}(N, \xi) = \tilde{g}(\tilde{P}N, \tilde{P}\xi) = 0,$$

since $\tilde{P}\xi \in \Gamma(S(T\dot{N}^{\perp}))$ and $\tilde{P}N \in \Gamma(S(T\dot{N}))$. Hence, $\tilde{P}N$ does not belong to $S(T\dot{N})$. We can also obtain that $\tilde{P}N$ does not belong to $RadT\dot{N}$. Then, from the decomposition of a lightlike submanifold, we conclude that $\tilde{P}N \in \Gamma(S(T\dot{N}^{\perp}))$. □

Definition 3.2. *Let $\dot{N}$ be a lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then we say that $\dot{N}$ is a screen transversal lightlike submanifold of $\tilde{N}$ if there exists a screen transversal bundle $\Gamma(S(T\dot{N}^{\perp}))$ such that*

$$\tilde{P}RadT\dot{N} \subset S(T\dot{N}^{\perp}).$$

From Definition 3.1 and Lemma 3.1, it follows that $\tilde{P}ltrT\dot{N} \subset S(T\dot{N}^{\perp})$.

Definition 3.3. *Let $\dot{N}$ be a screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then*

1. we say that $\dot{N}$ is a radical screen transversal lightlike submanifold of $\tilde{N}$ if $\tilde{P}(S(T\dot{N})) = S(T\dot{N})$,
2. we say that $\dot{N}$ is a screen transversal anti-invariant submanifold of $\tilde{N}$ if $\tilde{P}(S(T\dot{N})) \subset S(T\dot{N}^{\perp})$.

From Definition 3.2, if $\dot{N}$ is screen transversal anti-invariant lightlike submanifold, we have

$$S(T\dot{N}^{\perp}) = \tilde{P}RadT\dot{N} \oplus \tilde{P}ltrT\dot{N} \oplus \tilde{P}(S(T\dot{N}) \perp D_o.$$

In here D_o is a non-degenerate orthogonal complement distribution to

$$\tilde{P}RadT\dot{N} \oplus \tilde{P}ltrT\dot{N} \oplus \tilde{P}(S(T\dot{N})),$$

in $S(T\dot{N}^{\perp})$. For the distribution D_o, we have the following;

Proposition 3.4. *Let $\dot{N}$ be a screen transversal anti-invariant lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then the distribution D_o is invariant with respect to $\tilde{P}$.*

Proof. Using (16), we obtain

$$\tilde{g}(\tilde{P}U, \xi) = g(U, \tilde{P}\xi) = 0,$$

which show that $\tilde{P}U$ does not belong to $ltrT\dot{N}$,

$$\begin{aligned}
\tilde{g}(\tilde{P}U, N) &= \tilde{g}(U, \tilde{P}N) = 0,\\
\tilde{g}(\tilde{P}U, \tilde{P}\xi) &= \tilde{g}(U, \tilde{P}\xi) + \tilde{g}(U, \xi) = 0,\\
\tilde{g}(\tilde{P}U, \tilde{P}N) &= 0,\\
\tilde{g}(\tilde{P}U, W) &= \tilde{g}(U, \tilde{P}W) = 0,\\
\tilde{g}(\tilde{P}U, \tilde{P}W) &= 0,
\end{aligned}$$

for $U \in \Gamma(D_o)$, $\xi \in \Gamma(RadT\dot{N})$, $N \in \Gamma(ltrT\dot{N})$ and $W \in \Gamma(S(T\dot{N}))$. Therefore, the distribution D_o is invariant with respect to $\tilde{P}$. □

4. Screen Transversal Anti-Invariant Lightlike Submanifolds

In this section, we study screen transversal anti-invariant lightlike submanifolds of a Golden semi-Riemannian manifold. We investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on this submanifold to be a metric connection.

Let $\dot{N}$ be a screen transversal anti-invariant lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Let S and R be projection morphisms on $S(T\dot{N})$ and $RadT\dot{N}$, respectively. Then, for $W \in \Gamma(T\dot{N})$, we have

$$W = SW + RW. \tag{20}$$

On the other hand, if we apply $\tilde{P}$ to (20), we obtain

$$\tilde{P}W = S_1W + S_2W,$$

where $S_1W = \tilde{P}SW \in \Gamma(S(T\dot{N}^\perp))$, $S_2W = \tilde{P}RW \in \Gamma(S(T\dot{N}^\perp))$.

Let P_1, P_2, P_3, P_4 be the projection morphisms on $\tilde{P}RadT\dot{N}$, $\tilde{P}ltrT\dot{N}$, $\tilde{P}(S(T\dot{N}))$ and D_o respectively. Then, for $U \in \Gamma(S(T\dot{N}^\perp))$, we have

$$U = P_1U + P_2U + P_3U + P_4\,U. \tag{21}$$

If we apply $\tilde{P}$ to (21), then we can find

$$\tilde{P}U = \tilde{P}P_1U + \tilde{P}P_2U + \tilde{P}P_3U + \tilde{P}P_4\,U. \tag{22}$$

If we take $B_1 = \tilde{P}P_1$, $B_2 = \tilde{P}P_2$, $C_1 = \tilde{P}P_3$, $C_2 = \tilde{P}P_4$, we can rewrite (22) as follows:

$$\tilde{P}U = B_1U + B_2U + C_1U + C_2U.$$

In here, there are components of B_1U in $\Gamma(RadT\dot{N})$ with $\Gamma(S(T\dot{N}^\perp))$ and of B_2U in $\Gamma(S(T\dot{N}))$ with $\Gamma(S(T\dot{N}^\perp))$ and of C_1U in $\Gamma(ltrT\dot{N})$ with $\Gamma(S(T\dot{N}^\perp))$ and of C_2U in D_o with $\Gamma(S(T\dot{N}^\perp))$, namely $\tilde{P}U$ is belong to $T\tilde{N}\mid_{\dot{N}}$.

It is known that the induced connection on a screen transversal anti-invariant lightlike submanifold immersed in a semi-Riemannian manifold is not a metric connection. The condition under which the induced connection on the submanifold is a metric connection is given by the following theorem.

Theorem 4.1. *Let $\dot{N}$ be a screen transversal anti-invariant lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then, the induced connection ∇ on $\dot{N}$ is a metric connection if and only if*

$$K_1B_1\nabla^s_W\tilde{P}\xi = 0,$$

for $W \in \Gamma(T\dot{N})$ and $\xi \in \Gamma(RadT\dot{N})$.

Proof. From (17), we have

$$\tilde{\nabla}_W \tilde{P}U = \tilde{P}\tilde{\nabla}_W U.$$

if we take $U = \xi$, then we have

$$\tilde{\nabla}_W \tilde{P}\xi = \tilde{P}\tilde{\nabla}_W \xi$$

and from (4) and (6), we find

$$A_{\tilde{P}\xi}W + \nabla^s_W \tilde{P}\xi + D^l(W, \tilde{P}\xi) = \tilde{P}\left(\nabla_W \xi + h^l(W, \xi)\right) + h^s(W, \xi)).$$

Applying $\tilde{P}$ to the above equation, we find that

$$\begin{pmatrix} -\tilde{P}A_{\tilde{P}\xi}W + B_1\nabla^s_W \tilde{P}\xi + B_2\nabla^s_W \tilde{P}\xi \\ +C_1\nabla^s_W \tilde{P}\xi + C_2\nabla^s_W \tilde{P}\xi + \tilde{P}D^l(W, \tilde{P}\xi) \end{pmatrix} = \tilde{P}^2\left(\nabla_W \xi + h^l(W, \xi)\right) + h^s(W, \xi)),$$

in here, when we get, $K_1B_1\nabla^s_W \tilde{P}\xi$ is remaining part in $RadT\dot{N}$ of $B_1\nabla^s_W \tilde{P}\xi$ and $K_2B_2\nabla^s_W \tilde{P}\xi$ is the remaining part in $S(T\dot{N})$ of $B_2\nabla^s_W \tilde{P}\xi$, we can find

$$K_1B_1\nabla^s_W \tilde{P}\xi = 0.$$

□

Theorem 4.2. *Let $\dot{N}$ be a screen transversal anti-invariant lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then the radical distribution is integrable if and only if*

$$\nabla^s_{\xi_2}\tilde{P}\xi_1 = \nabla^s_{\xi_1}\tilde{P}\xi_2$$

for $\xi_1, \xi_2 \in \Gamma(RadT\dot{N})$.

Proof. From the definition of the screen transversal anti-invariant lightlike submanifold, the radical distribution is integrable if and only if

$$g\left([\xi_2, \xi_1], Z\right) = 0,$$

for $\xi_1, \xi_2 \in \Gamma(RadT\dot{N})$ and $Z \in \Gamma(S(T\dot{N}))$. Then, we have

$$0 = g(\tilde{\nabla}_{\xi_2}\tilde{P}\xi_1, \tilde{P}Z) - g(\tilde{\nabla}\xi_2\xi_1, \tilde{P}Z) - g(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, \tilde{P}Z) + g(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}Z),$$

because $\tilde{P}\xi_1, \tilde{P}\xi_2 \in \Gamma(S(T\dot{N}^{\perp}))$, from (4) and (5), we find

$$0 = g(\nabla^s_{\xi_2}\tilde{P}\xi_1 - \nabla^s_{\xi_1}\tilde{P}\xi_2, \tilde{P}Z).$$

Thus, the proof is completed. □

Theorem 4.3. *Let $\dot{N}$ be a screen transversal anti-invariant lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. In this case, the screen distribution is integrable if and only if*

$$\nabla^s_W \tilde{P}U - \nabla^s_U \tilde{P}W = h^s(W, U) - h^s(U, W),$$

for $W, U \in \Gamma(S(T\dot{N}))$.

By the definition of a screen transversal anti-invariant lightlike submanifold, the screen distribution is integrable if and only if

$$g\left([W, U], N\right) = 0,$$

for $W, U \in \Gamma(S(T\dot{N}))$ and $N \in \Gamma(ltrT\dot{N})$. In here, if we use (17) and (16), we find

$$\begin{aligned} 0 &= g(\tilde{\nabla}_W \tilde{P}U, \tilde{P}N) - g(\tilde{\nabla}_W U, \tilde{P}N) - g(\tilde{\nabla}_U \tilde{P}W, \tilde{P}N) + g(\tilde{\nabla}_U W, \tilde{P}N), \\ &= g(\nabla^s_W \tilde{P}U, \tilde{P}N) - g(\nabla^s_U \tilde{P}W, \tilde{P}N) - g(h^s(W, U), \tilde{P}N) + g(h^s(U, W), \tilde{P}N). \end{aligned}$$

From the last equation, we have

$$\nabla^s_W \tilde{P}U - \nabla^s_U \tilde{P}W = h^s(W, U) - h^s(U, W).$$

Thus, the proof is completed.

5. Radical Screen Transversal Lightlike Submanifolds Of Golden Semi-Riemannian Manifolds

In this section, we study radical screen transversal lightlike submanifolds. We investigate the integrability of distributions and give a necessary and sufficient condition for the induced connection to be a metric connection.

Theorem 5.1. *Let $\dot{N}$ be a radical screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. In this case, the screen distribution is integrable if and only if*

$$h^s(W,\tilde{P}U) = h^s(U,\tilde{P}W),$$

for $W,U \in \Gamma(S(T\dot{N}))$.

Proof. By the definition of a radical screen transversal lightlike submanifold, the screen distribution is integrable if and only if

$$g([W,U],N) = 0,$$

for $W,U \in \Gamma(S(T\dot{N}))$ and $N \in \Gamma(ltrT\dot{N})$. In here, using (16) and (17), we find

$$\begin{aligned} 0 &= g(\tilde{\nabla}_W\tilde{P}U,\tilde{P}N) - g(\tilde{\nabla}_W U,\tilde{P}N) - g(\tilde{\nabla}_U\tilde{P}W,\tilde{P}N) + g(\tilde{\nabla}_U W,\tilde{P}N), \\ &= g(h^s(W,\tilde{P}U) - h^s(U,\tilde{P}W) - h^s(W,U) + h^s(U,W),\tilde{P}N), \end{aligned}$$

hence, we have

$$h^s(W,\tilde{P}U) - h^s(U,\tilde{P}W) = h^s(W,U) - h^s(U,W).$$

Therefore, the proof is completed. □

Theorem 5.2. *Let $\dot{N}$ be a radical screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then the radical distribution is integrable if and only if*

$$A_{\tilde{P}\xi_2}\xi_1 - A_{\tilde{P}\xi_1}\xi_2 = A^*_{\xi_1}\xi_2 - A^*_{\xi_2}\xi_1,$$

for $\xi_1,\xi_2 \in \Gamma(RadT\dot{N})$.

Proof. From the definition of a radical screen transversal lightlike submanifold, the radical distribution is integrable if and only if

$$g([\xi_1,\xi_2],Z) = 0,$$

for $\xi_1,\xi_2 \in \Gamma(RadT\dot{N})$ and $Z \in \Gamma(S(T\dot{N}))$. Using (16) and (17), we have

$$0 = g(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2,\tilde{P}Z) - g(\tilde{\nabla}_{\xi_2}\tilde{P}\xi_1,\tilde{P}Z) - g(\tilde{\nabla}_{\xi_1}\xi_2,\tilde{P}Z) + g(\tilde{\nabla}_{\xi_2}\xi_1,\tilde{P}Z).$$

Since $\tilde{P}\xi_2,\tilde{P}\xi_1 \in \Gamma(S(T\dot{N}^\perp))$ and $Z \in \Gamma(S(T\dot{N}))$, if we use the equations (4) and (6), we obtain

$$0 = g(A_{\tilde{P}\xi_2}\xi_1 - A_{\tilde{P}\xi_1}\xi_2 - A^*_{\xi_1}\xi_2 + A^*_{\xi_2}\xi_1,\tilde{P}Z).$$

Hence, the proof is completed. □

Proposition 5.3. *Let $\dot{N}$ be a radical screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then the distribution D_o is invariant with respect to $\tilde{P}$.*

Proof. In a radical screen transversal lightlike submanifold, we know that

$$S(T\dot{N}^{\perp}) = \tilde{P}RadT\dot{N} \oplus \tilde{P}ltrT\dot{N} \perp D_o$$

and we have $\tilde{P}S(T\dot{N}) = S(T\dot{N})$. From here, for $W \in D_o$ and $U \in \Gamma(S(T\dot{N}))$, we find that

$$\begin{aligned} g(\tilde{P}W, \xi) &= g(W, \tilde{P}\xi) = 0, \\ g(\tilde{P}W, \tilde{P}\xi) &= 0, \\ g(\tilde{P}W, N) &= g(W, \tilde{P}N) = 0, \\ g(\tilde{P}W, \tilde{P}N) &= 0, \\ g(\tilde{P}W, U) &= g(W, \tilde{P}U) = 0, \\ g(\tilde{P}W, \tilde{P}U) &= 0. \end{aligned}$$

Therefore, we obtain that the distribution D_o is invariant with respect to $\tilde{P}$. □

Now, we investigate the geometry of the leaves of the distributions $RadT\dot{N}$ and $S(T\dot{N})$.

Theorem 5.4. *Let $\dot{N}$ be a radical screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. Then the screen distribution defines a totally geodesic foliation if and only if $h^s(W, \tilde{P}U) - h^s(W, U)$ has no component in $\tilde{P}ltrT\dot{N}$, for $W, U \in \Gamma(S(T\dot{N}))$.*

Proof. By the definition of a radical screen transversal lightlike submanifold, $S(T\dot{N})$ defines totally geodesic foliation if and only if

$$\tilde{g}\,(\nabla_W U, N) = 0,$$

where $W, U \in \Gamma(S(T\dot{N}))$ and $N \in \Gamma(ltrT\dot{N})$. From here, if we use (16) and (17), we can find

$$0 = \tilde{g}(\tilde{\nabla}_W \tilde{P}U, \tilde{P}N) - \tilde{g}(\tilde{\nabla}_W U, \tilde{P}N),$$

also from (4), we have

$$0 = g(h^s(W, \tilde{P}U) - h^s(W, U), \tilde{P}N).$$

Hence, we obtain that there is no component of $h^s(W, \tilde{P}U) - h^s(W, U)$ in $\tilde{P}ltrT\dot{N}$. □

Theorem 5.5. *Let $\dot{N}$ be a radical screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. The radical distribution defines a totally geodesic foliation if and only if there is no component of $A_{\tilde{P}\xi_2}\xi_1$ in $S(T\dot{N})$ and $A^*_{\xi_2}\xi_1 = 0$, for $\xi_1, \xi_2 \in \Gamma(RadT\dot{N})$.*

Proof. The radical distribution defines a totally geodesic foliation if and only if

$$\tilde{g}\,(\nabla_{\xi_1}\xi_2, Z) = 0,$$

for $\xi_1, \xi_2 \in \Gamma(RadT\dot{N})$ and $Z \in S(T\dot{N})$. From here, we have

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, Z) &= 0 \Leftrightarrow g(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, \tilde{P}Z) - g(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}Z) = 0 \\ &= g(-A_{\tilde{P}\xi_2}\xi_1 + A^*_{\xi_2}\xi_1, \tilde{P}Z). \end{aligned}$$

Therefore, the proof is completed. □

Theorem 5.6. *Let $\dot{N}$ be a radical screen transversal lightlike submanifold of a Golden semi-Riemannian manifold $\tilde{N}$. The induced connection ∇ on $\dot{N}$ is a metric connection if and only if there is no component of $h^s(U, W)$ in $\tilde{P}RadT\dot{N}$ or of $A_{\tilde{P}\xi}W$ in $S(T\dot{N})$, for $W, U \in S(T\dot{N})$.*

Proof. For $W \in S(T\dot{N})$ and $\xi \in \Gamma(RadT\dot{N})$, we have

$$\tilde{\nabla}_W \tilde{P}\xi = \tilde{P}\tilde{\nabla}_W \xi.$$

Taking the inner product of this equation with $U \in \Gamma(S(T\dot{N}))$, we obtain

$$\tilde{g}(\tilde{\nabla}_W \tilde{P}\xi, U) = \tilde{g}(\tilde{\nabla}_W \xi, \tilde{P}U).$$

From here, if we use the equation (6), we can find that

$$-g(A_{\tilde{P}\xi}W, U) = g(\nabla_W \xi, \tilde{P}U).$$

Here, we come to the conclusion that there is no component of $A_{\tilde{P}\xi}W$ in $S(T\dot{N})$, using the equation (7) in the last equation, we have

$$-g\left(h^s(U, W), U\right) = g(\nabla_W \xi, \tilde{P}U),$$

namely, there is no component of $h^s(U, W)$ in $\tilde{P}RadT\dot{N}$. $\square$

Now, let us give the relationship between the almost product structure and the Metallic structure. Then we associate the examples in [38] with the Metallic structure.

If F is an almost product structure on $\tilde{N}$, then

$$\tilde{P} = \frac{1}{\sqrt{2}}(I + \sqrt{5}F)$$

is a Golden structure on $\tilde{N}$. Conversely, if $\tilde{P}$ is a Golden structure on $\tilde{N}$, then

$$F = \frac{1}{\sqrt{5}}(2\tilde{P} - I)$$

is an almost product structure on $\tilde{N}$. We can give that following examples.

Example 5.7. *Let $\tilde{N} = \mathbb{R}^4_2 \times \mathbb{R}^4_2$ be a semi-Riemannian product manifold with the semi-Riemannian product metric tensor $\tilde{g} = \pi^* g_1 \otimes \sigma^* g_2$, i=1,2, where g_i denote standard metric tensor of $\mathbb{R}^4_2$. Let*

$$\begin{array}{rcl} f : \dot{N} & \rightarrow & \tilde{N}, \\ (x_1, x_2, x_3) & \rightarrow & (x_1,\ x_2 + x_3,\ x_1,\ 0, x_1,\ 0,\ x_2 - x_3,\ x_1). \end{array}$$

Then, we find

$$\begin{array}{rcl} Z_1 & = & \partial x_1 + \partial x_3 - \partial x_5 - \partial x_8, \\ Z_2 & = & \partial x_2 + \partial x_7, \\ Z_3 & = & \partial x_2 - \partial x_7. \end{array}$$

The radical bundle $RadT\dot{N}$ is spanned by Z_1 and $S(T\dot{N}) = Span\{Z_2, Z_3\}$ for $FZ_2 = Z_3$. If we choose

$$N = -\frac{1}{2}(\partial x_1 - \partial x_5),$$

we find

$$g(Z_1, N) = 1.$$

Also we obtain

$$\begin{array}{rcl} FZ_1 & = & \partial x_1 + \partial x_3 + \partial x_5 + \partial x_8 = W_1, \\ FN & = & -\frac{1}{2}(\partial x_1 + \partial x_5) = W_2. \end{array}$$

If we choose

$$W_3 = \partial x_4 + \partial x_6,$$

we obtain

$$FW_3 = W_4 = \partial x_4 - \partial x_6.$$

Thus, we have $FRadT\dot{N} \subset S(T\dot{N}^{\perp})$ *and* $FS(T\dot{N}) = S(T\dot{N})$. *From here, using* $\tilde{P} = \frac{1}{\sqrt{2}}(I + \sqrt{5}F)$, *we find that*

$$\begin{aligned}
\tilde{P}Z_1 &= \begin{pmatrix} \phi\sqrt{2}\partial x_1 + \phi\sqrt{2}\partial x_2 \\ +(\phi-1)\sqrt{2}\partial x_5 + (\phi-1)\sqrt{2}\partial x_8 \end{pmatrix} \in ST\dot{N}^{\perp}),\\
\tilde{P}N &= -\frac{1}{\sqrt{2}}(\phi\partial x_1 + (\phi-1)\partial x_5) \in S(T\dot{N}^{\perp}),\\
\tilde{P}Z_2 &= (\phi\sqrt{2}\partial x_2 + (\phi-1)\sqrt{2}\partial x_7) \in S(T\dot{N}),\\
\tilde{P}W_3 &= (\phi\sqrt{2}\partial x_4 + (\phi-1)\sqrt{2}\partial x_8) \in S(T\dot{N}).
\end{aligned}$$

Therefore, $\dot{N}$ *is a radical screen transversal lightlike submanifold of* $\tilde{N}$.

Example 5.8. *Let* $\tilde{N} = \mathbb{R}^4_2 \times \mathbb{R}^4_2$ *be a semi-Riemannian product manifold with the semi-Riemannian product metric tensor* $\tilde{g} = \pi^* g_1 \otimes \sigma^* g_2$, *i=1,2, where* g_i *denote standard metric tensor of* $\mathbb{R}^4_2$. *Let*

$$\begin{aligned}
f : \dot{N} &\rightarrow \tilde{N},\\
(x_1, x_2, x_3) &\rightarrow (-x_1,\ -x_2,\ x_1,\ \sqrt{2}x_2,\ x_1,\ 0,\ 0,\ x_1,\ -x_2).
\end{aligned}$$

Then, we find

$$\begin{aligned}
Z_1 &= -\partial x_1 + \partial x_3 + \partial x_5 + \partial x_7\\
Z_2 &= -\partial x_2 - \sqrt{2}\partial x_4 - \partial x_8.
\end{aligned}$$

The radical bundle $RadT\dot{N}$ *is spanned by* Z_1 *and* $S(T\dot{N}) = Span\{Z_2\}$. *If we choose*

$$N = \frac{1}{2}(\partial x_1 - \partial x_5),$$

we find

$$g(Z_1, N) = 1.$$

Thus, we obtain

$$\begin{aligned}
FZ_1 &= -\partial x_1 + \partial x_3 - \partial x_5 - \partial x_7 = W_1,\\
FZ_2 &= -\partial x_2 - \sqrt{2}\partial x_4 + \partial x_8 = W_2,\\
FN &= \frac{1}{2}(\partial x_1 + \partial x_5) = W_3.
\end{aligned}$$

If we choose

$$W_4 = -\sqrt{2}\partial x_2 - \sqrt{2}\partial x_4 + \partial x_6,$$

we obtain

$$FW_3 = W_4 = \partial x_4 - \partial x_6.$$

Thus, we have $FRadT\dot{N} \subset S(T\dot{N}^{\perp})$ *and* $FS(T\dot{N}) = S(T\dot{N}^{\perp})$. *From here, using* $\tilde{P} = \frac{1}{\sqrt{2}}(I + \sqrt{5}F)$, *we find that*

$$\begin{aligned}
\tilde{P}Z_1 &= \begin{pmatrix} -\phi\sqrt{2}\partial x_1 + \phi\sqrt{2}\partial x_2 \\ +(\phi-1)\sqrt{2}\partial x_5 + (\phi-1)\sqrt{2}\partial x_7 \end{pmatrix} \in S(T\dot{N}^{\perp}),\\
\tilde{P}Z_2 &= \begin{pmatrix} -\phi\sqrt{2}\partial x_2 - 2\phi\partial x_4 \\ +(\phi-1)\sqrt{2}\partial x_8 \end{pmatrix} \in S(T\dot{N}^{\perp}).
\end{aligned}$$

Therefore, $\dot{N}$ *is a radical transversal lightlike submanifold.*

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