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On Power Similarity of Complex Symmetric Operators

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Abstract. In this paper, we study properties of operators which are power similar to complex symmetric operators. In particular, we prove that if T is power similar to a complex symmetric operator, then T is decomposable modulo a closed set $S \subset \mathbb{C}$ if and only if R has the Bishop's property (β) modulo S. Using the results, we get some applications of such operators.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_a(T)$, $\sigma_{su}(T)$, and $\sigma_e(T)$ for the spectrum, the approximate point spectrum, the surjective spectrum, and the essential spectrum of T, respectively.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is a complex symmetric operator with a conjugation C. The terminology of complex symmetric operators was motivated by the antilinear eigenvalue problem $T\mathbf{x} = \lambda\bar{\mathbf{x}}$ where T is an $n\times n$ symmetric complex matrix and $\bar{\mathbf{x}}$ denotes the complex conjugation of the vector $\mathbf{x}$ in $\mathbb{C}^n$. In [18], T. Takagi noted that this equation gives information about eigenvalues of $|T| := (T^*T)^{\frac{1}{2}}$ and he obtained various results based on this observation. Indeed, complex symmetric operators have been studied for many years in the finite dimensional setting. In 2006, S. R. Garcia and M. Putinar ([4]) have proven interesting results for this class of operators in the infinite dimensional case. The class of complex symmetric operators includes all normal operators, Hankel operators, compressed Toeplitz operators, and the Volterra integration operator, and there are a lot of consequences and applications concerning complex symmetric operators (see [4], [5], [8]-[11], etc.).

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Definition 1.1. Let $R \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *power similar* to R if there exists a positive integer n such that T^n is similar to R^n. In this case, we use the notation $T \overset{ps}{\sim} R$.

For a fixed complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$, define the following subset of $\mathcal{L}(\mathcal{H})$:

$$PS_n(R) = \{T \in \mathcal{L}(\mathcal{H}) : T^n \text{ is similar to } R^n\}$$

where n is a positive integer. We observe that the following relations hold:

$$PS_1(R) \subset PS_n(R) \subset PS_{n^2}(R) \subset PS_{n^3}(R) \subset \cdots$$

for each positive integer n. Set

$$PS(R) := \cup_{n=1}^{\infty} PS_n(R) = \{T \in \mathcal{L}(\mathcal{H}) : T \overset{ps}{\sim} R\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathcal{H}$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known from [13] that

$$\text{Bishop's property } (\beta) \Rightarrow \text{SVEP}.$$

It can be shown that the converse implication does not hold in general as can be seen from [13].

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in \mathcal{L}(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$ind(T) := \alpha(T) - \beta(T)$$

where $\alpha(T) := dim\ker(T)$ and $\beta(T) := dim\ker(T^*)$, respectively. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent, respectively.

In 2015, S. Jung, E. Ko, and M. Lee ([12]) studied operators which are power similar to hyponormal operators. Recently, S. Zhu and J. Zhao ([19]) considered similarity orbits of complex symmetric operators. From the main results of [12] and [19], we study operators which are power similar to complex symmetric operators.

The outline of the paper organizes the followings. In section 2, we investigate examples of an operator T which belongs to $PS_n(R)$ and basic properties of such operators. In section 3, we prove that if $T \in PS_n(R)$ for a complex symmetric operator R, then T is decomposable modulo a closed set $S \subset \mathbb{C}$ if and only if R has the Bishop's property (β) modulo S.

2. Examples and basic properties

Let C be a fixed conjugation on $\mathcal{H}$. For a complex symmetric operator R with respect to a conjugation C, we set $PS_n(R) = \{T \in \mathcal{L}(\mathcal{H}) : T^n \text{ is similar to } R^n\}$ for some positive integer n and $PS(R) = \cup_{n=1}^{\infty} PS_n(R)$. In this section, we investigate examples of an operator T which belongs to $PS_n(R)$ and basic properties of such operators. Even if R is complex symmetric, then $T \in PS_n(R)$ may not be complex symmetric. In the following example, we know that complex symmetry is not invariant under power similarity.

Example 2.1. Let $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \lambda I$ on $\mathbb{C}^3$ for some nonzero $\lambda \in \mathbb{C}$. Then R is a complex symmetric operator from [4, Example 6]. Since $\sigma(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \cap \sigma(\lambda I) = \emptyset$ from [16, Corollary 0.15], it follows that R is similar to $T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ by [6, Corollary 3.22]. Therefore T belongs to $PS_1(R)$. But T is not complex symmetric from [19, Lemma 2.16].

Example 2.2. Let $R = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}$ where A is a complex symmetric operator with a conjugation C. Then R^2 is a complex symmetric operator with a conjugation $C \oplus C$. If $T \in PS_2(R)$ such that $T^2 = X^* R^2 X$ for some unitary operator X, then T^2 is a complex symmetric operator with a conjugation $X^*(C \oplus C)X$.

In general, if C is a conjugation on $\mathcal{H}$, then U^*CU is a conjugation where U is unitary from [4]. But, $X^{-1}CX$ may not be a conjugation as in the following example.

Example 2.3. On $\mathbb{C}^2$, define $X : \mathbb{C}^2 \to \mathbb{C}^2$ as $X(a, b) = (2a, b)$. Define $C : \mathbb{C}^2 \to \mathbb{C}^2$ as $C(a, b) = (\overline{b}, \overline{a})$. Then X is invertible and C is a conjugation on $\mathbb{C}^2$. Moreover, since

$$X^{-1}CX(a, b) = X^{-1}C(2a, b) = X^{-1}(\overline{b}, 2\overline{a}) = (\frac{1}{2}\overline{b}, 2\overline{a}),$$

it follows that $(X^{-1}CX)^2(a, b) = X^{-1}CX(\frac{1}{2}\overline{b}, 2\overline{a}) = (\frac{1}{2}2\overline{\overline{a}}, 2\frac{1}{2}\overline{\overline{b}}) = (a, b)$, which means that $(X^{-1}CX)^2 = I$. On the other hand for $x = (a, b)$ and $y = (c, d)$, we have

$$\langle (X^{-1}CX)x, y\rangle = \langle (X^{-1}CX)(a, b), (c, d)\rangle = \langle (\frac{1}{2}\overline{b}, 2\overline{a}), (c, d)\rangle = \frac{1}{2}\overline{b}\overline{c} + 2\overline{a}\overline{d}$$

and

$$\langle (X^{-1}CX)y, x\rangle = \langle (X^{-1}CX)(c, d), (a, b)\rangle = \langle (\frac{1}{2}\overline{d}, 2\overline{c}), (a, b)\rangle = \frac{1}{2}\overline{d}\overline{a} + 2\overline{c}\overline{b}.$$

Thus $X^{-1}CX$ is not isometric. Hence $X^{-1}CX$ is not a conjugation.

Now, we state some conditions for $X^{-1}CX$ to be a conjugation in the following lemma.

Lemma 2.4. *Let C be a conjugation on $\mathcal{H}$ and X be an invertible operator. If $X^{-1}CX$ is isometric, i.e., $\langle X^{-1}CXx, y\rangle = \langle X^{-1}CXy, x\rangle$ for all $x, y \in \mathcal{H}$, then $X^{-1}CX$ is a conjugation on $\mathcal{H}$.*

Proof. Let C be a conjugation on $\mathcal{H}$ and X be an invertible operator. Then $X^{-1}CX$ is clearly antilinear and $(X^{-1}CX)^2 = X^{-1}C^2X = I$. Since $\langle X^{-1}CXx, y\rangle = \langle X^{-1}CXy, x\rangle$ for all $x, y \in \mathcal{H}$, it follows that $X^{-1}CX$ is a conjugation on $\mathcal{H}$. □

Example 2.5. On $\mathbb{C}^2$, define $X : \mathbb{C}^2 \to \mathbb{C}^2$ as $X(a, b) = (\lambda a, \lambda b)$. Define $C : \mathbb{C}^2 \to \mathbb{C}^2$ as $C(a, b) = (\overline{b}, \overline{a})$. Then X is invertible and X^{-1}CX is a conjugation on $\mathbb{C}^2$. Indeed, since

$$X^{-1}CX(a, b) = X^{-1}C(\lambda a, \lambda b) = X^{-1}(\overline{\lambda b}, \overline{\lambda a}) = (\frac{\overline{\lambda}}{\lambda}\overline{b}, \frac{\overline{\lambda}}{\lambda}\overline{a}),$$

it follows that $(X^{-1}CX)^2(a, b) = (a, b)$, which means that $(X^{-1}CX)^2 = I$. On the other hand, for $x = (a, b)$ and $y = (c, d)$, we have

$$\langle (X^{-1}CX)x, y\rangle = \langle (X^{-1}CX)(a, b), (c, d)\rangle = \langle (\frac{\overline{\lambda}}{\lambda}\overline{b}, \frac{\overline{\lambda}}{\lambda}\overline{a}), (c, d)\rangle = \frac{\overline{\lambda}}{\lambda}\overline{b}\overline{c} + \frac{\overline{\lambda}}{\lambda}\overline{a}\overline{d}$$

and

$$\langle (X^{-1}CX)y, x\rangle = \langle (X^{-1}CX)(c,d),(a,b)\rangle = \langle (\frac{\overline{\lambda}}{\lambda}\overline{d}, \frac{\overline{\lambda}}{\lambda}\overline{c}),(a,b)\rangle = \frac{\overline{\lambda}}{\lambda}\overline{d}\overline{a} + \frac{\overline{\lambda}}{\lambda}\overline{c}\overline{b}.$$

Thus $X^{-1}CX$ is isometric. Hence $X^{-1}CX$ is a conjugation.

Proposition 2.6. *Let $R \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator with a conjugation C. If $T \in PS_n(R)$, that is, $T^n = X^{-1}R^nX$ for an invertible operator X, then $(X^{-1}CX)T^n(X^{-1}CX) = (X^*X)^{-1}T^{n*}(X^*X)$. In particular, if $\langle X^{-1}CXx, y\rangle = \langle X^{-1}CXy, x\rangle$ for all $x, y \in \mathcal{H}$, then T^n is complex symmetric with the conjugation $X^{-1}CX$ if and only if $[T^n, |X|^2] = 0$ for all $n \in \mathbb{N}$.*

Proof. Since R is a complex symmetric operator with a conjugation C, so is R^n with a conjugation C. Since $T \in PS_n(R)$, there exists an integer $n > 0$ such that T^n is similar to R^n, i.e., $T^n = X^{-1}R^nX$ for some invertible $X \in \mathcal{L}(\mathcal{H})$. Then

$$\begin{aligned}(X^{-1}CX)T^n(X^{-1}CX) &= X^{-1}CX(X^{-1}R^nX)X^{-1}CX\\ &= X^{-1}C(R^n)CX = X^{-1}(R^n)^*X\\ &= X^{-1}(XT^nX^{-1})^*X\\ &= X^{-1}((X^{-1})^*T^{*n}X^*)X\\ &= (X^*X)^{-1}T^{n*}(X^*X).\end{aligned} \tag{1}$$

In particular, if $\langle X^{-1}CXx, y\rangle = \langle X^{-1}CXy, x\rangle$ for all $x, y \in \mathcal{H}$, then $X^{-1}CX$ is a conjugation from Lemma 2.4. If $[T^n, |X|^2] = 0$, then (1) gives that

$$(X^{-1}CX)T^n(X^{-1}CX) = (X^*X)^{-1}T^{n*}(X^*X) = (T^n)^*.$$

Similarly, the converse statement holds. Hence T^n is complex symmetric with the conjugation $X^{-1}CX$ if and only if $[T^n, |X|^2] = 0$ for all $n \in \mathbb{N}$. □

For $u, v \in \mathcal{H}$, let $u \otimes v$ denote the operator given by $(u \otimes v)f = \langle f, v\rangle u$ for $f \in \mathcal{H}$, which has rank one when $u, v \neq 0$. Note that the operator $R = u \otimes v$ satisfies $R = CR^*C$ if and only if R is a constant multiple of $u \otimes Cu$ (see [4, Lemma 2]).

Example 2.7. Let $T \in PS_2(R)$ where $R = u \otimes Cu$. Then

$$R^2 = (u \otimes Cu)(u \otimes Cu) = \langle Cu, u\rangle(u \otimes Cu)$$

and so R^2 is clearly complex symmetric from [4, Lemma 2]. Since $T^2 = X^{-1}R^2X$ where X is invertible with $X^{-1} \neq CX^*C$, we get that $T^2 = \langle Cu, u\rangle(X^{-1}u \otimes X^*Cu)$. On the other hand,

$$CT^2 = \langle u, Cu\rangle(CX^{-1}u \otimes X^*Cu) \text{ and } T^{*2}C = \langle u, Cu\rangle(X^*Cu \otimes CX^{-1}u).$$

Thus $CT^2 \neq T^{*2}C$ in general. So T^2 is not complex symmetric.

Let $R \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator with a conjugation C. For some $n \in \mathbb{N}$, set

$$CPS_n(R) := \{T \in \mathcal{L}(\mathcal{H}) : T^n \text{ is similar to } R^n \text{ and } T^n \text{ is complex symmetric}\}.$$

Then from Examples 2.2 and 2.7, $CPS_n(R)$ is a nonempty proper subset of $PS_n(R)$.

Proposition 2.8. *Let $R \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator with a conjugation C. If $T \in CPS_n(R)$, then the following properties hold:*

(i) *T is left invertible if and only if T is right invertible. Hence T is left or right invertible if and only if T is invertible.*

(ii) $\ker(T)$ *is trivial if and only if* $\operatorname{ran}(T)$ *is dense in $\mathcal{H}$.*

(iii) *If R is Fredholm, then T is Weyl. Conversely, if T is Fredholm, then R is Weyl.*

Proof. (i) If T is left invertible, then there exists an operator $A \in \mathcal{L}(\mathcal{H})$ such that $AT = I$. Thus $A^nT^n = I$, which means that T^n is left invertible. Since T^n is a complex symmetric operator, it follows from [4, Proposition 1] that T^n is left invertible if and only if T^n is right invertible. Hence T^n is right invertible. Since there exists an operator $B \in \mathcal{L}(\mathcal{H})$ such that $T(T^{n-1}B) = T^nB = I$, it follows that T is right invertible. The converse implication holds by similar methods.

(ii) Assume that $\ker(T)$ is trivial. Then $\ker(T^n)$ is trivial. Since T^n is a complex symmetric operator, it follows from [4, Proposition 1] that $\ker(T^n)$ is trivial if and only if $\operatorname{ran}(T^n)$ is dense in $\mathcal{H}$. Moreover, since $\overline{\operatorname{ran}(T^n)} \subset \overline{\operatorname{ran}(T)} \subset \mathcal{H}$, it follows that $\operatorname{ran}(T)$ is dense in $\mathcal{H}$. Similarly, the converse implication holds.

(iii) Since R is Fredholm, R^n is also Fredholm. Moreover, since $T^n = X^{-1}R^nX$, it follows that T^n is Fredholm. Hence T is Fredholm by [17, Theorem 3 (b)]. Since T^n is complex symmetric, $ind(T^n) = n \cdot ind(T) = 0$ by [4, Proposition 1]. Hence T is Weyl. Conversely, if T is Fredholm, then T^n is Fredholm. Since $T \in CPS_n(R)$, R^n is Fredholm and so R is also Fredholm from [17]. Moreover, since R is a complex symmetric operator, $ind(R) = 0$ from [4, Proposition 1]. Hence R is Weyl. □

We next focus on the stability of complex symmetry for $T \in PS_n(R)$ for some symmetric operator R. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially compact* if $p(T)$ is compact for some nonzero polynomial $p(z)$. In [15], C. L. Olsen proved that each polynomially compact operator is the sum of an algebraic operator and a compact one. So, if T is polynomially compact, then $\sigma_e(T)$ is finite. Let CSO denote the set of all complex symmetric operators. Let $R \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator and let $T \in PS_n(R)$. If $\sigma(R)$ is connected, then the following theorem says that $\sigma(T^n)$ is a singleton. Moreever, if $T \in PS_n(R), n \geq 1$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$ and $\sigma(R)$ consists of two components, then T may not be an algebraic operator of order 2. For example, if N is a normal operator with $\sigma(N) = \{z \in \mathbb{C} : |z-2| \leq 1\}$. Define $T = R = N \oplus (-N)$. Then T and R are complex symmetric, $T \in PS_n(R)$, and $\sigma(T) = \sigma(R)$ consists of two components. Hence T is not algebraic.

Proposition 2.9. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. If $\sigma(R)$ is connected, then $\sigma(T^n)$ is a singleton.*

Proof. We want to show that $\sigma(T^n)$ is a singleton. If not, then $\sigma(T^n)$ is an infinite connected set. Then $\partial\sigma(T^n)$ is infinite. By [3, Chapter XI. 6.8 and 6.9], $\sigma_e(T^n) \supset \partial\sigma(T)$ is infinite. Then T^n is not polynomially compact and so R^n is not polynomially compact. By [19, Lemma 2.18], R^n is not complex symmetric, which is a contradiction. Since $\sigma(T^n)$ is connected, it follows that $\sigma(T^n)$ is a singleton. □

3. Local spectral properties

In this section, we focus on the local spectral properties of operators $T \in PS_n(R)$ for a complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. Recall an operator $T \in \mathcal{L}(\mathcal{H})$ has the single valued extension property, respectively, Bishop's property (β) modulo a closed set $S \subset \mathbb{C}$ if for all open subsets $V \subseteq \mathbb{C} \setminus S$ the mapping

$$\mathcal{O}(V,\mathcal{H}) \to \mathcal{O}(V,\mathcal{H}), \quad f \mapsto (T-z)f$$

is injective, respectively, injective with closed range on the space $\mathcal{O}(V,\mathcal{H})$ of all analytic functions on V with values in $\mathcal{H}$. Fix an arbitrary open set $V \subseteq \mathbb{C} \setminus S$ and let now X be the quotient of the space $w(\mathbb{N},\mathcal{O}(V,\mathcal{H}))$ of all sequences in $\mathcal{O}(V,\mathcal{H})$ modulo the subspace $c_0(\mathbb{N},\mathcal{O}(V,\mathcal{H}))$ of all sequences that tend to 0 in $\mathcal{O}(V,\mathcal{H})$. If these conditions are satisfied with $S = \emptyset$, the T will be said to possess the single valued extension property or Bishop's property (β), respectively. By Theorems 8 and 21 in [2], an operator $T \in \mathcal{L}(\mathcal{H})$ is decomposable modulo a closed set $S \subseteq \mathbb{C}$ if and only if T and its adjoint $T^* \in \mathcal{L}(\mathcal{H}^*)$ both have the Bishop's property (β) modulo S. Hence we get the following theorem.

Theorem 3.1. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. Then T is decomposable modulo a closed set $S \subset \mathbb{C}$ if and only if R has the Bishop's property (β) modulo S.*

Proof. We first show that if R has the Bishop's property (β) modulo a closed set $S \subset \mathbb{C}$, then T has. Suppose that R has the Bishop's property (β) modulo a closed set $S \subset \mathbb{C}$. Let $z_0 \in \mathbb{C} \setminus S$ and let G be an open set in $\mathbb{C} \setminus S$ and let $\{f_k\}_{k=1}^{\infty}$ be any $\mathcal{H}$-valued analytic function on G such that

$$\lim_{k\to\infty} \|(T-z)f_k(z)\| = 0$$

uniformly on compact subsets K of G, which implies that

$$\lim_{k\to\infty} \|(T^n-z^n)f_k(z)\| = 0$$

uniformly on compact subsets K of G. Since $T^n = X^{-1}R^nX$, it follows that

$$\lim_{k\to\infty} \|(R^n-z^n)Xf_k(z)\| = \lim_{k\to\infty} \|(R-z)Q(R,z)Xf_k(z)\| = 0 \tag{2}$$

uniformly on compact subsets K of G. Here, by the fundamental theorem of algebra,

$$Q(\lambda, z) = (\lambda - p_1 z)\cdots(\lambda - p_{n-1}z)$$

where $p_1 z, \cdots, p_{n-1}z$ list the zeros of $Q(\lambda, z)$ by multiplicities. Set $p_n = 1$. Since each p_j is nonzero, we obtain from (2) that

$$\lim_{k\to\infty} \|\prod_{j=1}^{n}(\frac{1}{p_j}R - z)Xf_k(z)\| = 0 \tag{3}$$

uniformly on compact subsets K of G.

We claim that it holds for $r = 1, 2, \cdots, n$ that

$$\lim_{k\to\infty} \|\prod_{j=r}^{n}(\frac{1}{p_j}R - z)Xf_k(z)\| = 0 \tag{4}$$

uniformly on compact subsets K of G. We will prove the induction on r. If $r = 1$, then the claim holds clearly by (3). Suppose that (4) is true for some $r = t < n$, that is,

$$\lim_{k\to\infty} \|(\frac{1}{p_t}R - z)\prod_{j=t+1}^{n}(\frac{1}{p_j}R - z)Xf_k(z)\| = 0$$

uniformly on compact subsets K of G. Since $\frac{1}{p_t}R$ has the Bishop's property (β) modulo S,

$$\lim_{k\to\infty} \|\prod_{j=t+1}^{n}(\frac{1}{p_j}R - z)Xf_k(z)\| = 0 \tag{5}$$

uniformly on compact subsets K of G.

From (4) with $r = n$, we have

$$\lim_{k\to\infty} \|(R-z)Xf_k(z)\| = 0$$

uniformly on compact subsets K of G. Since R has the Bishop's property (β) modulo S, it follows that $\lim_{k\to\infty} \|Xf_k(\lambda)\| = 0$ uniformly on compact subsets K of G. Moreover, since X is invertible, it holds that

$$\lim_{k\to\infty} \|f_k(z)\| = 0 \tag{6}$$

uniformly on compact subsets K of G. Hence T has the Bishop's property (β) modulo S. Since R is a complex symmetric operator, R^* has also the Bishop's property (β) modulo S by [11, Theorem 2.1]. Therefore, T^* has the Bishop's property (β) modulo S. Hence T is decomposable modulo S from [13]. Conversely, if T is decomposable modulo S, then T has the Bishop's property (β) modulo S. Since $T \in PS_n(R)$, R has the Bishop's property (β) modulo S. □

Recall that a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called an *invariant subspace* for an operator $T \in \mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$. We say that $\mathcal{M} \subset \mathcal{H}$ is a *hyperinvariant subspace* for $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{M}$ is an invariant subspace for every $S \in \mathcal{L}(\mathcal{H})$ commuting with T. As some applications of Theorem 3.1, we provide several useful results.

Corollary 3.2. *Let $T \in PS_n(R)$ for some operator $R \in \mathcal{L}(\mathcal{H})$ which has the Bishop's property (β). If $T \neq \lambda$ for any $\lambda \in \mathbb{C}$ and $\sigma_T(x) \subsetneq \sigma(T)$ for some $x \in \mathcal{H} \setminus \{0\}$, then T and T^* have a nontrivial hyperinvariant subspace.*

Proof. Since R has the Bishop's property (β), T is decomposable from Theorem 3.1. The proof follows from similar arguments in [12]. $\square$

An operator T in $\mathcal{L}(\mathcal{H})$ is called *quasitriangular* if T can be written as sum $T = T_0 + K$, where T_0 is a triangular operator (i.e., there exists an orthonormal basis for $\mathcal{H}$ with respect to which the matrix for T_0 has upper triangular form) and $K \in \mathcal{K}(\mathcal{H})$. We say that T is *biquasitriangular* if both T and T^* are quasitriangular (see [14] for more details).

Corollary 3.3. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. Then the following statements hold:*
(i) *If R has the single-valued extension property, then*

$$\sigma_a(T^*) = \sigma(T^*) = \sigma_a(T)^* = \sigma(T)^* = \sigma_{su}(T)^* = \sigma_{su}(T^*).$$

Moreover, in this case, T is biquasitriangular.
(ii) *If R has finite ascent of order $m(\geq 2)$, then T and T^* are finite ascent of order n. Moreover, in this case,* $\ker(T) \cap \operatorname{ran}(T^n) = \{0\}$ *and* $\ker(T^*) \cap \operatorname{ran}(T^{*n}) = \{0\}$ *for some $n \in \mathbb{N}$.*

Proof. (i) Since R is complex symmetric and it has the single-valued extension property, it follows from Theorem 3.1 and [8] that T and T^* have the single-valued extension property. Then $\sigma(T^*) = \sigma_a(T^*)$ and $\sigma(T) = \sigma_a(T)$ (see [1] or [13]). For any $T \in \mathcal{L}(\mathcal{H})$, $\sigma_a(T^*) = \sigma_{su}(T)^*$. Hence it holds that

$$\sigma_a(T^*) = \sigma(T^*) = \sigma_a(T)^* = \sigma(T)^* = \sigma_{su}(T)^* = \sigma_{su}(T^*).$$

Moreover, since T and T^* have the single-valued extension property, we conclude from [14] that T is biquasitriangular.

(ii) Assume that $\ker(R^m) = \ker(R^{m+1})$ for some $m \geq 2$. If $T \in PS_n(R)$, then $T^n = X^{-1}R^nX$ for some positive integer n. It suffices to show the inclusion $\ker(T^{n+1}) \subset \ker(T^n)$. If $x \in \ker(T^{n+1})$, then $T^{n+1}x = 0$, i.e., $T^{mn}x = 0$. Since $T \in PS_n(R)$, it follows that $T^{mn} = X^{-1}R^{mn}X$, which implies that $T^{mn}x = X^{-1}R^{mn}Xx = 0$ and so $R^{mn}Xx = 0$. Since $mn \geq n+1$, $R^nXx = 0$ and so $T^nx = 0$. Hence $x \in \ker(T^n)$. Thus $\ker(T^{n+1}) \subset \ker(T^n)$. So T has finite ascent. Moreover, since R is a complex symmetric operator and R has finite ascent, it follows that R^* has also finite ascent by [11, Lemma 4.2]. Hence T^* has finite ascent using the above similar way.

By the above statement, T has finite ascent. If $y \in \ker(T) \cap ran(T^n)$, then $Ty = 0$ and $y = T^nx$ for some $x \in \mathcal{H}$ which implies that $T^{n+1}x = Ty = 0$. Since $x \in \ker(T^{n+1}) = \ker(T^n)$, we have $y = T^nx = 0$. Hence $\ker(T) \cap \operatorname{ran}(T^n) = \{0\}$. Moreover, since T^* has also finite ascent, we get that $\ker(T^*) \cap \operatorname{ran}(T^{*n}) = \{0\}$ using the above similar method. $\square$

Recall that for $T \in \mathcal{L}(\mathcal{H})$, we define the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

It is evident that

$$\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T).$$

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in iso(\sigma(T)) : 0 < dim\ker(T-\lambda) < \infty\}$ and $iso(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$.

We define the definitions of some spectra;

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the Browder essential approximate point spectrum.

We say that

(i) *a-Browder's theorem holds for T* if $\sigma_{ea}(T) = \sigma_{ab}(T)$;

(ii) *a-Weyl's theorem holds for T* if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$,
where $\pi_{00}^a(T) := \{\lambda \in \text{iso}\,\sigma_a(T) : 0 < dim\,\ker(T-\lambda) < \infty\}$.

It is known that

$$\begin{array}{ccc} \text{Browder's theorem} & \Longleftarrow & a\text{-Browder's theorem} \\ \Uparrow & & \Uparrow \\ \text{Weyl's theorem} & \Longleftarrow & a\text{-Weyl's theorem.} \end{array}$$

Corollary 3.4. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. Suppose that R has the single-valued extension property. Then the following statements hold:*
(i) *Weyl's theorem holds for T if and only if a-Weyl's theorem holds for T.*
(ii) *Weyl's theorem holds for T^* if and only if a-Weyl's theorem holds for T^*.*
(iii) *Browder's theorem holds for T and T^*.*
(iv) *a-Browder's theorem holds for T and T^*.*

Proof. (i) Since R has the single-valued extension property, it follows from Corollary 3.3 that $\sigma(T) = \sigma_a(T)$ and $\sigma(T^*) = \sigma_a(T^*)$. Hence $\pi_{00}^a(T) = \pi_{00}(T)$ and $\pi_{00}^a(T^*) = \pi_{00}(T^*)$. Moreover, since T^* has the single-valued extension property by Theorem 3.1 and [8], it follows [1, Corollary 3.53] from that $\sigma_w(T) = \sigma_{ea}(T)$. If Weyl's theorem holds for T, then $\pi_{00}^a(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{ea}(T)$. Hence a-Weyl's theorem holds for T. The converse implication is trivial.

The statement (ii) holds by similar methods (i). The statement (iii) and (iv) hold by [1] and (i). □

For an operator $T \in \mathcal{L}(\mathcal{H})$, the *commutant* of T, denoted by $\{T\}'$, is the collection of all $S \in \mathcal{L}(\mathcal{H})$ commuting with T. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ has *the property* (E) if there exist sequences $\{B_n\} \subset \mathcal{L}(\mathcal{H})$ and $\{K_n\} \subset \mathcal{K}(\mathcal{H})$ such that $\|B_n - T\| \to 0$, $K_nB_n = B_nK_n$ for each $n \in \mathbb{N}$ and $\{K_n\}$ is a nontrivial sequence of compact operators. An operator T in $\mathcal{L}(\mathcal{H})$ will be said to have *the property* (PS) if there exist sequences $\{S_n\} \subset \{T\}'$ and $\{K_n\} \subset \mathcal{K}(\mathcal{H})$ such that $\|S_n - K_n\| \to 0$ and $\{K_n\}$ is a nontrivial sequence of compact operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property* (A) provided that for every (not necessarily strict) contraction S, every operator X with dense range such that $TX = XS$, and every vector $x \in \mathcal{H}$, there exists a nonzero polynomial $p(z)$ such that $p(T)x$ belongs to ran(X). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property* (K) if for every $\lambda \in \sigma(T)$ and for every $\epsilon > 0$, there exists a unit vector $x_{\lambda,\epsilon}$ in $\mathcal{H}$ such that $\limsup_{n\to\infty} \|(T-\lambda)^n x_{\lambda,\epsilon}\|^{\frac{1}{n}} < \epsilon$ (see [7] for more details).

Proposition 3.5. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. If R has the property P where P is (E), (PS), or (K), then T^n and $(T^n)^*$ have the property P for some $n \in \mathbb{N}$.*

Proof. (i) If R has the property (E), then there exist sequences $\{B_j\} \subset \mathcal{L}(\mathcal{H})$ and $\{K_j\} \subset \mathcal{K}(\mathcal{H})$ such that $\|B_j - R\| \to 0$, $K_jB_j = B_jK_j$ for each $j \in \mathbb{N}$, and $\{K_j\}$ is a nontrivial sequence of compact operators. Since $T \in PS_n(R)$, it follows that $\{X^{-1}B_j^nX\} \subset \mathcal{L}(\mathcal{H})$ and $\{X^{-1}K_j^nX\} \subset \mathcal{K}(\mathcal{H})$ such that $(X^{-1}K_j^nX)(X^{-1}B_j^nX) = (X^{-1}B_j^nX)(X^{-1}K_j^nX)$ for each $j \in \mathbb{N}$, and $\{X^{-1}K_j^nX\}$ is a nontrivial sequence of compact operators. Furthermore,

$$\|(X^{-1}B_jX)^n - T^n\| = \|X^{-1}B_j^nX - T^n\| \to 0 \tag{7}$$

as $j \to 0$. Hence T^n has the property (E).

(ii) If R has the property (PS), then there exist sequences $\{S_j\} \subset \{R\}'$ and $\{K_j\} \subset \mathcal{K}(\mathcal{H})$ such that $\|S_j - K_j\| \to 0$ for each $j \in \mathbb{N}$, and $\{K_j\}$ is a nontrivial sequence of compact operators. Since $T \in PS_n(R)$, it follows that $\{X^{-1}S_j^nX\} \subset \{T^n\}'$ and $\{X^{-1}K_j^nX\} \subset \mathcal{K}(\mathcal{H})$ such that

$$\|(X^{-1}S_jX)^n - (X^{-1}K_jX)^n\| = \|X^{-1}S_j^nX - X^{-1}K_j^nX\| \to 0 \tag{8}$$

as $j \to 0$ and $\{X^{-1}K_j^nX\}$ is a nontrivial sequence of compact operators. Then we get that T^n has the property (PS).

(iii) Suppose that R has the property (K). This means that for every $\lambda \in \sigma(R)$ and every $\epsilon > 0$, there exists a unit vector $x_{\lambda,\epsilon}$ in $\mathcal{H}$ such that

$$\limsup_{j\to\infty} \|(R-\lambda)^j x_{\lambda,\epsilon}\|^{\frac{1}{j}} < \epsilon. \tag{9}$$

For every $\mu \in \sigma(T^n) = \sigma(R^n) = \sigma(R)^n$ and every $\epsilon > 0$, there exists a $\lambda \in \sigma(R)$ such that $\mu = \lambda^n$. Since $T \in PS_n(R)$, it follows that

$$\begin{aligned} \|(T^n-\mu)^j y_{\mu,\epsilon}\| &= \|X^{-1}(R^n-\mu)^j X y_{\mu,\epsilon}\| \\ &\le \frac{\|X^{-1}\|}{\|X^{-1}x_{\lambda,\epsilon}\|}\|(R^n-\mu)^j x_{\lambda,\epsilon}\| \end{aligned} \tag{10}$$

where $y_{\mu,\epsilon} := \frac{X^{-1}x_{\lambda,\epsilon}}{\|X^{-1}x_{\lambda,\epsilon}\|}$ is a unit vector. Hence

$$\begin{aligned} \|(T^n-\mu)^j y_{\mu,\epsilon}\|^{\frac{1}{j}} &\le \Big(\frac{\|X^{-1}\|}{\|X^{-1}x_{\lambda,\epsilon}\|}\Big)^{\frac{1}{j}} \|(R^n-\lambda^n)^j x_{\lambda,\epsilon}\|^{\frac{1}{j}} \\ &\le \Big(\frac{\|X^{-1}\|}{\|X^{-1}x_{\lambda,\epsilon}\|}\Big)^{\frac{1}{j}} \|(\sum_{i=0}^{n-1}\lambda^{n-1-i}R^i)^j\|^{\frac{1}{j}} \|(R-\lambda)^j x_{\lambda,\epsilon}\|^{\frac{1}{j}}. \end{aligned} \tag{11}$$

Taking limsup in both sides of (11),

$$\limsup_{j\to\infty} \|(T^n-\mu)^j y_{\mu,\epsilon}\|^{\frac{1}{j}} \le \limsup_{j\to\infty} \|(R-\lambda)^j x_{\lambda,\epsilon}\|^{\frac{1}{j}} < \epsilon$$

from (9). Hence T^n has the property (K).

For the remaining parts, if R is complex symmetric and it has the property P, then it follows that R^* has the property P from [10, Proposition 3.12]. By similar arguments of (i)-(iii), we get that $(T^n)^*$ has the property P. □

Corollary 3.6. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. Then the following statements hold.*
(i) *If R has the property (E), then there exist sequences $\{C_j\} \subset \mathcal{L}(\mathcal{H})$ and $\{E_j\} \subset \mathcal{K}(\mathcal{H})$ such that $\|C_j - T\| \to 0$, $E_jC_j = C_jE_j$ for each $j \in \mathbb{N}$, and $\{E_j\}$ is a nontrivial sequence of compact operators where $\sigma(C_j^n)$ does not separate 0 from ∞ for sufficiently large j. In this case, T has the property (E).*
(ii) *If R has the property (PS), then there exist sequences $\{D_j\} \subset \mathcal{L}(\mathcal{H})$ and $\{E_j\} \subset \mathcal{K}(\mathcal{H})$ such that $\|D_j - E_j\| \to 0$ and $\{E_j\}$ is a nontrivial sequence of compact operators where $\sigma(D_j^n)$ does not separate 0 from ∞ for sufficiently large j.*

Proof. (i) From (7) in Proposition 3.5, set $C_j = X^{-1}B_jX$. Then we have

$$\limsup_{j\to\infty} \sigma(C_j^n) \subset \sigma(T^n).$$

For sufficiently large j, $\sigma(C_j^n) \subset (\sigma(T^n))_\epsilon$ where $(\sigma(T^n))_\epsilon$ is an open set containing $\sigma(T^n)$. Then $\sigma(C_j^n)$ and $\sigma(T^n)$ separate 0 from ∞ for sufficiently large j. So we can define $log(C_j^n)$ and $log(T^n)$ by using the Riesz-Dunford functional calculus and choosing an analytic branch of the function $logz$. Furthermore, for $t \geq 0$, the operator $T^t := e^{tlog(T)}$ is well-defined. From (7), we get that

$$n\|log(C_j) - log(T)\| = \|log(C_j)^n - log(T^n)\| \to 0$$

as $j \to 0$. Therefore $\|log(C_j) - log(T)\| \to 0$ as $j \to 0$. Since e^z is an entire function, it follows that $\|C_j - T\| = \|e^{log(C_j)} - e^{log(T)}\| \to 0$ as $j \to 0$. Set $E_j = X^{-1}K_jX$, it is clear that $E_jC_j = C_jE_j$ for each $j \in \mathbb{N}$, and $\{E_j\}$ is a nontrivial sequence of compact operators. Thus T has the property (E).

(ii) From (8), set $D_j = X^{-1}S_jX$ and $E_j = X^{-1}K_jX$, we have

$$\limsup_{j\to\infty} \sigma(D_j^n) \subset \sigma(E_j^n).$$

For sufficiently large j, $\sigma(D_j^n) \subset (\sigma(E_j^n))_\epsilon$ where $(\sigma(E_j^n))_\epsilon$ is an open set containing $\sigma(E_j^n)$. If $(\sigma(E_j^n))_\epsilon$ separates 0 from ∞, then $\sigma(D_j^n)$ and $\sigma(E_j^n)$ separate 0 from ∞ for sufficiently large j. From (8), we get that

$$n\|log(D_j) - log(E_j)\| = \|log(D_j)^n - log(E_j)^n\| \to 0$$

as $j \to 0$. Therefore $\|log(D_j) - log(E_j)\| \to 0$ as $j \to 0$. Since e^z is an entire function, it follows that $\|D_j - E_j\| = \|e^{log(D_j)} - e^{log(E_j)}\| \to 0$ as $j \to 0$ for each $j \in \mathbb{N}$ and $\{E_j\}$ is a nontrivial sequence of compact operators. □

Corollary 3.7. *Let $T \in PS_n(R)$ for some complex symmetric operator $R \in \mathcal{L}(\mathcal{H})$. Then the following statements hold:*
(i) *If $T \neq \lambda$ and R has the property (PS), then T^n and $(T^n)^*$ have nontrivial hyperinvariant subspace.*
(ii) *If R has either the property (PS) or (E), then T^n and $(T^n)^*$ have the property (A).*

Proof. (i) It follows from Proposition 3.5 and [7, Theorem 1.4].
(ii) It follows from Proposition 3.5 and [7, Proposition 2.1]. □

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