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Energy and Laplacian Energy of Unitary Addition Cayley Graphs

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Abstract. In this paper, we obtain the eigenvalues and Laplacian eigenvalues of the unitary addition Cayley graph G_n and its complement. Moreover, we compute the bounds for energy and Laplacian energy for G_n and its complement. In addition, we prove that G_n is hyperenergetic if and only if n is odd other than the prime number and power of 3 or n is even and has at least three distinct prime factors. It is also shown that the complement of G_n is hyperenergetic if and only if n has at least two distinct prime factors and $n \neq 2p$.

1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph with $|V(G)| = n$ and $|E(G)| = m$, where $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G)$ are the vertex set and edge set of G. The complement of a graph G, denoted by G^c, is the graph with the same vertex set as G such that two vertices of G^c are adjacent if and only if they are not adjacent in G. We use the following definitions from [[1], [2], [3], [5], [7], [10], [12]].

The adjacency matrix of G is the $n \times n$ symmetric matrix $A(G) = (a_{ij})$ such that $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise.

The eigenvalues of a graph G are defined to be the eigenvalues of its adjacency matrix $A(G)$. Collection of the eigenvalues of G is called the spectrum of G.

The energy of a graph G, $\mathbb{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of $A(G)$, $\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of G. Energy of a complete graph K_n of order n is $2(n-1)$.

A graph G is said to be hyperenergetic if $\mathbb{E}(G) > \mathbb{E}(K_n) = 2(n-1)$ and non-hyperenergetic if $\mathbb{E}(G) \leq 2(n-1)$.

The Laplacian matrix of G is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the degree matrix of G. The Laplacian energy, $LE(G)$, is defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, where $\mu_1, \mu_2, \cdots, \mu_n$ are the Laplacian eigenvalues of G.

Let Γ be a multiplicative group with identity 1. For $S \subseteq \Gamma, 1 \notin S, S^{-1} = \{s^{-1} \mid s \in S\} = S$ the *Cayley graph* $X = Cay(\Gamma, S)$ is the undirected graph having vertex set $V(X) = \Gamma$ and edge set $E(X) = \{(a, b) \mid ab^{-1} \in S\}$. The Cayley graph X is a regular graph of degree $|S|$.

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For a positive integer $n > 1$ the *unitary Cayley graph* $X_n = Cay(Z_n, U_n)$ is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a-b \in U_n$. The graph X_n is regular of degree $|U_n| = \phi(n)$, where $\phi(n)$ denotes the Euler pi-function.

For a positive integer $n > 1$, the *unitary addition Cayley graph* G_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes the set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a + b \in U_n$. The unitary addition Cayley graph G_n is also defined as, $G_n = Cay^+(Z_n, U_n)$.The unitary addition Cayley graph G_n is *regular* if n is even and *semi regular* if n is odd.

Walter Klotz and Torsten Sander[10], proved that the eigenvalues of unitary Cayley graph X_n are $\lambda_r = \sum_{1\leq j<n, gcd(j,n)=1} \omega^{rj} = c(r, n), 0 \leq r \leq n-1$, where ω is a complex primitive n-th root of unity. The arithmetic function $c(r, n)$ is a Ramanujan sum[11] and is defined by $c(r, n) = \mu(t_r)\frac{\phi(n)}{\phi(t_r)}$, $t_r = \frac{n}{gcd(r,n)}$, where μ denotes the Mobius function.
The right circulant matrix $C_R(\bar{c})$ associated to the vector $\bar{c} = (c_0, c_1, \cdots, c_{n-1}) \in R^n$ is

$$C_R(\bar{c}) = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}.$$

The left circulant matrix $C_L(\bar{c})$ associated to the vector $\bar{c} = (c_0, c_1, \cdots, c_{n-1}) \in R^n$ is

$$C_L(\bar{c}) = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_0 & \cdots & c_{n-2} \end{bmatrix}.$$

In [9], Herbert Karner, et al. have shown that $C_L(\bar{c}) = \Pi C_R(\bar{c})$, where Π is the orthogonal cyclic shift matrix given by

$$\Pi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{bmatrix}.$$

In the same paper, they also proved that the eigenvalues of left circulant matrix $C_L(\bar{c})$ are $\lambda_0, \pm|\lambda_1|, \cdots, \pm|\lambda_{(n-1)/2}|$ if n is odd and $\lambda_0, \lambda_{n/2}, \pm|\lambda_1|, \cdots, \pm|\lambda_{(n-2)/2}|$ if n is even, where $\lambda_k' s$ are the eigenvalues of right circulant matrix $C_R(\bar{c})$.

Throughout this paper, we use p for a prime number. Also an integer n greater than 1 can be written in the form $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \cdots, p_r$ are distinct prime numbers and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers.

There is a vast literature devoted to regular Cayley graphs but only a few papers addressed to the irregular ones to the best of our knowledge. In this paper we attempt to compute the spectrum and energy of irregular addition Cayley graphs namely unitary addition Cayley graphs. Unitary addition Cayley graphs is an addition Cayley graphs on Z_n together with a generating set U_n.

The concept of graph energy arose in theoretical chemistry and was first defined by Gutman in 1978. In theoretical chemistry, the π-electron energy of a conjugated carbon molecule, computed using Hückel theory, coincides with the energy of its "molecular"graph. In 2006, Gutman and Zhou[7] was defined the Laplacian energy of a graph.

In [6], Gutman stated a conjecture "If G is an n-vertex graph, $G \ncong K_n$,then $\mathbb{E}(G) < 2n-2$". But later D. Cvetković and Gutman was shown that this conjecture was false with counterexamples. All most all

graphs are non-hyperenergetic.
Compairing the degree of hyperenergeticity of G_n and X_n, G_n is more hyperenergetic than X_n.

This paper is organized as follows. In the second section, we consider a few preliminary results which are useful to prove the results obtained in this paper. The third section contains the computation of eigenvalues of G_n and energy of G_n. Moreover we prove that unitary addition Cayley graph $G_n, n > 3$, is hyperenergetic if and only if n is odd other than the prime number and power of 3 or n is even and has at least three distinct prime factors. Fourth section deals with the bounds on the spectrum and energy of the complement of unitary addition Cayley graph G_n and also we show that complement of unitary addition Cayley graph G_n is hyperenergetic if and only if n has at least two distinct prime factors and $n \neq 2p$. In the fifth section we present the Laplacian energy of unitary addition Cayley graph G_n for all n. The Laplacian energy of the complement of G_n is determined in the final section.

2. Preliminaries

Theorem 2.1. [12] *The unitary addition Cayley graph G_n is isomorphic to the unitary Cayley graph X_n if and only if n is even.*

The following remark is obtained by using Theorem 2.1 and the result of eigenvalues of X_n [10].

Remark 2.2. *Let n be even. Then eigenvalues of the unitary addition Cayley graph G_n are $\lambda_r = \mu(t_r)\frac{\phi(n)}{\phi(t_r)}, 0 \leq r \leq n-1$.*

Theorem 2.3. [8] *The energy of unitary Cayley graph X_n equals $2^r\phi(n)$, where r is the number of distinct prime factors dividing n.*

Theorem 2.4. [8] *The unitary Cayley graph X_n is hyperenergetic if and only if $r > 2$ or $r = 2$ and $p_1 > 2$.*

Theorem 2.5. [8] *The eigenvalues of the complement of unitary Cayley graph X_n are $\lambda_0 = n - 1 - \phi(n)$ and $\lambda_r = -1 - \mu(t_r)\frac{\phi(n)}{\phi(t_r)}$, where $t_r = \frac{n}{gcd(r,n)}$.*

Theorem 2.6. [8] *Let $s = p_1p_2\cdots p_r$ be the largest square-free number that divides n. The energy of the complement of unitary Cayley graph X_n equals*

$$\mathbb{E}(X_n^c) = 2n - 2 + (2^r - 2)\phi(n) - s + \prod_{i=1}^{r}(2 - p_i).$$

Theorem 2.7. [8] *The complement of unitary Cayley graph X_n is hyperenergetic if and only if n has at least two distinct prime factors and $n \neq 2p$.*

Theorem 2.8. [4] *Let A, A_1, A_2 be three $n \times n$ real symmetric matrices such that $A = A_1 + A_2$. The eigenvalues of these matrices satisfy the following inequalities:*
for $1 \leq i \leq n$ and $0 \leq j \leq min\{i-1, n-i\}$, $\lambda_{i-j}(A_1) + \lambda_{1+j}(A_2) \geq \lambda_i(A) \geq \lambda_{i+j}(A_1) + \lambda_{n-j}(A_2)$.

Theorem 2.9. [13] *Let G be a connected graph. Then*

$$LE(G) \leq \mathbb{E}(G) + \sum_{i=1}^{n}\left|d_i - \frac{2m}{n}\right|.$$

In this paper, we attempt to find the energy and Laplacian energy of all unitary addition Cayley graph G_n and its complements. From Theorems 2.1 and 2.3 we obtain energy of unitary addition Cayley graph G_n is $2^r\phi(n)$. In the following section, we show that how to compute the energy of G_n for odd n.

3. Spectrum of unitary addition Cayley graphs

In this section, we compute the spectrum and energy of G_n when n is a power of a prime, $n = p^m, m \geq 1$. In addition, we calculate the bounds of the spectrum and energy of the unitary addition Cayley graph G_n for all odd n.

Theorem 3.1. *Let n be a power of a prime number, $n = p^m, m \geq 1$. Then spectrum of the unitary addition Cayley graph G_n is*

$$\begin{pmatrix} -1-p^{m-1} & \frac{x-y}{2} & -1 & 0 & p^{m-1}-1 & \frac{x+y}{2} \\ \frac{p-3}{2} & 1 & (p-1)(p^{m-1}-1) & p^{m-1}-1 & \frac{p-1}{2} & 1 \end{pmatrix},$$

where $x = p^m - 2p^{m-1} - 1$ and $y = \sqrt{(p^m-1)^2 + 4p^{m-1}}$.

Proof. Let $A = \begin{bmatrix} B & C & \cdots & C \\ C & B & \cdots & C \\ \vdots & \vdots & \ddots & \vdots \\ C & C & \cdots & B \end{bmatrix}$ be the adjacency matrix of G_n of order $k = p^{m-1}$,

where $B = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 0 & \cdots & 1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{p\times p}$ and $C = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & 1 & 1 \end{bmatrix}_{p\times p}$.

The matrix A is permutationally similar to $\hat{A} = \begin{bmatrix} O & J & J & \cdots & J & J & \cdots & J & J \\ J & J-I & J & \cdots & J & J & \cdots & J & O \\ J & J & J-I & \cdots & J & J & \cdots & O & J \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J & J & J & \cdots & J-I & O & \cdots & J & J \\ J & J & J & \cdots & O & J-I & \cdots & J & J \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J & J & O & \cdots & J & J & \cdots & J-I & J \\ J & O & J & \cdots & J & J & \cdots & J & J-I \end{bmatrix}_{p\times p}$,

where J is a matrix of order k with all entries are 1 and O is a null matrix of order k.
Then $A = P\hat{A}P^{-1}$, $P = \begin{bmatrix} A_{11}^T & A_{12}^T & \cdots & A_{1p^{m-2}}^T & A_{21}^T & A_{22}^T & \cdots & A_{2p^{m-2}}^T & \cdots & A_{p1}^T & A_{p2}^T & \cdots & A_{pp^{m-2}}^T \end{bmatrix}^T$
is a permutation matrix of order p^m, $m \geq 2$, $A_{ij} = (a_{\alpha\beta})$, $a_{\alpha\beta} = \begin{cases} 1 & \text{if } (\alpha,\beta) \in H_{ij}, \\ 0 & \text{otherwise.} \end{cases}$ where $\alpha = 1, 2, \ldots, p$,
$\beta = 1, 2, \ldots, p^m$ and $H_{ij} = \{(1, i+(j-1)p), (2, i+(j-1)p+p^{m-1}), \ldots, (p, i+(j-1)p+(p-1)p^{m-1})\}$, $i = 1, 2, \ldots, p$,

$j = 1, 2, \ldots, p^{m-2}$. If $m = 1$, then $P = I$.

$$det(\hat{A}-\lambda I) = \begin{vmatrix} -\lambda I & J & J & \cdots & J & J & \cdots & J & J \\ J & J-I-\lambda I & J & \cdots & J & J & \cdots & J & O \\ J & J & J-I-\lambda I & \cdots & J & J & \cdots & O & J \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J & J & J & \cdots & J-I-\lambda I & O & \cdots & J & J \\ J & J & J & \cdots & O & J-I-\lambda I & \cdots & J & J \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J & J & O & \cdots & J & J & \cdots & J-I-\lambda I & J \\ J & O & J & \cdots & J & J & \cdots & J & J-I-\lambda I \end{vmatrix}$$

$$= det(J-I-\lambda I)^{\frac{p-1}{2}} \begin{vmatrix} -\lambda I & J+\lambda I & J+\lambda I & \cdots & J+\lambda I \\ 2J & -J-I-\lambda I & O & \cdots & O \\ 2J & O & -J-I-\lambda I & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2J & O & O & \cdots & -J-I-\lambda I \end{vmatrix}$$

$$= det(J-I-\lambda I)^{\frac{p-1}{2}} \begin{vmatrix} -\lambda I & \left(\frac{p-1}{2}\right)(J+\lambda I) & J+\lambda I & \cdots & J+\lambda I \\ 2J & -J-I-\lambda I & O & \cdots & O \\ O & O & -J-I-\lambda I & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & -J-I-\lambda I \end{vmatrix}$$

$$= det(J-I-\lambda I)^{\frac{p-1}{2}} det(-J-I-\lambda I)^{\frac{p-3}{2}} \begin{vmatrix} -\lambda I & \left(\frac{p-1}{2}\right)(J+\lambda I) \\ 2J & -J-I-\lambda I \end{vmatrix}$$

$$= (\lambda)^{k-1}(-1-\lambda)^{(p-2)(k-1)}(1+\lambda)^{k-1}(k-1-\lambda)^{\frac{p-1}{2}}(k+1+\lambda)^{\frac{p-3}{2}}$$
$$\times [\lambda^2 + \lambda(1+2k-kp) + k^2(1-p)].$$

Thus, the spectrum of G_n is

$$\begin{pmatrix} -1-p^{m-1} & \frac{x-y}{2} & -1 & 0 & p^{m-1}-1 & \frac{x+y}{2} \\ \frac{p-3}{2} & 1 & (p-1)(p^{m-1}-1) & p^{m-1}-1 & \frac{p-1}{2} & 1 \end{pmatrix},$$

where $x = p^m - 2p^{m-1} - 1$ and $y = \sqrt{(p^m-1)^2 + 4p^{m-1}}$. $\square$

The following tables gives some information about some eigenvalues of G_n for $n = p^m$.

	1^{st} largest eigenvalue	2^{nd} largest eigenvalue	3^{rd} largest eigenvalue
G_p	$\frac{p-3+\sqrt{(p-1)^2+4}}{2}$	0	0
$G_{p^m}, m>1, p=3$	$\frac{p^m-2p^{m-1}-1+\sqrt{(p^m-1)^2+4p^{m-1}}}{2}$	$p^{m-1}-1$	0
$G_{p^m}, m>1, p\neq 3$	$\frac{p^m-2p^{m-1}-1+\sqrt{(p^m-1)^2+4p^{m-1}}}{2}$	$p^{m-1}-1$	$p^{m-1}-1$

Table 1: First three largest eigenvalues of $G_n, n = p^m$

	1^{st} least eigenvalue	2^{nd} least eigenvalue	3^{rd} least eigenvalue
$G_p, p = 3$	$-\sqrt{2}$	0	$\sqrt{2}$
$G_p, p = 5$	-2	$1 - \sqrt{5}$	0
$G_p, p = 7$	-2	-2	$2 - \sqrt{10}$
$G_p, p > 7$	-2	-2	-2
$G_{p^m}, m > 1, p = 3$	$\frac{p^m-2p^{m-1}-1-\sqrt{(p^m-1)^2+4p^{m-1}}}{2}$	-1	-1
$G_{p^m}, m > 1, p = 5$	$-1 - p^{m-1}$	$\frac{p^m-2p^{m-1}-1-\sqrt{(p^m-1)^2+4p^{m-1}}}{2}$	-1
$G_{p^m}, m > 1, p = 7$	$-1 - p^{m-1}$	$-1 - p^{m-1}$	$\frac{p^m-2p^{m-1}-1-\sqrt{(p^m-1)^2+4p^{m-1}}}{2}$
$G_{p^m}, m > 1, p > 7$	$-1 - p^{m-1}$	$-1 - p^{m-1}$	$-1 - p^{m-1}$

Table 2: First three smallest eigenvalues of $G_n, n = p^m$

Corollary 3.2. *Let n be a power of a prime number, $n = p^m, m \geq 1$. Then energy of the unitary addition Cayley graph G_n is $2p^m - 3p^{m-1} - p + \sqrt{(p^m-1)^2 + 4p^{m-1}}$.*

Corollary 3.3. *Let n be a power of a prime number, $n = p^m, p > 3$ and $m \geq 2$. Then unitary addition Cayley graph G_n is hyperenergetic.*

Compute the exact eigenvalues of unitary addition Cayley graph G_n is very tedious when n is odd. However, in this section we obtain some bounds on the eigenvalues of the unitary addition Cayley graphs.
A lower bound for E(Gn) as given below, can be obtained from Theorems 5.7 and 2.9.

Theorem 3.4. *Let n be odd. Then $\mathbb{E}(G_n) \geq \phi(n)\left[\frac{n(2^r-1)-s-1+2\phi(n)}{n}\right]$.*

Corollary 3.5. *Let n be odd and has at least two distinct prime factors. Then $\phi(n)\left[\frac{n(2^r-1)-s-1+2\phi(n)}{n}\right] > 2n-2$ except $n = 15, 21, 33$.*

Observation 3.6. *The energy of G_n for $n = 15, 21$ and 33 are 30.4446, 46.5331 and 78.9425 respectively. But the corresponding values for $2n-2$ in G_n are 28, 40 and 64 respectively. So G_n is hyperenergetic for these values.*

The following theorem gives more information about the bounds of the eigenvalues of the unitary addition Cayley graph G_n if n is odd.

Theorem 3.7. *Let n be odd. Then eigenvalues of the unitary addition Cayley graph G_n satisfy the following inequalities:*
$\mu(t_k)\frac{\phi(n)}{\phi(t_k)} - 1 \leq \lambda_k \leq \mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ *for* $0 \leq k \leq (n-1)/2$ *and*
$-\mu(t_k)\frac{\phi(n)}{\phi(t_k)} - 1 \leq \lambda_k \leq -\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ *for* $(n+1)/2 \leq k \leq n-1$.

Proof. Let $A = (a_{ij}), 0 \leq i, j \leq n-1$, be the adjacency matrix of G_n, where

$$a_{ij} = \begin{cases} 1 & \text{if gcd(i+j,n)=1 and i} \neq \text{j,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $A = B + C$, where
$B = (b_{ij}), 0 \leq i, j \leq n-1, b_{ij} = \begin{cases} 1 & \text{if gcd(i+j,n)=1,} \\ 0 & \text{otherwise,} \end{cases}$
and
$C = (c_{ij}), 0 \leq i, j \leq n-1, c_{ij} = \begin{cases} -1 & \text{if gcd(i+j,n)=1 and i=j,} \\ 0 & \text{otherwise.} \end{cases}$

From the definition of B, it is a left circulant matrix with first row $(c_0, c_1, \cdots, c_{n-1})$ where $c_j = \begin{cases} 1 & \text{if gcd(j,n)=1,} \\ 0 & \text{otherwise.} \end{cases}$

So eigenvalues of B are $\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $0 \leq k \leq (n-1)/2$ and $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $(n+1)/2 \leq k \leq n-1$. Eigenvalues of C are $0^{n-\phi(n)}, -1^{\phi(n)}$, since $x \in U_n$ implies $2x \in U_n$ and $y \in V(G_n) - U_n$ implies $2y \in V(G_n) - U_n$.
Thus, the result follows from the eigenvalues of B, C and Theorem 2.8. □

Corollary 3.8. *Let n be odd. Then $2^r\phi(n) - \frac{s+1}{2} \le \mathbb{E}(G_n) \le 2^r\phi(n) + \frac{2n-s-1}{2}$, where $s = p_1p_2\cdots p_r$ is the maximal square-free divisor of n.*

Remark 3.9. *Let n be odd and square free number. Then $\phi(n)\left[\frac{n(2^r-1)-s-1+2\phi(n)}{n}\right] > 2^r\phi(n) - \frac{s+1}{2}$.*

Obviously, the lower bound for $\mathbb{E}(G_n)$ in Theorem 3.4 and Corollary 3.8 are comparable. From this a better lower bound for $\mathbb{E}(G_n)$ is obtained from the following theorem.

Theorem 3.10. *Let n be odd and square free number. Then $\phi(n)\left[\frac{n(2^r-1)-s-1+2\phi(n)}{n}\right] \le \mathbb{E}(G_n) \le 2^r\phi(n) + \frac{2n-s-1}{2}$.*

By making use of the Theorem 2.4, Corollaries 3.3 and 3.5 and Observation 3.6, we now characterise the hyperenergeticity of unitary addition Cayley graphs.

Theorem 3.11. *The unitary addition Cayley graph G_n is hyperenergetic if and only if n is odd other than the prime number and power of 3 or n is even and has at least three distinct prime factors.*

4. Spectrum of Complement of unitary addition Cayley graphs

This section deals with spectrum and energy of the complement of unitary addition Cayley graph G_n when n is a power of a prime, $n = p^m, m \ge 1$.

Theorem 4.1. *Let $n = p^m, m \ge 1$. Then spectrum of the complement of unitary addition Cayley graph G_n is*

$$\begin{pmatrix} -p^{m-1} & -1 & 0 & p^{m-1}-1 & p^{m-1} \\ \frac{p-1}{2} & p^{m-1}-1 & (p-1)(p^{m-1}-1) & 1 & \frac{p-1}{2} \end{pmatrix}.$$

Proof. Let $A^c = \begin{bmatrix} B & C & \cdots & C \\ C & B & \cdots & C \\ \vdots & \vdots & \ddots & \vdots \\ C & C & \cdots & B \end{bmatrix}$ be the adjacency matrix of G_n^c of order $k = p^{m-1}$, where $B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}_{p\times p}$,

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}_{p\times p}.$$

The matrix A^c is permutationally similar to $\hat{A}^c = \begin{bmatrix} J-I & O & \cdots & O & O \\ O & O & \cdots & O & J \\ O & O & \cdots & J & O \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ O & J & \cdots & O & O \end{bmatrix}_{p\times p}$.

where J is a matrix of order k with all entries are 1 and O is a null matrix of order k.
Then $A^c = P\hat{A}^cP^{-1}$, where P is a matrix given in the proof of Theorem 3.1.
Let $\bar{\lambda}$ be the eigenvalue of the complement of unitary addition Cayley graph G_n.

$$\begin{aligned} det(\hat{A}^c - \bar{\lambda}I) &= det(J - I - \bar{\lambda})det(J-\bar{\lambda})^{\frac{p-1}{2}}det(-J-\bar{\lambda})^{\frac{p-1}{2}} \\ &= (-1-\bar{\lambda})^{k-1}(k-1-\bar{\lambda})(k-\bar{\lambda})^{\frac{p-1}{2}}(\bar{\lambda})^{\frac{(p-1)(k-1)}{2}}(-k-\bar{\lambda})^{\frac{p-1}{2}}(-\bar{\lambda})^{\frac{(p-1)(k-1)}{2}} \\ &= (-k-\bar{\lambda})^{\frac{p-1}{2}}(-1-\bar{\lambda})^{k-1}(-\bar{\lambda})^{(p-1)(k-1)}(k-1-\bar{\lambda})(k-\bar{\lambda})^{\frac{p-1}{2}}. \end{aligned}$$

Thus, the spectrum of G_n^c is

$$\begin{pmatrix} -p^{m-1} & -1 & 0 & p^{m-1}-1 & p^{m-1} \\ \frac{p-1}{2} & p^{m-1}-1 & (p-1)(p^{m-1}-1) & 1 & \frac{p-1}{2} \end{pmatrix}.$$

□

Corollary 4.2. *Let $n = p^m, m \geq 1$. Then energy of the complement of unitary addition Cayley graph G_n is $p^m + p^{m-1} - 2$.*

Corollary 4.3. *The complement of unitary addition Cayley graph G_n is non-hyperenergetic if $n = p^m, m \geq 1$.*

Next we have to find the bounds of the eigenvalues of the complement of unitary addition Cayley graph G_n when n is odd.
The following facts are needed for the computation of the eigenvalues of G_n.
The adjacency matrix $A(X_n^c)$ of the complement of unitary Cayley graph X_n is a right circulant matrix with first row $(c_0, c_1, \cdots, c_{n-1})$, where $c_0 = 0$ and

$$c_j = \begin{cases} 1 & \text{if gcd(j, n)} \neq 1,\ 1 \leq \text{j} \leq \text{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvalues of $A(X_n^c)$ are $\lambda_0 = n - 1 - \phi(n)$ and $\lambda_r = -1 - \mu(t_r)\frac{\phi(n)}{\phi(t_r)}$.
The eigenvalues of $A(X_n^c) + I$ are $\lambda_0 = n - \phi(n)$ and $\lambda_r = -\mu(t_r)\frac{\phi(n)}{\phi(t_r)}$, $1 \leq k \leq n-1$.
Let $E = (e_{ij})$, $0 \leq i, j \leq n-1$, where

$$e_{ij} = \begin{cases} 1 & \text{if gcd(i+j, n)} \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is a left circulant matrix with first row $(c_0, c_1, \cdots, c_{n-1})$, where

$$c_j = \begin{cases} 1 & \text{if gcd(j, n)} \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd, then eigenvalues of E are $n - \phi(n)$, $\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $1 \leq k \leq (n-1)/2$ and $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $(n+1)/2 \leq k \leq n-1$.
By Theorems 6.6 and 2.9, a lower bound for $\mathbb{E}(G_n^c)$ is given in the next theorem.

Theorem 4.4. *Let n be odd. Then lower bound for energy of the complement of unitary addition Cayley graph G_n is* $(n-s)\left[1 - \frac{\phi(n)}{n}\right] + (2^r - 3)\phi(n) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n) + 2\frac{\phi(n)^2}{n}$.

Corollary 4.5. *Let n be odd and has at least two distinct prime factors. Then* $(n-s)\left[1 - \frac{\phi(n)}{n}\right] + (2^r - 3)\phi(n) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n) + 2\frac{\phi(n)^2}{n} > 2n - 2$ *except $n = 3p$ and $n = 35, 45, 63$.*

Observation 4.6. *Let $n = 3p$ and $p > 3$. Then eigenvalues of the complement of unitary addition Cayley graph G_n are $-2.4142, -1.7321, -1, 0.4142, 1, 1.7321$ with multiplicity $\frac{p-1}{2}, \frac{p-3}{2}, \frac{p-1}{2}, \frac{p-1}{2}, \frac{p-3}{2}, \frac{p-3}{2}$ and simple eigenvalues lies in $(-p, 1-p), (-2, -1), (0, 1), (1, 2), (p-2, p-1), (p+1, p+2)$ respectively. So lower bound for energy of the complement of unitary addition Cayley graph G_n is $7.1463q - 8.6105$ and this value is greater than $6p - 2$ if $p \geq 7$.*

If $n = 15, 35, 45$ and 63, then energy of the complement of unitary addition Cayley graph G_n are $29.4788, 81.0735, 105.1123$ and 151.1216 respectively. But the corresponding values of $2n - 2$ in G_n^c are $28, 68, 88$ and 124 respectively. So G_n^c is hyperenergetic for these values.
The following theorem gives more information about the bounds of the eigenvalues of the complement of unitary addition Cayley graph G_n when n is odd.
Let $\bar{\lambda}_k$, $0 \leq k \leq n-1$, denotes the eigenvalues of the complement of unitary addition Cayley graph G_n.

Theorem 4.7. *Let n be odd. Then eigenvalues of the complement of unitary addition Cayley graph G_n, satisfy the following inequalities:*
$\mu(t_k)\frac{\phi(n)}{\phi(t_k)} - 1 \leq \bar{\lambda}_k \leq \mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ *for* $1 \leq k \leq (n-1)/2$ *and*
$-\mu(t_k)\frac{\phi(n)}{\phi(t_k)} - 1 \leq \bar{\lambda}_k \leq -\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ *for* $(n+1)/2 \leq k \leq n-1$ *and*
$n - \phi(n) - 1 \leq \bar{\lambda}_0 \leq n - \phi(n)$.

Proof. Let $A^c = (d_{ij})$, $0 \leq i, j \leq n-1$, be the adjacency matrix of the complement of unitary addition Cayley graph G_n, where

$$d_{ij} = \begin{cases} 1 & \text{if gcd(i+j,n)}\neq 1 \text{ and i}\neq \text{j}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A^c = E + F$, where
$E = (e_{ij})$, $0 \leq i, j \leq n-1$, $e_{ij} = \begin{cases} 1 & \text{if gcd(i+j,n)}\neq 1, \\ 0 & \text{otherwise,} \end{cases}$
and
$F = (f_{ij})$, $0 \leq i, j \leq n-1$, $f_{ij} = \begin{cases} -1 & \text{if gcd(i+j,n)}\neq 1 \text{ and i=j}, \\ 0 & \text{otherwise.} \end{cases}$
E is a left circulant matrix, so eigenvalues of E are $n - \phi(n)$, $\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $1 \leq k \leq (n-1)/2$ and $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $(n+1)/2 \leq k \leq n-1$. Eigenvalues of F are $0^{\phi(n)}, -1^{n-\phi(n)}$, since $x \in U_n$ implies $2x \in U_n$ and $y \in V(G_n) - U_n$ implies $2y \in V(G_n) - U_n$.
Thus, the result is obtained from the eigenvalues of E, F and Theorem 2.8. □

Corollary 4.8. *Let n be odd. Then* $(2^r - 2)\phi(n) + \frac{2n-s-1}{2} \leq \mathbb{E}(G_n^c) \leq (2^r-2)\phi(n) + \frac{4n-s-1}{2}$, *where* $s = p_1 p_2 \cdots p_r$ *is the maximal square-free divisor of n.*

Remark 4.9. *Let n be odd and square free number. Then*
$(n-s)\left[1 - \frac{\phi(n)}{n}\right] + (2^r - 3)\phi(n) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n) + 2\frac{\phi(n)^2}{n} > (2^r-2)\phi(n) + \frac{2n-s-1}{2}$.

For n is odd and square free number, a better lower bound of $\mathbb{E}(G_n^c)$ is obtained from Theorem 4.4 and Remark 4.9. Now we can rewrite Corollary 4.8 as follows.

Theorem 4.10. *Let n be odd and square free number. Then*
$(n-s)\left[1 - \frac{\phi(n)}{n}\right] + (2^r - 3)\phi(n) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n) + 2\frac{\phi(n)^2}{n} \leq \mathbb{E}(G_n^c) \leq (2^r-2)\phi(n) + \frac{4n-s-1}{2}$.

Combining Theorem 2.7, Corollaries 4.3 and 4.5, and Observation 4.6, we can prove the following theorem.

Theorem 4.11. *The complement of unitary addition Cayley graph G_n is hyperenergetic if and only if n has at least two distinct prime factors and $n \neq 2p$.*

5. Laplacian energy of unitary addition Cayley graphs

In this section, we discuss the Laplacian energy of the unitary addition Cayley graph G_n.

Theorem 5.1. *Let n be even. Then Laplacian eigenvalues of the unitary addition Cayley graph G_n are* $\mu_k = \phi(n) - \mu(t_k)\frac{\phi(n)}{\phi(t_k)}, 0 \leq k \leq n-1$.

Proof. Let $L(G_n) = (l_{ij})$, $0 \leq i, j \leq n-1$, be the Laplacian matrix of G_n, where

$$l_{ij} = \begin{cases} -1 & \text{if gcd(i+j,n)=1 and i}\neq \text{j}, \\ \phi(n) & \text{if i= j}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(G_n) = \phi(n)I - A(X_n)$.

Therefore Laplacian eigenvalues of the unitary addition Cayley graph G_n are
$\mu_k = \phi(n) - \mu(t_k)\frac{\phi(n)}{\phi(t_k)}, 0 \le k \le n-1$. □

Corollary 5.2. *Let n be even. Then Laplacian energy of the unitary addition Cayley graph G_n is $2^r\phi(n)$, where r is the number of distinct prime divisors of n.*

Theorem 5.3. *Let n be odd. Then Laplacian eigenvalues of the unitary addition Cayley graph G_n are $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}+\phi(n)$ for $0 \le k \le \frac{n-1}{2}$ and $\mu(t_k)\frac{\phi(n)}{\phi(t_k)} + \phi(n)$ for $\frac{n+1}{2} \le k \le n-1$.*

Proof. Let $L(G_n) = (l_{ij}), 0 \le i, j \le n-1$, be the Laplacian matrix of G_n, where

$$l_{ij} = \begin{cases} -1 & \text{if gcd(i+j,n)=1 and i} \neq \text{j}, \\ \phi(n) & \text{if gcd(i+j,n)} \neq \text{1 and i= j}, \\ \phi(n)-1 & \text{if gcd(i+j,n)=1 and i= j}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(G_n) = M + \phi(n)I$, where
$M = (m_{ij}), 0 \le i, j \le n-1, m_{ij} = \begin{cases} -1 & \text{if gcd(i+j,n)=1}, \\ 0 & \text{otherwise.} \end{cases}$
It is a left circulant matrix with first row $(c_0, c_1, \cdots, c_{n-1})$, where

$$c_j = \begin{cases} -1 & \text{if gcd(j, n)=1}, \\ 0 & \text{otherwise.} \end{cases}$$

So eigenvalues of M are $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $0 \le k \le (n-1)/2$ and $\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $(n+1)/2 \le k \le n-1$. Thus Laplacian eigenvalues of the unitary addition Cayley graph G_n are $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}+\phi(n)$ for $0 \le k \le \frac{n-1}{2}$ and $\mu(t_k)\frac{\phi(n)}{\phi(t_k)} + \phi(n)$ for $\frac{n+1}{2} \le k \le n-1$. □

Corollary 5.4. *Let n be odd. Then Laplacian spectral radius of unitary addition Cayley graph G_n is $\phi(n) + \frac{\phi(n)}{\phi(p_1)}$.*

Corollary 5.5. *Let n be odd. Then algebraic connectivity of unitary addition Cayley graph G_n is $\phi(n) - \frac{\phi(n)}{\phi(p_1)}$.*

Corollary 5.6. *Let n be odd. Then $\phi(n) - \frac{\phi(n)}{\phi(p_1)} \le \kappa(G_n)$, where $\kappa(G_n)$ is vertex connectivity of G_n.*

Now we can set our main result.

Theorem 5.7. *Let n be odd. Then Laplacian energy of the unitary addition Cayley graph G_n is*

$$LE(G_n) = \phi(n)\left[\frac{n(1+2^r)-s-1}{n}\right],$$

where $s = p_1p_2\cdots p_r$ is the maximal square-free divisor of n.

Proof. Laplacian energy of G_n is,

$$\begin{aligned} LE(G_n) &= \sum_{i=0}^{n-1}\left|\mu_i - \frac{2m}{n}\right| \\ &= \sum_{0\le i\le \frac{n-1}{2}}\left|\frac{-\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| + \sum_{\frac{n+1}{2}\le i\le n-1}\left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right|. \end{aligned} \tag{1}$$

Divide the sum in equation (1) into two parts,

1. $\frac{n}{gcd(i,n)}$ is a non square-free number
2. $\frac{n}{gcd(i,n)}$ is a square-free number(SF)

Then $LE(G_n) = S_1 + S_2 + S_3$, where

$$S_1 = \sum_{0\leq i\leq n-1,\, \frac{n}{gcd(i,n)}\notin SF} \left|\phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right|,$$

$$S_2 = \sum_{0\leq i\leq \frac{n-1}{2},\, \frac{n}{gcd(i,n)}\in SF} \left|\frac{-\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| \text{and}$$

$$S_3 = \sum_{\frac{n+1}{2}\leq i\leq n-1,\, \frac{n}{gcd(i,n)}\in SF} \left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right|. \tag{2}$$

Part I: Suppose $\frac{n}{gcd(i,n)}$ is a non square-free number.
We know that number of solutions of the equation $\mu\left(\frac{n}{gcd(i,n)}\right) = 0$ is $n - s$.
Therefore

$$S_1 = (n-s)\left[\phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right]. \tag{3}$$

Part II: Suppose $\frac{n}{gcd(i,n)}$ is square-free number.
In S_2 and S_3 two possibility arises one is $\frac{n}{gcd(i,n)}$ has an even number of distinct prime divisors and the other one is $\frac{n}{gcd(i,n)}$ has an odd number of distinct prime divisors. We can denote the corresponding sum in S_2 by S_4 and S_5 and S_3 by S_6 and S_7. Now we have two sub cases.
Sub case I:

$$\begin{aligned}
\text{Assume } S_2 &= \sum_{S_4\cup S_5} \left|\frac{-\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| \\
&= \sum_{S_4} \left|\frac{-\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| + \sum_{S_5} \left|\frac{-\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| \\
&= \sum_{S_4} \left|\frac{-\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| + \sum_{S_5} \left|\frac{\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right| \\
&= \phi(n)\sum_{S_4} \left|\frac{-1}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \left(\frac{n-1}{n}\right)\right| + \phi(n)\sum_{S_5} \left|\frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \left(\frac{n-1}{n}\right)\right| \\
&= \phi(n)\sum_{S_4} \left[\frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} - 1 + \left(\frac{n-1}{n}\right)\right] + \phi(n)\sum_{S_5} \left[\frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \left(\frac{n-1}{n}\right)\right] \\
&= \phi(n)\sum_{S_4} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} - \phi(n)\sum_{S_4} 1 + \left(\frac{n-1}{n}\right)\phi(n)\sum_{S_4} 1 + \phi(n)\sum_{S_5} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} \\
&+ \phi(n)\sum_{S_5} 1 - \left(\frac{n-1}{n}\right)\phi(n)\sum_{S_5} 1 \\
&= \phi(n)\sum_{S_4\cup S_5} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} - \phi(n)\left[\sum_{S_4} 1 - \sum_{S_5} 1\right] + \left(\frac{n-1}{n}\right)\phi(n)\left[\sum_{S_4} 1 - \sum_{S_5} 1\right].
\end{aligned} \tag{4}$$

Sub Case II:

$$\text{Suppose } S_3 = \sum_{S_6 \cup S_7} \left| \frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n) - \left(\frac{n-1}{n}\right)\phi(n) \right|$$

$$= \phi(n) \sum_{S_6 \cup S_7} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \phi(n)\left[\sum_{S_6} 1 - \sum_{S_7} 1\right] - \left(\frac{n-1}{n}\right)\phi(n)\left[\sum_{S_6} 1 - \sum_{S_7} 1\right]. \tag{5}$$

From (2), (3), (4) and (5),

$$\begin{aligned} LE(G_n) &= (n-s)\left[\phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right] + \phi(n)2^r - \phi(n)\left[\sum_{S_4} 1 - \sum_{S_5} 1\right] + \left(\frac{n-1}{n}\right)\phi(n)\left[\sum_{S_4} 1 - \sum_{S_5} 1\right] \\ &\quad + \phi(n)\left[\sum_{S_6} 1 - \sum_{S_7} 1\right] - \left(\frac{n-1}{n}\right)\phi(n)\left[\sum_{S_6} 1 - \sum_{S_7} 1\right] \\ &= (n-s)\left[\phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right] + \phi(n)2^r - \phi(n)\left[\sum_{S_4} 1 - \sum_{S_5} 1 - \sum_{S_6} 1 + \sum_{S_7} 1\right] \\ &\quad + \left(\frac{n-1}{n}\right)\phi(n)\left[\sum_{S_4} 1 - \sum_{S_5} 1 - \sum_{S_6} 1 + \sum_{S_7} 1\right] \\ &= (n-s)\left[\phi(n) - \left(\frac{n-1}{n}\right)\phi(n)\right] + \phi(n)2^r - \phi(n) + \left(\frac{n-1}{n}\right)\phi(n). \end{aligned}$$

Thus, Laplacian energy of the unitary addition Cayley graph G_n is

$$LE(G_n) = \phi(n)\left[\frac{n(1+2^r) - s - 1}{n}\right],$$

where $s = p_1 p_2 \cdots p_r$ is the maximal square-free divisor of n. □

6. Laplacian energy of complement of unitary addition Cayley graphs

This section covers Laplacian energy of the complement of unitary addition Cayley graph G_n for all n.

Theorem 6.1. *Let n be even. Then Laplacian eigenvalues of the complement of unitary addition Cayley graph G_n are 0 and $n - \phi(n) + \mu(t_k)\frac{\phi(n)}{\phi(t_k)}, 1 \leq k \leq n-1$.*

Proof. Let $L(G_n^c) = (l_{ij}^c)$, $0 \leq i, j \leq n-1$, be the Laplacian matrix of G_n^c, where

$$l_{ij}^c = \begin{cases} -1 & \text{if gcd(i+j,n)} \neq 1 \text{ and i} \neq \text{j}, \\ n - 1 - \phi(n) & \text{if i= j}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(G_n^c) = (n - 1 - \phi(n))I - A(X_n^c)$.

The eigenvalues of $-A(X_n^c)$ are $\phi(n) + 1 - n$ and $\mu(t_k)\frac{\phi(n)}{\phi(t_k)} + 1$ for $1 \leq k \leq n-1$. Thus Laplacian eigenvalues of the complement of unitary addition Cayley graph G_n are 0 and $n - \phi(n) + \mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $1 \leq k \leq n-1$. □

Corollary 6.2. *Let n be even. Then Laplacian energy of the complement of unitary addition Cayley graph G_n is*

$$2n - 2 + (2^r - 2)\,\phi(n) - s,$$

where $s = p_1 p_2 \cdots p_r$ is the maximal square-free divisor of n.

In order to find the Laplacian energy of the complement of unitary addition Cayley graph G_n we need to know the following facts.
From Theorem 2.5 eigenvalues of X_n^c are $\lambda_0 = n-1-\phi(n)$ and $\lambda_r = -1-\mu(t_r)\frac{\phi(n)}{\phi(t_r)}$.
Therefore eigenvalues of $-A(X_n^c)-I$ are $\lambda_0 = \phi(n)-n$ and $\lambda_r = \mu(t_r)\frac{\phi(n)}{\phi(t_r)}, 1 \leq r \leq n-1$.
Consider $N = (n_{ij}), 0 \leq i, j \leq n-1$, where

$$n_{ij} = \begin{cases} -1 & \text{if gcd(i+j, n)}\neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is a left circulant matrix with first row $(c_0, c_1, \cdots, c_{n-1})$, where

$$c_j = \begin{cases} -1 & \text{if gcd(j, n)}\neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd, then eigenvalues of N are $\phi(n)-n$ and $\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $1 \leq k \leq (n-1)/2$ and $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $(n+1)/2 \leq k \leq n-1$.

Theorem 6.3. *Let n be odd. Then Laplacian eigenvalues of the complement of unitary addition Cayley graph G_n are 0 and $n+\mu(t_k)\frac{\phi(n)}{\phi(t_k)}-\phi(n)$ for $1 \leq k \leq \frac{n-1}{2}$ and $n-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}-\phi(n)$ for $\frac{n+1}{2} \leq k \leq n-1$.*

Proof. Let $L(G_n^c) = (l_{ij}^c), 0 \leq i, j \leq n-1$, be the Laplacian matrix of G_n^c, where

$$l_{ij}^c = \begin{cases} -1 & \text{if gcd(i+j,n)}\neq 1 \text{ and i}\neq \text{j}, \\ n-1-\phi(n) & \text{if gcd(i+j,n)}\neq 1 \text{ and i= j}, \\ n-\phi(n) & \text{if gcd(i+j,n)=1 and i= j}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(G_n^c) = N + (n-\phi(n))I$, where $N = (n_{ij}), 0 \leq i, j \leq n-1$, $n_{ij} = \begin{cases} -1 & \text{if gcd(i+j,n)}\neq 1, \\ 0 & \text{otherwise.} \end{cases}$

From the above discussion we get eigenvalues of N are $\phi(n)-n$ and $\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $1 \leq k \leq (n-1)/2$ and $-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}$ for $(n+1)/2 \leq k \leq n-1$.
Therefore Laplacian eigenvalues of the complement of unitary addition Cayley graph G_n are 0 and $n+\mu(t_k)\frac{\phi(n)}{\phi(t_k)}-\phi(n)$ for $1 \leq k \leq \frac{n-1}{2}$ and $n-\mu(t_k)\frac{\phi(n)}{\phi(t_k)}-\phi(n)$ for $\frac{n+1}{2} \leq k \leq n-1$. □

Corollary 6.4. *Let n be odd. Then Laplacian spectral radius of the complement of unitary addition Cayley graph G_n is $n+\frac{\phi(n)}{\phi(p_1)}-\phi(n)$.*

Corollary 6.5. *Let n be odd. Then algebraic connectivity of the complement of unitary addition Cayley graph G_n is $n-\frac{\phi(n)}{\phi(p_1)}-\phi(n)$.*

Theorem 6.6. *Let n be odd. Then Laplacian energy of the complement of unitary addition Cayley graph G_n is*

$$LE(G_n^c) = (n-s)\left[1-\frac{\phi(n)}{n}\right] + (2^r-1)\phi(n) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n),$$

where $s = p_1p_2\cdots p_r$ is the maximal square-free divisor of n.

Proof. Laplacian energy of G_n^c is,

$$LE(G_n^c) = \sum_{i=1}^{n-1}\left|n-\mu_i-\frac{2\overline{m}}{n}\right| + \frac{2\overline{m}}{n} + \sum_{i=1}^{n-1}\left|n-\mu_i-n+1+\phi(n)-\frac{\phi(n)}{n}\right| + \frac{2\overline{m}}{n}, \text{ where } \overline{m} = E(G_n^c)$$

$$LE(G_n^c) = \sum_{1\leq i\leq \frac{n-1}{2}}\left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)}+1-\frac{\phi(n)}{n}\right| + \sum_{\frac{n+1}{2}\leq i\leq n-1}\left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)}+1-\frac{\phi(n)}{n}\right| + n-1-\left(\frac{n-1}{n}\right)\phi(n). \tag{6}$$

Divide the sum in equation (6) into two parts,

1. $\frac{n}{gcd(i,n)}$ is a non square-free number
2. $\frac{n}{gcd(i,n)}$ is a square-free number(SF)

Then $LE(G_n^c) = S_1 + S_2 + S_3 + n - 1 - \left(\frac{n-1}{n}\right)\phi(n)$, where

$$S_1 = \sum_{1\leq i\leq n-1,\ \frac{n}{gcd(i,n)}\notin SF} \left|1 - \frac{\phi(n)}{n}\right|,$$

$$S_2 = \sum_{1\leq i\leq \frac{n-1}{2},\ \frac{n}{gcd(i,n)}\in SF} \left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right| \text{and}$$

$$S_3 = \sum_{\frac{n+1}{2}\leq i\leq n-1,\ \frac{n}{gcd(i,n)}\in SF} \left|-\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right|. \tag{7}$$

Part I: Suppose $\frac{n}{gcd(i,n)}$ is a non square-free number.
We know that number of solutions of the equation $\mu\left(\frac{n}{gcd(i,n)}\right) = 0$ is $n - s$.
Therefore

$$S_1 = (n - s)\left[1 - \frac{\phi(n)}{n}\right]. \tag{8}$$

Part II: Suppose $\frac{n}{gcd(i,n)}$ is square-free number.
In S_2 and S_3 two possibility arises one is $\frac{n}{gcd(i,n)}$ has an even number of distinct prime divisors and the other one is $\frac{n}{gcd(i,n)}$ has an odd number of distinct prime divisors. We can denote the corresponding sum in S_2 by S_4 and S_5 and S_3 by S_6 and S_7. Now we have two sub cases.
Sub case I:

$$\begin{aligned}
\text{Assume } S_2 &= \sum_{S_4\cup S_5} \left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right| \\
&= \sum_{S_4} \left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right| + \sum_{S_5} \left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right| \\
&= \sum_{S_4} \left|\frac{\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right| + \sum_{S_5} \left|-\frac{\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right| \\
&= \sum_{S_4} \left[\frac{\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n}\right] + \sum_{S_5} \left[\frac{\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} - 1 + \frac{\phi(n)}{n}\right]
\end{aligned}$$

$$\begin{aligned}
&= \phi(n)\sum_{S_4} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \sum_{S_4} 1 - \frac{\phi(n)}{n}\sum_{S_4} 1 + \phi(n)\sum_{S_5} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} - \sum_{S_5} 1 + \frac{\phi(n)}{n}\sum_{S_5} 1 \\
&= \phi(n)\sum_{S_4\cup S_5} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} + \left[\sum_{S_4} 1 - \sum_{S_5} 1\right] - \frac{\phi(n)}{n}\left[\sum_{S_4} 1 - \sum_{S_5} 1\right].
\end{aligned} \tag{9}$$

Sub case II:

$$\text{Suppose } S_3 = \sum_{S_6 \cup S_7} \left| -\frac{\mu\left(\frac{n}{gcd(i,n)}\right)\phi(n)}{\phi\left(\frac{n}{gcd(i,n)}\right)} + 1 - \frac{\phi(n)}{n} \right|$$
$$= \phi(n) \sum_{S_6 \cup S_7} \frac{1}{\phi\left(\frac{n}{gcd(i,n)}\right)} - \left[\sum_{S_6} 1 - \sum_{S_7} 1\right] + \frac{\phi(n)}{n}\left[\sum_{S_6} 1 - \sum_{S_7} 1\right]. \tag{10}$$

From (7), (8), (9) and (10),

$$LE(G_n^c) = (n-s)\left(1 - \frac{\phi(n)}{n}\right) + \phi(n)\,(2^r - 1) + \left[\sum_{S_4} 1 - \sum_{S_5} 1 - \sum_{S_6} 1 + \sum_{S_7} 1\right]$$
$$- \frac{\phi(n)}{n}\left[\sum_{S_4} 1 - \sum_{S_5} 1 - \sum_{S_6} 1 + \sum_{S_7} 1\right] + n - 1 - \left(\frac{n-1}{n}\right)\phi(n)$$
$$= (n-s)\left(1 - \frac{\phi(n)}{n}\right) + \phi(n)\,(2^r - 1) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n).$$

Thus, Laplacian energy of the complement of unitary addition Cayley graph G_n is

$$LE(G_n^c) = (n-s)\left[1 - \frac{\phi(n)}{n}\right] + (2^r - 1)\phi(n) + n - 1 - \left(\frac{n-1}{n}\right)\phi(n),$$

where $s = p_1 p_2 \cdots p_r$ is the maximal square-free divisor of n. □

7. Conclusion

In this paper we obtain the following results based on different types of eigenvalues of G_n and G_n^c, where n is an odd integer. μ_k and $\bar{\mu}_k$ denotes the Laplacian eigenvalues of G_n and G_n^c.
1) $\lambda_k < \mu_k, 1 \le k \le n-1$ and $\lambda_0 > \mu_0$.
2) $\lambda_k < \bar{\mu}_k, 1 \le k \le n-1$ and $\lambda_0 > \bar{\mu}_0$.
3) $\bar{\lambda}_k < \mu_k, 1 \le k \le n-1$ and $\bar{\lambda}_0 > \mu_0$.
4) $\bar{\lambda}_k < \bar{\mu}_k, 1 \le k \le n-1$ and $\bar{\lambda}_0 > \bar{\mu}_0$.
5) $\bar{\mu}_k > \mu_k, 1 \le k \le n-1$ if $n > 2\phi(n)$ and $\bar{\mu}_0 = 0 = \mu_0$.
6) $\bar{\mu}_k < \mu_k, 1 \le k \le n-1$ if $n < 2\phi(n)$ and $\bar{\mu}_0 = 0 = \mu_0$.

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