



Pseudo-B-Fredholm Operators, Poles of the Resolvent and Mean Convergence in the Calkin Algebra

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Abstract. We define here a pseudo B-Fredholm operator as an operator such that 0 is isolated in its essential spectrum, then we prove that an operator T is pseudo- B-Fredholm if and only if $T = R + F$ where R is a Riesz operator and F is a B-Fredholm operator such that the commutator $[R, F]$ is compact. Moreover, we prove that 0 is a pole of the resolvent of an operator T in the Calkin algebra if and only if $T = K + F$, where K is a power compact operator and F is a B-Fredholm operator, such that the commutator $[K, F]$ is compact. As an application, we characterize the mean convergence in the Calkin algebra.

1. Introduction

Let X be an infinite dimensional Banach space and let $L(X)$ be the Banach algebra of bounded linear operators acting on X . The Calkin algebra over X is the quotient algebra $C(X) = B(X)/K(X)$, where $K(X)$ is the closed ideal of compact operators on X . For $T \in L(X)$, let $\text{Ker}(T)$ denote the null-space and $R(T)$ the range of T . An operator $T \in L(X)$ is Fredholm if $\dim \text{Ker}(T) < \infty$ and $\text{codim} R(T) < \infty$. For $T \in L(X)$ the Fredholm spectrum, is defined by:

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Fredholm operator}\}$$

Recall that the class of linear bounded B-Fredholm operators were defined in [5]. If $F_0(X)$ is the ideal of finite rank operators in $L(X)$ and $\pi : L(X) \rightarrow A$ is the canonical homomorphism, where $A = L(X)/F_0(X)$, it is well known by the Atkinson's theorem [3, Theorem 0.2.2, p.4], that $T \in L(X)$ is a Fredholm operator if and only if its projection $\pi(T)$ in the algebra A is invertible. Similarly, the following result established an Atkinson-type theorem for B-Fredholm operators.

Theorem 1.1. [7, Theorem 3.4] *Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in the algebra $L(X)/F_0(X)$.*

We conclude from the Atkinson's theorem and the previous theorem, that invertibility in the algebra $A = L(X)/F_0(X)$ give rises to Fredholm operators, while Drazin invertibility in this algebra give rises to B-Fredholm operators.

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Recall that an element of a unital algebra A is called generalized Drazin invertible if 0 is not an accumulation point of its spectrum. Then it is natural to ask what are the properties of those operators whose image under the canonical homomorphism $\pi : L(X) \rightarrow A$ is generalized Drazin invertible?

Such operators will be called pseudo B-Fredholm operators, and will be studied in the second section. The scheme Fredholmness-Invertibility, B-Fredholmness- Drazin invertibility is completed naturally by the couple pseudo-B-Fredholmness- Generalized Drazin invertibility.

In a recent works, among them [8], [20] and [22], several authors studied pseudo-B-Fredholm operators as the direct sum of a Fredholm operator and a quasi-nilpotent operator. As will be seen, this definition is a particular case of our new definition, and by an example we prove the class of pseudo-B-Fredholm operators we study here contains strictly the class of operators studied by the authors cited above.

In the main results of the second section we prove that the set $pBF(X)$ of pseudo-B-Fredholm operators in $L(X)$ is a regularity. Thus, by usual properties of regularities, this implies that the pseudo-B-Fredholm spectrum $\sigma_{pBF}(T)$ satisfies the spectral mapping theorem. Then we show that $T \in L(X)$ is a pseudo-B-Fredholm operator if and only if $T = R + F$ where R is a Riesz operator, F is a B-Fredholm operator such that the commutator $[R, F]$ is compact or an inessential operator, and if and only if T is a compact perturbation of a direct sum of a Fredholm operator and a Riesz operator.

In the third section, we will answer successively the following three questions. The first one is: given a bounded linear operator T , when 0 is a pole of its resolvent in the Calkin algebra $C(X)$? We show that this holds if and only if $T = K + B$, where B is a B-Fredholm operator and K is a power compact one, such that the commutator $[K, B]$ is compact. The second question is about the relation between the order of 0 as a pole of the resolvent of $\Pi(T)$ for a B-Fredholm operator T and the essential ascent $a_e(T)$ and the essential descent $d_e(T)$ of T , where $\Pi : L(X) \rightarrow L(X)/K(X)$ is the canonical homomorphism. We show that if 0 is a pole of order n , then $n \leq a_e(T) = d_e(T)$. Moreover we prove that $n = a_e(T) = d_e(T)$ if and only if $R(T^n)$ is closed. The third question is: if 0 is a pole of the resolvent of $\Pi(T)$ of order n , when T is then a B-Fredholm operator with $n = a_e(T) = d_e(T)$? The answer is that this happens if and only if $R(T^n)$ and $R(T^{n+1})$ are closed. With the answer to those questions, we retrieve in particular some similar results established in the case of Hilbert spaces in [4].

As an application, we characterize mean ergodic convergence in the Calkin algebra. Precisely, we show that the sequence $(\Pi(M_n(T)))_n$ converges in the Calkin algebra if and only if $\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and there exists a power compact operator K such that $a_e(I - T + K)$ and $d_e(I - T + K)$ are both finite and the commutator $[T, K]$ is compact, where $M_n(T) = \frac{1+T+T^2+\dots+T^n}{n}$, $n \in \mathbb{N}$ and $T \in L(X)$.

We define now some tools that will be needed later. For $n \in \mathbb{N}$ and $T \in L(X)$, we set $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. From [15, Lemmas 3.1 and 3.2] it follows that $c_n(T) = \text{codim}(R(T) + N(T^n))$ and $c'_n(T) = \dim(N(T) \cap R(T^n))$. Obviously, the sequences $(c_n(T))_n$ and $(c'_n(T))_n$ are decreasing. The *descent* $\delta(T)$ and the *ascent* $a(T)$ of T are defined by $\delta(T) = \inf\{n \in \mathbb{N} : c_n(T) = 0\} = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ and $a(T) = \inf\{n \in \mathbb{N} : c'_n(T) = 0\} = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$. We set formally $\inf \emptyset = \infty$.

The *essential descent* $\delta_e(T)$ and the *essential ascent* $a_e(T)$ of T are defined by $\delta_e(T) = \inf\{n \in \mathbb{N} : c_n(T) < \infty\}$ and $a_e(T) = \inf\{n \in \mathbb{N} : c'_n(T) < \infty\}$.

Given a Banach algebra A and an element a of A , the left multiplication operator $L_a : A \rightarrow A$ is defined by $L_a(x) = ax$, for all $x \in A$. It is well known that the spectrum of a is equal to the spectrum of L_a . We are particularly interested in the case when A is the Calkin algebra and $a = \Pi(T)$ for $T \in L(X)$.

The ascent and the descent of a Banach algebra element $a \in A$ are defined respectively as the ascent and the descent of the operator L_a .

Now we give the definition of operators of topological uniform descent, studied in [11].

Definition 1.2. Let $T \in L(X)$ and let $d \in \mathbb{N}$. Then T has a uniform descent for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geq d$.

If in addition $R(T) + N(T^d)$ is closed, then T is said to have a topological uniform descent for $n \geq d$.

The *radical* of a unital Banach algebra A is the set:

$$\{d \in A : 1 - ad \text{ is invertible for all } a \in A\} = \{d \in A : 1 - da \text{ is invertible for all } a \in A\}.$$

The set of all operators $A \in L(X)$ satisfying $\Pi(A) \in \text{Rad}(C(X))$, is the set of *inessential* operators, denoted by $I(X)$.

For more details about those definitions, we refer the reader to [1].

2. Pseudo-B-Fredholm operators

Definition 2.1. [17] Let A be an algebra over the field of complex numbers with a unit e . A non-empty subset \mathbf{R} of A is called a *regularity* if it satisfies the following conditions:

- If $a \in A$ and $n \geq 1$ is an integer, then $a \in \mathbf{R}$ if and only if $a^n \in \mathbf{R}$,
- If $a, b, c, d \in A$ are mutually commuting elements satisfying $ac + bd = e$, then $ab \in \mathbf{R}$ if and only if $a, b \in \mathbf{R}$.

Recall also that an element $a \in A$ is said to be Drazin invertible if there exists $b \in A$ such that $bab = b, ab = ba$ and $aba - a$ is a nilpotent element in A .

Definition 2.2. An element a of a Banach algebra A will be said to be *generalized Drazin invertible* if there exists $b \in A$ such that $bab = b, ab = ba$ and $aba - a$ is a quasinilpotent element in A .

Koliha [16] proved that $a \in A$ is generalized Drazin invertible if and only if there exists $\epsilon > 0$, such that for all λ such that $0 < |\lambda| < \epsilon$, the element $a - \lambda e$ is invertible.

In the case of a general unital algebra, not necessarily a normed algebra, we adopt this characterization as the definition of generalized Drazin invertibility in such algebra. This is in particular the case of the algebra $A = L(X)/F_0(X)$.

Proposition 2.3. Let $T \in L(X)$. Then $\pi(T)$ is generalized Drazin invertible in the algebra $A = L(X)/F_0(X)$ if and only if $\Pi(T)$ is generalized Drazin invertible in the Calkin algebra $C(X)$.

Proof. This is a direct consequence of the well known characterization of Fredholm operators. \square

Definition 2.4. Let $T \in L(X)$. Then T is said to be a *pseudo-B-Fredholm operator* if $\pi(T)$ is generalized Drazin invertible in the algebra $L(X)/F_0(X)$.

If $K \subset \mathbb{C}$, then $\text{acc } K$ is the set of accumulation points of K .

Proposition 2.5. Let $T \in L(X)$. Then T is a pseudo-B-Fredholm operator if and only if $0 \notin \text{acc } \sigma_F(T)$.

Proof. This is a direct consequence of the Definition 2.4, the characterisation of generalized Drazin invertible operators and the characterization of Fredholm operators. \square

It is proved in [18] that the set of generalized Drazin invertible elements in a unital Banach algebra is a regularity, from Proposition 2.3 we obtain immediately the following result.

Theorem 2.6. The set $pBF(X)$ of pseudo-B-Fredholm operators in $L(X)$ is a regularity.

Let $\sigma_{pBF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a pseudo-B-Fredholm operator}\}$ be the spectrum generated by the regularity $pBF(X)$, for $T \in L(X)$, then we have the following spectral mapping theorem.

Theorem 2.7. If f an analytic function in a neighborhood of the usual spectrum $\sigma(T)$ of an operator T in $L(X)$, which is non-constant on any connected component of $\sigma(T)$, then $f(\sigma_{pBF}(T)) = \sigma_{pBF}(f(T))$.

Proof. This is a direct consequence of the properties of regularities. \square

Remark 2.8. We say that an operator $T \in L(X)$ is polynomially Riesz if there exists a non-zero complex polynomial $P(z)$ such that $P(T)$ is a Riesz operator. Every polynomially Riesz operator in $L(X)$ is a pseudo-B-Fredholm operator. Indeed if T is polynomially Riesz, then $P(T)$ is Riesz for a non-zero complex polynomial $P(z)$. As it is well known that the Fredholm spectrum satisfies the spectral mapping theorem, then we have $P(\sigma_F(T)) = \sigma_F(P(T)) = \{0\}$. Hence $\sigma_F(T)$ is finite because a polynomial has a finite set of roots. So it has no accumulation points and from Proposition 2.5, T is pseudo-B-Fredholm.

For $T \in L(X)$, we will say that a subspace M of X is T -invariant if $T(M) \subset M$. We define $T|_M : M \rightarrow M$ as $T|_M(x) = T(x)$, $x \in M$. If M and N are two closed T -invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in \text{Red}(T)$. In this case we write $T = T|_M \oplus T|_N$ and say that T is a *direct sum* of $T|_M$ and $T|_N$.

It is said that $T \in L(X)$ admits a generalized Kato decomposition, abbreviated as GKD, if there exists $(M, N) \in \text{Red}(T)$ such that $T|_M$ is Kato and $T|_N$ is quasinilpotent. Recall that an operator $T \in L(X)$ is *Kato* if $R(T)$ is closed and $\text{Ker}(T) \subset R(T^n)$ for every $n \in \mathbb{N}$.

Definition 2.9. $T \in L(X)$ is called a *Riesz-Fredholm operator* if there exists $(M, N) \in \text{Red}(T)$ such that $T|_M$ is a Riesz operator and $T|_N$ is a Fredholm operator.

It is known that the sum of a Fredholm operator and a Riesz operator whose commutator is compact (or only an inessential operator) is again a Fredholm operator. In the following theorem we show that an operator $T \in L(X)$ is pseudo-B-Fredholm if and only if it is the sum of a B-Fredholm operator and a Riesz operator whose commutator is a compact (an inessential) operator.

Theorem 2.10. Let $T \in L(X)$. Then the following properties are equivalent:

1. T is a pseudo-B-Fredholm operator.
2. T is a compact perturbation of a Riesz-Fredholm operator.
3. $T = R + B$ where R is a Riesz operator, B is a B-Fredholm operator such that the commutator $[R, B]$ is compact.
4. $T = R + B$ where R is a Riesz operator, B is a B-Fredholm operator such that the commutator $[R, B]$ is an inessential operator.

Proof. 1) \Leftrightarrow 2) Suppose that T is a pseudo-B-Fredholm operator. If T is Fredholm, then the statement 2 holds. Further, suppose that T is not Fredholm, then 0 is an isolated point of $\sigma(\Pi(T))$. Let $R \in C(X)$ be the spectral idempotent of $\Pi(T)$ corresponding to $\lambda = 0$, then $R \neq 0$, $\Pi(T)$ and R commute, $\Pi(T)R$ is quasinilpotent and $\Pi(T) + R$ is invertible according to [16, Theorem 3.1]. From [2, Lemma 1] we know that there exists an idempotent $P \in L(X)$ such that $\Pi(P) = R$. Therefore, $\Pi(TP)$ is quasinilpotent and $\Pi(T + P)$ is invertible. Since $\Pi(T)$ and $\Pi(P)$ commutes, we have that $\Pi(PTP) = \Pi(TP)$ and $\Pi((I - P)T(I - P)) = \Pi(T(I - P))$. It follows that PTP is Riesz, $TP = PTP + K_1$, $T(I - P) = (I - P)T(I - P) + K_2$, where $K_1, K_2 \in K(X)$, and so,

$$T = TTP + T(I - P) = PTP + (I - P)T(I - P) + K,$$

where $K = K_1 + K_2 \in K(X)$. We have that $(R(P), R(I - P)) \in \text{Red}(PTP)$, $(R(P), R(I - P)) \in \text{Red}((I - P)T(I - P))$,

$$PTP = (PTP)|_{R(P)} \oplus (PTP)|_{R(I-P)} = (PTP)|_{R(P)} \oplus 0$$

and

$$\begin{aligned} (I - P)T(I - P) &= ((I - P)T(I - P))|_{R(P)} \oplus ((I - P)T(I - P))|_{R(I-P)} \\ &= 0 \oplus ((I - P)T(I - P))|_{R(I-P)}. \end{aligned}$$

Therefore,

$$T = (PTP)|_{R(P)} \oplus ((I - P)T(I - P))|_{R(I-P)} + K. \tag{1}$$

It's easily seen that $(PTP)_{|R(P)}$ is Riesz and further we prove that $((I - P)T(I - P))_{|R(I-P)}$ is Fredholm. Since $\Pi(T + P)$ is invertible there exists $S \in L(X)$ such that $\Pi(S)\Pi(T + P) = \Pi(T + P)\Pi(S) = \Pi(I)$. As $\Pi(P)$ and $\Pi(T + P)$ commute, then $\Pi(P)$ and $\Pi(S)$ commute and hence

$$\begin{aligned} (I - P)(T + P)(I - P)(I - P)S(I - P) &= I - P + F_1, \\ (I - P)S(I - P)(I - P)(T + P)(I - P) &= I - P + F_2, \end{aligned}$$

where F_1 and F_2 are compact. As $I - P$ is the identity on $R(I - P)$, it follows that $((I - P)T(I - P))_{|R(I-P)} = ((I - P)(T + P)(I - P))_{|R(I-P)}$ is a Fredholm operator. According to (1), we see that T is a compact perturbation of a Riesz-Fredholm operator.

Conversely let $T = T_1 \oplus T_2 + K$ where T_1 is Riesz, T_2 Fredholm and $K \in K(X)$.

It's clear that 0 is not an accumulation point of $\sigma_F(T_1 \oplus T_2) = \sigma_F(T)$ and according to Proposition 2.5, we get that T is a pseudo-B-Fredholm operator.

1) \Rightarrow 3) If T is a pseudo-B-Fredholm operator, then

$$\begin{aligned} T &= (PTP)_{|R(P)} \oplus ((I - P)T(I - P))_{|R(I-P)} + K \\ &= [((PTP)_{|R(P)} \oplus 0) + K] + [0 \oplus ((I - P)T(I - P))_{|R(I-P)}], \end{aligned}$$

where $[((PTP)_{|R(P)} \oplus 0) + K]$ is a Riesz operator and from [5, Theorem 2.7], $[0 \oplus ((I - P)T(I - P))_{|R(I-P)}]$ is a B-Fredholm operator, here P is the same idempotent as in the previous part of the proof. It is clear that the commutator of $[((PTP)_{|R(P)} \oplus 0) + K]$ and $[0 \oplus ((I - P)T(I - P))_{|R(I-P)}]$ is compact.

3) \Rightarrow 4) It follows from the inclusion $K(X) \subset I(X)$.

4) \Rightarrow 1) Let $T = R + B$, where R is a Riesz operator and B is a B-Fredholm operator with $[R, B]$ is an inessential operator. From [24, Theorem 10.1] it follows that

$$\sigma_F(T) = \sigma_F(B). \tag{2}$$

Since B is B-Fredholm, according to [6, Remark A (iii)] there exists $\epsilon > 0$, such that if $0 < |\lambda| < \epsilon$, we have that $B - \lambda I$ is Fredholm which together with (2) gives that $\lambda \notin \sigma_F(T)$. So $0 \notin \text{acc } \sigma_F(T)$ and thus T is a pseudo-B-Fredholm operator by Proposition 2.5. \square

We mention that Boasso considered in [8, Theorem 5.1] isolated points of the spectrum of $\Pi(T)$ for $T \in L(X)$ and he concluded the equivalence $(1) \iff (2)$ by studying generalized Drazin invertible elements in the range of a Banach algebra homomorphism [8, Theorem 3.2], though our proof is more direct.

Corollary 2.11. *Let H be a Hilbert space and $T \in L(H)$. Then T is a pseudo-B-Fredholm operator if and only if $T = K + Q + B$, where K is compact, Q quasi-nilpotent, B B-Fredholm, with K and $[Q, B]$ compact operators.*

Proof. In the case of a Hilbert space, using the West decomposition [21] for a Riesz operator $R \in L(H)$ we have $R = K + Q$ with K compact and Q quasi-nilpotent. Thus, according to Theorem 2.10, $T \in L(H)$ is a pseudo-B-Fredholm operator if and only if $T = K + Q + B$, where Q quasi-nilpotent, B B-Fredholm, K and $[Q, B]$ compact operators. \square

Remark 2.12. In the recent works [8], [20] and [22], the authors studied pseudo-B-Fredholm operators as the direct sum of a Fredholm operator and a quasi-nilpotent one. In [10, Theorem 3.4] it is proved that

$$\begin{aligned} T \text{ is the direct sum of a Fredholm operator and a quasi - nilpotent one} \\ \iff \\ T \text{ admits a GKD and } 0 \notin \text{acc } \sigma_F(T). \end{aligned} \tag{3}$$

However there exists operators which are pseudo-B-Fredholm operators in the sense of Definition 2.4, but do not have a decomposition as the direct sum of a Fredholm operator and a quasi-nilpotent operator as seen by the following example.

Let T be a compact operator having infinite spectrum. Since $\Pi(T) = 0$, then T is a pseudo-B-Fredholm operator in the sense of Definition 2.4. We prove that T cannot be written as the direct sum of a Fredholm operator and a quasi-nilpotent one. Assume the contrary. We observe first that T is not quasinilpotent, because it has non-zero spectrum. Also T is not Fredholm because is compact on the infinite dimensional space X .

Assume that there exists a pair M, N of closed T -invariant subspaces of X such that $T = T_1 \oplus T_2$ where $T_1 = T|_M$ is a quasi-nilpotent operator and $T_2 = T|_N$ is a Fredholm operator. Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, it follows that $\sigma(T_2)$ is infinite. Therefore N is infinite-dimensional and hence $\sigma_F(T_2) \neq \emptyset$. As T_2 is Fredholm, it follows that $0 \notin \sigma_F(T_2)$. Then then there exists $\lambda \neq 0$ such that $\lambda \in \sigma_F(T_2)$. But $\sigma_F(T) = \sigma_F(T_1) \cup \sigma_F(T_2)$, $\sigma_F(T) = \{0\}$ and we get a contradiction.

Our example shows that the condition that T admits a GKD cannot be removed from the equivalence (3), in the other words the condition $0 \notin \text{acc } \sigma_F(T)$ does not imply the condition that T admits a GKD.

3. Poles of the resolvent in the Calkin algebra

Theorem 3.1. *Let $T \in L(X)$. Then the following properties are equivalent:*

- 1- 0 is a pole of the resolvent of $\Pi(T)$ in the Calkin algebra.
- 2- T is the sum of a B-Fredholm operator B and a power compact operator K , such that the commutator $[K, B]$ is compact.
- 3- There exists a power compact operator K such that $a_e(T + K)$ and $d_e(T + K)$ are both finite, such that the commutator $[K, T]$ is compact.

Proof. 1 \Rightarrow 2) Suppose that 0 is pole of the resolvent of $\Pi(T)$ in the Calkin algebra. Let $R \in C(X)$ be the spectral idempotent of $\Pi(T)$ corresponding to $\lambda = 0$. Then $\Pi(T)$ and R commute, $\Pi(T)R$ is nilpotent and $\Pi(T) + R$ is invertible. From [2, Lemma 1] it follows that there exists an idempotent $P \in L(X)$ such that $\Pi(P) = R$. Since $\Pi(TP) = \Pi(PTP)$, it follows that PTP is a power compact operator. As in the proof of Theorem 2.10, we get that there is $K' \in K(X)$ such that

$$T = (PTP)_{|R(P)} \oplus ((I - P)T(I - P))_{|R(I-P)} + K'.$$

Set $K = ((PTP)_{|R(P)} \oplus 0) + K' = PTP + K'$ and $B = 0 \oplus ((I - P)T(I - P))_{|R(I-P)}$. Then K is clearly a power compact operator, B is a B-Fredholm operator by [5, Theorem 2.7] and the commutator $[K, B]$ is compact.

2) \Rightarrow 3) Assume that T is the sum of a B-Fredholm operator B and a power compact operator K' such that the commutator $[K', B]$ is compact. Let $K = -K'$, then $B = T + K$, and from [6, Theorem 3.1], $a_e(T + K)$ and $d_e(T + K)$ are both finite. Moreover, the commutator $[K, T]$ is compact.

3) \Rightarrow 1) Assume that there exists a power compact operator K such that $a_e(T + K)$ and $d_e(T + K)$ are both finite and the commutator $[K, T]$ is compact. Then from [6, Theorem 3.1], $T + K$ is a B-Fredholm operator. Hence $\Pi(T + K)$ is Drazin invertible in the Calkin algebra. As the commutator $[K, T]$ is compact, then $\Pi(T)\Pi(K) = \Pi(K)\Pi(T)$. Since K is power compact, then $\Pi(K)$ is nilpotent. From [23, Theorem 3] we know that Drazin invertibility is stable under nilpotent commuting perturbations. Thus it follows that $\Pi(T) = \Pi(T + K) - \Pi(K)$ is Drazin invertible in the Calkin algebra and 0 is pole of $\Pi(T)$.

□

As a consequence, in the case of Hilbert spaces, we recover [4, Theorem 2.2].

Corollary 3.2. *Let H be a Hilbert space and $T \in L(H)$. The following properties are equivalent:*

- 1- 0 is pole of the resolvent of $\Pi(T)$ in the Calkin algebra.
- 2- There exist a compact operator K such that $T + K$ is a B-Fredholm operator.
- 3- There exist a compact operator K such that $a_e(T + K)$ and $d_e(T + K)$ are both finite.

Proof. 1) \Rightarrow 2) As in the proof of Theorem 3.1, there exist an idempotent $P \in L(H)$ and $K \in K(H)$ such that $T = (PTP)_{|R(P)} \oplus ((I - P)T(I - P))_{|R(I - P)} + K$, where $((I - P)T(I - P))_{|R(I - P)}$ is a Fredholm operator and PTP is a power compact operator, which implies that $(PTP)_{|R(P)}$ is a power compact operator. From [13, Lemma 5], there exists a nilpotent operator N_1 and a compact operator K_1 defined on $R(P)$ such that $(PTP)_{|R(P)} = N_1 + K_1$. Hence

$$T = [N_1 \oplus ((I - P)T(I - P))_{|R(I - P)}] + [(K_1 \oplus 0) + K],$$

$(K_1 \oplus 0) + K$ is clearly a compact operator and by [5, Theorem 2.7] the operator $N_1 \oplus ((I - P)T(I - P))_{|R(I - P)}$ is a B-Fredholm operator.

The proof of the other implications are similar to the corresponding implications in Theorem 3.1. \square

Recall that from [12, Theorem 5.3], if $T \in L(X)$ has finite essential ascent $a_e(T)$ and finite essential descent $d_e(T)$, then they are equal. In the following result, for a B-Fredholm operator T , we compare the order of 0 as a pole of the resolvent of $\Pi(T)$ and the common value of its essential ascent and its essential descent.

Theorem 3.3. *Let $T \in L(X)$ be a B-Fredholm operator. Then $\Pi(T)$ is Drazin invertible in the Calkin algebra and if n is the order of 0 as a pole of the resolvent of $\Pi(T)$, then $n \leq a_e(T) = d_e(T)$. Moreover $n = a_e(T) = d_e(T)$ if and only if $R(T^n)$ is closed*

Proof. Let $d = a_e(T) = d_e(T)$ and assume that $d < n$. Then from [6, Theorem 3.1] $R(T^d)$ is closed and the operator $T_d : R(T^d) \rightarrow R(T^d)$ is a Fredholm operator. Thus there exists a compact operator K_d in $L(R(T^d))$, an operator R_d in $L(R(T^d))$ such that $R_d T_d = I_d + K_d$, where I_d is the identity of $L(R(T^d))$. Thus $T^d - R_d T^{d+1}$ is a compact operator.

Let $V \in L(X)$ such that $T^n V$ is compact. Then

$$T^{n-1} V = (T^d - R_d T^{d+1}) T^{n-d-1} V + R_d T^n V$$

is a compact operator. Hence $a(\Pi(T)) \leq n - 1 < n$ and this is a contradiction. Thus $n \leq a_e(T) = d_e(T)$.

If $n = a_e(T) = d_e(T)$, then from [6, Theorem 3.1], $R(T^n)$ is closed. Conversely if $R(T^n)$ is closed, let $d = a_e(T) = d_e(T)$. Since $\Pi(T)$ is Drazin invertible in the Calkin algebra, there exists $S \in L(X)$, such that the operators $TS - ST, STS - S, T^n ST - T^n$ are all compact operators. Let $K = ST^{n+1} - T^n$, then K is a compact operator and $\text{Ker}(T) \cap R(T^n) \subset R(K)$. As $\text{Ker}(T) \cap R(T^n)$ is closed, then $\text{Ker}(T) \cap R(T^n)$ is finite dimensional. If $n < d$, then $n \leq d - 1$ and $\text{Ker}(T) \cap R(T^{d-1}) \subset \text{Ker}(T) \cap R(T^n)$. Hence $\text{Ker}(T) \cap R(T^{d-1})$ is finite dimensional and consequently $a_e(T) \leq d - 1$, which is a contradiction. Hence $n \geq d$. As we know already that $n \leq a_e(T) = d_e(T)$, then $n = a_e(T) = d_e(T)$. \square

Remark 3.4. Without the hypothesis of the closedness of the range $R(T^n)$, Theorem 3.3 may be false. For example let K be a nilpotent compact operator with infinite dimensional range, then it is easily seen that the order of 0 as a pole of $\Pi(K)$ is equal to one, while K has a finite essential ascent and descent strictly greater than 1, because the range $R(K)$ of K is not closed.

In the following result, we give a sufficient condition which implies the equality of the order and the common value of the essential ascent and descent. In the case of Hilbert spaces, this result was proved in [4, proposition 3.3]. While in [4, proposition 3.3], the proof is based on Sadosvskii essential enlargement of an operator [19], our proof is based directly on the definition of the Drazin inverse.

Theorem 3.5. *Let $T \in L(X)$ be a B-Fredholm operator with finite essential ascent and finite essential descent equaling d . If $d = 0$ or $R(T^{d-1})$ is closed, then 0 is a pole of the resolvent of $\Pi(T)$ of order d .*

Proof. If $d = 0$ then T is a Fredholm operator. So $\Pi(T)$ is invertible in the Calkin algebra and 0 is a pole of the resolvent of $\Pi(T)$ of order d .

Assume now that $d > 0$ and $R(T^{d-1})$ is closed. Since T is a B-Fredholm operator, then $\Pi(T)$ is Drazin invertible in the Calkin algebra. Let $n = a(\Pi(T)) = d(\Pi(T))$. Then there exists $S \in L(X)$ such that the operators $TS - ST, STS - S, T^n ST - T^n$, are all compact operators. Let $K = ST^{n+1} - T^n$, then K is a compact operator and $\text{Ker}(T) \cap R(T^n) \subset R(K)$.

We already know that $n \leq d$. If $n < d$, then $n \leq d - 1$ and $\text{Ker}(T) \cap R(T^{d-1}) \subset \text{Ker}(T) \cap R(T^n) \subset R(K)$. As $\text{Ker}(T) \cap R(T^{d-1})$ is closed, then $\text{Ker}(T) \cap R(T^{d-1})$ is finite dimensional. Thus $a_e(T) \leq d - 1$. Contradiction. Hence $n = d$.

□

Now we give necessary and sufficient conditions to lift a Drazin invertible element in the Calkin algebra as a B-Fredholm operator. The sufficient condition in the case Hilbert spaces had been proved in [4, Theorem 3.2], where the proof is based also on Sadovskii essential enlargement of an operator [19], while we use here properties of B-Fredholm operators.

Theorem 3.6. *Let $T \in L(X)$ such that $\Pi(T)$ is Drazin invertible in the Calkin algebra and 0 is a pole of the resolvent of $\Pi(T)$ of order n . Then T is a B-Fredholm operator with $a_e(T) = d_e(T) = n$ if and only if $R(T^n)$ and $R(T^{n+1})$ are closed.*

Proof. If T is a B-Fredholm operator with $a_e(T) = d_e(T) = n$, then from [6, Theorem 3.1], $R(T^n)$ and $R(T^{n+1})$ are closed.

Conversely assume that $R(T^n)$ and $R(T^{n+1})$ are closed. Since 0 is a pole of the resolvent of $\Pi(T)$ of order n , then there exists an operator S in $L(X)$, such that $TS - ST, STS - S, T^n ST - T^n$ are all compact operators. Let $K = ST^{n+1} - T^n$, then K is a compact operator. If $y \in N(T) \cap R(T^n)$, then $y \in R(K)$. Since $N(T) \cap R(T^n)$ is closed and K is compact, then $N(T) \cap R(T^n)$ is of finite dimension. Let $T_n : R(T^n) \rightarrow R(T^n)$ be the operator induced by T . Then T_n is an upper semi-Fredholm operator and so, T is a semi-B-Fredholm operator. In particular and from [11] it is an operator of topological uniform descent. Since 0 is a pole of the resolvent of $\Pi(T)$ of order n , if $|\lambda|$ is small enough and $\lambda \neq 0$, then $T - \lambda I$ is a Fredholm operator. From [11, Theorem 4.7] it follows that T has a finite essential ascent and finite essential descent. Thus T is a B-Fredholm operator and $a_e(T) = d_e(T) \leq n$. As T is a B-Fredholm operator, then from Theorem 3.3, we have $n \leq a_e(T) = d_e(T)$ and so, $a_e(T) = d_e(T) = n$.

□

Application

We give now an application of the previous results for the study of the mean convergence in the Calkin algebra. For the uniform ergodic theorem, we refer the reader to [9, Theorem 1.5] and the references cited there. Here, using Theorem 3.1, we obtain easily a general characterization of the convergence of the sequence $(\Pi(M_n(T)))_n$ in the Calkin algebra. Let us mention that by [14, Proposition 7], 0 is a pole of $\Pi(T)$ if and only if 0 is a pole of the left multiplication by $\Pi(T)$ in the Calkin algebra.

Theorem 3.7. *Let $T \in L(X)$ and let $M_n(T) = \frac{1+T+T^2+\dots+T^n}{n}$, $n \in \mathbb{N}^*$. Then following conditions are equivalent:*

- 1- *The sequence $(\Pi(M_n(T)))_n$ converges in the Calkin algebra.*
- 2- *$\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and there exists a power compact operator K such that $I - T + K$ is a B-Fredholm operator, and the commutator $[T, K]$ is compact.*
- 3- *$\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and there exists a power compact operator K such that $a_e(I - T + K)$ and $d_e(I - T + K)$ are both finite and the commutator $[T, K]$ is compact.*
- 4- *$\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and 1 is a pole of the resolvent of $\Pi(T)$.*

Proof. 1) \Rightarrow 2) Assume that the sequence $(\Pi(M_n(T)))_n$ converges in the Calkin algebra. Then from [9, Theorem 1.5], it follows that $\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and 0 is a pole of the resolvent of $\Pi(I - T)$. From Theorem 3.1, there exists a power compact operator K such that $I - T + K$ is a B-Fredholm operator and the commutator $[T, K]$ is compact.

2) \Rightarrow 3) Assume $\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and there exists a power compact operator K such that $I - T + K$ is a B-Fredholm operator. Then $a_e(I - T + K)$ and $d_e(I - T + K)$ are both finite and the commutator $[T, K]$ is compact.

3) \Rightarrow 4) Assume that $\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and there exists a power compact operator K such that $a_e(I - T + K)$ and $d_e(I - T + K)$ are both finite, and the commutator $[T, K]$ is compact. Then from Theorem 3.1, 1 is a pole of the resolvent of $\Pi(T)$.

4) \Rightarrow 1) Assume that $\frac{\|\Pi(T)^n\|}{n} \rightarrow 0$ as $n \rightarrow \infty$ and 1 is a pole of the resolvent of $\Pi(T)$. Using [9, Theorem 1.5], it follows that the sequence $(\Pi(M_n(T)))_n$ converges in the Calkin algebra. \square

Note that from [9, Theorem 1.5], if T satisfies one of the conditions of Theorem 3.7, then 1 is a pole of the resolvent of $\Pi(T)$ of order less or equal to 1.

Remark 3.8. In the case of a Hilbert space, and from Corollary 3.2, the operator K of Theorem 3.7 can be chosen to be compact.

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