Filomat 33:12 (2019), 3759–3771
https://doi.org/10.2298/FIL1912759S

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

L^1-Convergence of Double Trigonometric Series

Karanvir Singh[a], Kanak Modi[b]

[a]*Department of Applied Mathematics, GZS Campus College of Engineering and Technology, Maharaja Ranjit Singh Punjab Technical University Bathinda, Punjab, India*
[b]*Department of Mathematics, Amity University of Rajasthan, Jaipur, India*

Abstract. In this paper we study the pointwise convergence and convergence in L^1-norm of double trigonometric series whose coefficients form a null sequence of bounded variation of order $(p,0)$, $(0,p)$ and (p,p) with the weight $(jk)^{p-1}$ for some integer $p>1$. The double trigonometric series in this paper represents double cosine series, double sine series and double cosine sine series. Our results extend the results of Young [9], Kolmogorov [4] in the sense of single trigonometric series to double trigonometric series and of Móricz [6, 7] in the sense of higher values of p.

1. Introduction

Consider the double trigonometric series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}\ \psi_j(x)\ \psi_k(y) \tag{1.1}$$

on positive quadrant $T=[0,\pi]\times[0,\pi]$ of the two dimensional torus.
The double trigonometric series (1.1) represents

(a) double cosine series $\sum\limits_{j=0}^{\infty}\sum\limits_{k=0}^{\infty}\lambda_j\lambda_k a_{jk}\cos jx\cos ky$ where $\lambda_0=\frac{1}{2}$ and $\lambda_j=1$ for $j=1,2,3,\dots$.

(b) double sine series $\sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{\infty} a_{jk}\ \sin jx\ \sin ky$

(c) double cosine-sine series $\sum\limits_{j=0}^{\infty}\sum\limits_{k=1}^{\infty}\lambda_j a_{jk}\cos jx\sin ky$ where $\lambda_0=\frac{1}{2}$ and $\lambda_j=1$ for $j=1,2,3,\dots$.

The rectangular partial sums $\psi_{mn}(x,y)$ and the *Cesàro* means $\sigma_{mn}(x,y)$ of the series (1.1) are defined as

$$\psi_{mn}(x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}\ \psi_j(x)\ \psi_k(y),$$

2010 *Mathematics Subject Classification*. 42A20, 42A32
Keywords. L^1-convergence, *Cesàro* means, monotone sequences
Received: 18 May 2018; Revised: 31 August 2018; Accepted: 14 September 2018
Communicated by Ivana Djolović
Email addresses: karanvir@mrsptu.ac.in (Karanvir Singh), kmodi@jpr.amity.edu (Kanak Modi)

$$\sigma_{mn}(x,y)=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\psi_{jk}(x,y)\quad (m,n>0)$$

and for $\lambda > 1$, the truncated *Cesàro* means are defined by

$$V_{mn}^{\lambda}(x,y)=\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}\psi_{jk}(x,y).$$

Assuming the coefficients $\{a_{jk} : j,k \geq 0\}$ in (1.1) be a double sequence of real numbers which satisfy the following conditions which may be called as conditions of bounded variation for some positive integer p:

$$|a_{jk}|(jk)^{p-1}\to 0 \quad as \quad max\{j,k\}\to\infty, \tag{1.2}$$

$$\lim_{k\to\infty}\sum_{j=0}^{\infty}|\triangle_{p0}a_{jk}|(jk)^{p-1}=0, \tag{1.3}$$

$$\lim_{j\to\infty}\sum_{k=0}^{\infty}|\triangle_{0p}a_{jk}|(jk)^{p-1}=0, \tag{1.4}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\triangle_{pp}a_{jk}|(jk)^{p-1}<\infty. \tag{1.5}$$

For some integers p and q, the finite order differences $\triangle_{pq}a_{jk}$ are defined by

$$\triangle_{00}a_{jk}=a_{jk};$$
$$\triangle_{pq}a_{jk}=\triangle_{p-1,q}a_{jk}-\triangle_{p-1,q}a_{j+1,k}\quad (p\geq 1, q\geq 0);$$
$$\triangle_{pq}a_{jk}=\triangle_{p,q-1}a_{jk}-\triangle_{p,q-1}a_{j,k+1}\quad (p\geq 0, q\geq 1).$$

Also a double induction argument gives

$$\triangle_{pq}a_{jk}=\sum_{s=0}^{p}\sum_{t=0}^{q}(-1)^{s+t}\binom{p}{s}\binom{q}{t}a_{j+s,\,k+t}.$$

The above mentioned (1.2)-(1.5) conditions generalise the concept of monotone sequences. Also any sequence satisfying (1.5) with $p = 2$ is called a quasi-convex sequence [4, 7]. Clearly the conditions (1.2) and (1.5) implies (1.3) and (1.4) for $p = 1$ and moreover for $p = 1$, the conditions (1.2) and (1.5) reduce to

$$|a_{jk}|\to 0 \text{ as } max\{j,k\}\to\infty \quad \text{and} \quad \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\triangle_{11}a_{jk}|<\infty.$$

Generally the pointwise convergence of the series (1.1) is defined in Pringsheim's sense ([10],vol. 2, ch. 17). Let the sum of the series (1.1) be denoted by $f(x,y)$ (provided it exists).

Also let $\|f\|$ denotes the $L^1(T^2)$-norm, i.e, $\|f\|=\int_0^{\pi}\int_0^{\pi}\left|f(x,y)\right|dxdy$

Many authors like Móricz [6, 7], Chen [2], K. Kaur et al. [3] and Krasniqi [5] studied integrability and L^1-convergence of double trigonometric series under different classes of coefficients. In [7], Móricz studied both double cosine series and double sine series as far as their integrability and convergence in L^1-norm is concerned where as in [6] he studied complex double trigonometric series under coefficients of bounded variation.

These authors mainly discussed the case for $p = 1$ or $p = 2$ and preferred the condition of bounded variation on coefficients. Our aim in this paper is to extend the above results from $p = 1$ or $p = 2$ to general cases for double trigonometric series of all types as mentioned above.

For convenience, we write $\lambda_n = [\lambda n]$ where n is a positive integer, $\lambda > 1$ is a real number and [ ] means greatest integral part and in the results, C_p denote constants which may not be the same at each occurrence.

Our first main result is as follows:

Theorem 1.1. *Assume that conditions* (1.2) − (1.5) *are satisfied for some integer* $p \geq 1$*, then*
(i) $\psi_{mn}(x, y)$ *converges pointwise to* $f(x, y)$ *for every* $(x, y) \in T^2 \setminus \{(0, 0)\}$*;*
(ii) $\|\psi_{mn}(x, y) - f(x, y)\| = o(1)$ *as* $\min(m, n) \to \infty$.

The results mentioned in above theorem has been proved by Móricz [6, 7] for $p = 1$ and $p = 2$ using suitable estimates for Dirichlet's kernel $D_j(x)$ and Fejér kernel $K_j(x)$ where as in the case of a single series for $p = 2$, the results regarding convergence have been proved by Kolmogorov [4].

Obviously, condition (1.5) implies any of the following conditions:

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{j=0}^{\infty} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| (jk)^{p-1} = 0; \tag{1.6}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m\to\infty}} \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{\infty} \frac{\lambda_m - j + 1}{\lambda_m - m} |\triangle_{pp} a_{jk}| (jk)^{p-1} = 0. \tag{1.7}$$

We introduce the following three sums for $m, n \geq 0$ and $\lambda > 1$:

$$S_{10}^{\lambda}(m, n, x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m}\, a_{jk}\, \psi_j(x)\, \psi_k(y);$$

$$S_{01}^{\lambda}(m, n, x, y) = \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n}\, a_{jk}\, \psi_j(x)\, \psi_k(y);$$

$$S_{11}^{\lambda}(m, n, x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m}\, \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk}\, \psi_j(x)\, \psi_k(y)$$

and we have

$$S_{11}^{\lambda}(m, n; x, y) = \frac{1}{(\lambda_m - m)} \sum_{u=m+1}^{\lambda_m} \Big(S_{01}^{\lambda}(u, n; x, y) - S_{01}^{\lambda}(m, n; x, y) \Big);$$

$$S_{11}^{\lambda}(m, n; x, y) = \frac{1}{(\lambda_n - n)} \sum_{v=n+1}^{\lambda_n} \Big(S_{10}^{\lambda}(m, v; x, y) - S_{10}^{\lambda}(m, n; x, y) \Big).$$

This implies

$$S_{11}^{\lambda}(m, n; x, y) \leq \left\{ \begin{array}{c} 2 \sup\limits_{m \leq u \leq \lambda m} \Big(|S_{01}^{\lambda}(u, n; x, y)| \Big) \\ 2 \sup\limits_{n \leq v \leq \lambda n} \Big(|S_{10}^{\lambda}(m, v; x, y)| \Big) \end{array} \right\} \tag{1.8}$$

The second result of this paper is the following theorem:

Theorem 1.2. *Let* $E \subset T^2$*. Assume that the following conditions are satisfied:*

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \Big(\sup_{(x,y)\in E} |S_{10}^{\lambda}(m, n; x, y)| \Big) = 0; \tag{1.9}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \Big(\sup_{(x,y)\in E} |S_{01}^{\lambda}(m, n; x, y)| \Big) = 0. \tag{1.10}$$

If $V_{mn}^{\lambda}(x, y)$ *converges uniformly on E to* $f(x, y)$ *as* $\min(m, n) \to \infty$*, then so does* ψ_{mn}.

We will also prove the following theorem:

Theorem 1.3. *Assume that the conditions (1.2)-(1.4) and (1.6)-(1.7) are satisfied for some integer $p \geq 1$, then (i) if $V^{\lambda}_{mn}(x, y)$ converges uniformly to $f(x, y)$ as $\min(m, n) \to \infty$ then ψ_{mn} will also converge uniformly to $f(x, y)$ as $\min(m, n) \to \infty$. (ii) If $\|V^{\lambda}_{mn} - f\| \longrightarrow 0$ then $\|\psi_{mn} - f\| \longrightarrow 0$ as $\min(m, n) \to \infty$.*

2. Notations and formulas

The Cesàro sums of order α of the sequence $\{\psi_j(t)\}$ for any real number α are denoted by $\psi^{\alpha}_j(t)$. Thus we have

$$\psi^{\alpha}_j(t) = \sum_{s=0}^{j} \psi^{\alpha-1}_s(t) \quad (\alpha \geq 1, j \geq 0) \tag{2.1}$$

In this paper $\psi_j{}^1(t)$ either represents $D_j(t)$ or $\tilde{D}_j(t)$ where $D_j(t)$ and $\tilde{D}_j(t)$ represents Dirichlet and conjugate Dirichlet Kernels respectively. Also from [8], we have following estimates

$$(i)\ |\psi^{\alpha}_j(x)| = O\Big((j+1)^{\alpha}\Big) \ for\ all\ \alpha \geq 1,\ -\pi \leq x \leq \pi. \tag{2.2}$$

$$(ii)\ \psi^{p}_j(x) = O\Big(\frac{1}{x^p}\Big) \text{ for all } p \geq 2,\ (0 < x \leq \pi) \tag{2.3}$$

3. Lemmas

We require the following lemmas for the proof of our results:

Lemma 3.1. *For $m, n \geq 0$ and $p > 1$, the following representation holds:*

$$\psi_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk}\ \psi_j(x)\ \psi_k(y)$$

$$= \sum_{j=0}^{m} \sum_{k=0}^{n} \triangle_{pp} a_{jk} \psi^{p-1}_j(x) \psi^{p-1}_k(y) + \sum_{j=0}^{m} \sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} \psi^{p-1}_j(x) \psi^{t}_n(y)$$

$$+ \sum_{k=0}^{n} \sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} \psi^{s}_m(x) \psi^{p-1}_k(y) + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} \psi^{s}_m(x) \psi^{t}_n(y).$$

Lemma 3.2. *[2] For $m, n \geq 0$ and $\lambda > 1$, the following representation holds:*

$$\begin{aligned}\psi_{mn} - \sigma_{mn} &= \frac{\lambda_m + 1}{\lambda_m - m}\ \frac{\lambda_n + 1}{\lambda_n - n}\ (\sigma_{\lambda_m,\lambda n} - \sigma_{\lambda_m,n} - \sigma_{m,\lambda n} + \sigma_{mn}) \\ &+ \frac{\lambda_m + 1}{\lambda_m - m}\ (\sigma_{\lambda_m,n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m,\lambda_n} - \sigma_{mn}) \\ &- S^{\lambda}_{11}(m, n, x, y) - S^{\lambda}_{10}(m, n, x, y) - S^{\lambda}_{01}(m, n, x, y).\end{aligned}$$

Lemma 3.3. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$V^{\lambda}_{mn} - \psi_{mn} = S^{\lambda}_{11}(m, n, x, y) + S^{\lambda}_{10}(m, n, x, y) + S^{\lambda}_{01}(m, n, x, y).$$

Proof. We have

$$V_{mn}^{\lambda}(x,y) = \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \psi_{jk}(x,y)$$

Now we can write

$$\frac{1}{(\lambda_m - m)} \sum_{j=m+1}^{\lambda_m} \psi_{jk}(x,y) = \frac{1}{(\lambda_m - m)} \Big[\sum_{j=0}^{\lambda_m} \psi_{jk}(x,y) - \sum_{j=0}^{m} \psi_{jk}(x,y) \Big]$$

$$= \frac{\lambda_m + 1}{(\lambda_m - m))} \Big[\frac{1}{\lambda_m + 1} \sum_{j=0}^{\lambda_m} \psi_{jk}(x,y) \Big] - \frac{m+1}{(\lambda_m - m))} \Big[\frac{1}{m+1} \sum_{j=0}^{m} \psi_{jk}(x,y) \Big]$$

Thus

$$V_{mn}^{\lambda}(x,y) = \frac{1}{(\lambda_n - n)} \sum_{k=n+1}^{\lambda_n} \Big[\frac{1}{(\lambda_m - m)} \sum_{j=m+1}^{\lambda_m} \psi_{jk}(x,y) \Big]$$

$$= \frac{1}{(\lambda_n - n)} \sum_{k=n+1}^{\lambda_n} \Big[\frac{\lambda_m + 1}{(\lambda_m - m)} \frac{1}{\lambda_m + 1} \sum_{j=0}^{\lambda_m} \psi_{jk}(x,y) - \frac{m+1}{(\lambda_m - m)} \frac{1}{m+1} \sum_{j=0}^{m} \psi_{jk}(x,y) \Big]$$

$$= \frac{1}{(\lambda_n - n)} \frac{\lambda_m + 1}{(\lambda_m - m)} \frac{1}{\lambda_m + 1} \sum_{j=0}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \psi_{jk}(x,y) - \frac{1}{(\lambda_n - n)} \frac{m+1}{(\lambda_m - m)} \frac{1}{m+1} \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \psi_{jk}(x,y)$$

= S11 + S22

$$\text{Now} \quad S11 = \frac{1}{(\lambda_n - n)} \frac{\lambda_m + 1}{(\lambda_m - m)} \frac{1}{\lambda_m + 1} \Big[\sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x,y) - \sum_{j=0}^{\lambda_m} \sum_{k=0}^{n} \psi_{jk}(x,y) \Big]$$

$$= \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{\lambda_m, \lambda n} - \frac{\lambda_m + 1}{\lambda_m - m} \frac{n+1}{\lambda_n - n} \sigma_{\lambda_m, n}$$

Similarly we get

$$S22 = \frac{m+1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{m, \lambda n} - \frac{m+1}{\lambda_m - m} \frac{n+1}{\lambda_n - n} \sigma_{mn}$$

Thus we have

$$V_{mn}^{\lambda} = \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{\lambda_m, \lambda n} - \frac{\lambda_m + 1}{\lambda_m - m} \frac{n+1}{\lambda_n - n} \sigma_{\lambda_m, n} - \frac{m+1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{m, \lambda n} + \frac{m+1}{\lambda_m - m} \frac{n+1}{\lambda_n - n} \sigma_{mn}$$

$$= \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{\lambda_m, \lambda n} - \sigma_{\lambda_m, n} - \sigma_{m, \lambda n} + \sigma_{mn}) + \frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m, n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \sigma_{mn}.$$

(by rearrangement of terms)

The use of Lemma 3.2 gives

$$V_{mn}^{\lambda} - \psi_{mn} = \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \psi_j(x) \psi_k(y)$$

$$+ \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} \psi_j(x) \psi_k(y) + \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \psi_j(x) \psi_k(y).$$

□

Lemma 3.4. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$\begin{aligned} S_{10}^{\lambda}(m,n;x,y) &= \sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\, a_{jk}\psi_j(x)\psi_k(y) \\ &= \sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\,\triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y) + \sum_{j=m+1}^{\lambda_m}\sum_{t=0}^{p-1}\frac{\lambda_m-j+1}{\lambda_m-m}\,\triangle_{pt}a_{j,n+1}\psi_j^{p-1}(x)\psi_n^{t}(y) \\ &+\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{s=0}^{p-1}\sum_{k=0}^{n}\triangle_{sp}a_{j+1,k}\psi_j^{s}(x)\psi_k^{p-1}(y) + \frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{j+1,n+1}\psi_j^{s}(x)\psi_n^{t}(y) \\ &- \sum_{s=0}^{p-1}\sum_{k=0}^{n}\triangle_{sp}a_{m+1,k}\psi_m^{s}(x)\psi_k^{p-1}(y) - \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\psi_m^{s}(x)\psi_n^{t}(y). \end{aligned}$$

Proof. We have by summation by parts,

$$\begin{aligned} S_{10}^{\lambda}(m,n;x,y) &= \sum_{k=0}^{n}\psi_k(y)\Big(\sum_{j=m+1}^{\lambda_m}\frac{\lambda_m-j+1}{\lambda_m-m}\, a_{jk}\psi_j(x)\Big) \\ &= \sum_{k=0}^{n}\psi_k(y)\Big(\sum_{j=m+1}^{\lambda_m}\frac{\lambda_m-j+1}{\lambda_m-m}\,\triangle_{p0}a_{jk}\psi_j^{p-1}(x) + \frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{s=0}^{p-1}\triangle_{s0}a_{j+1,k}\psi_j^{s}(x) - \sum_{s=0}^{p-1}\triangle_{s0}a_{m+1,k}\psi_m^{s}(x)\Big) \\ &= \sum_{j=m+1}^{\lambda_m}\frac{\lambda_m-j+1}{\lambda_m-m}\,\psi_j^{p-1}(x)\Big(\sum_{k=0}^{n}\triangle_{p0}a_{jk}\psi_k(y)\Big) + \frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{s=0}^{p-1}\Big(\sum_{k=0}^{n}\triangle_{s0}a_{j+1,k}\psi_k(y)\Big)\psi_j^{s}(x) \\ &- \sum_{s=0}^{p-1}\Big(\sum_{k=0}^{n}\triangle_{s0}a_{m+1,k}\psi_k(y)\Big)\psi_m^{s}(x) \\ &= \sum_{j=m+1}^{\lambda_m}\frac{\lambda_m-j+1}{\lambda_m-m}\,\psi_j^{p-1}(x)\Big(\sum_{k=0}^{n}\triangle_{pp}a_{jk}\psi_k^{p-1}(y) + \sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}\psi_n^{t}(y)\Big) \\ &+\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{s=0}^{p-1}\Big(\sum_{k=0}^{n}\triangle_{sp}a_{j+1,k}\psi_k^{p-1}(y) + \sum_{t=0}^{p-1}\triangle_{st}a_{j+1,n+1}\psi_n^{t}(y)\Big)\psi_j^{s}(x) \\ &- \sum_{s=0}^{p-1}\Big(\sum_{k=0}^{n}\triangle_{sp}a_{m+1,k}\psi_k^{p-1}(y) + \sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\psi_n^{t}(y)\Big)\psi_m^{s}(x) \end{aligned}$$

Similarly we can have representation for $S_{01}^{\lambda}(m,n;x,y)$. □

4. Proof of Theorems

Proof of Theorem 1.1

For $m, n \geq 0$ and $p > 1$, we have from Lemma 3.1

$$\psi_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n}\triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y) + \sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}\psi_j^{p-1}(x)\psi_n^{t}(y)$$

$$+\sum_{k=0}^{n}\sum_{s=0}^{p-1}\triangle_{sp}a_{m+1,k}\psi_m^s(x)\psi_k^{p-1}(y)+\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\psi_m^s(x)\psi_n^t(y)=\sum_1+\sum_2+\sum_3+\sum_4.$$

Using (2.3) ,that is, $\psi_j^p(x)=O\Big(\frac{1}{x^p}\Big)$ for all $p\geq 2$, $(0<x\leq\pi)$ etc, we have for $(0<x,y\leq\pi)$,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y)|<\infty \qquad \text{(by (1.2))}$$

and also by (1.3) - (1.5), we have

$$\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}\leq\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\Big(\sum_{j=0}^{m}|\triangle_{p0}a_{j,n+v+1}|\Big)$$

$$\leq\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\leq\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\to 0$$

$$as\ \min(m,n)\to\infty$$

Thus $\displaystyle\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}\psi_j^{p-1}(x)\psi_n^t(y)\to 0$ as $\min(m,n)\to\infty$.

and similarly

$$\sum_{s=0}^{p-1}\sum_{k=0}^{n}\triangle_{sp}a_{m+1,k}\leq\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\Big(\sum_{k=0}^{n}|\triangle_{0p}a_{m+u+1,k}|\Big)$$

$$\leq\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\leq\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\to 0$$

$as\ \min(m,n)\to\infty$.

Thus $\displaystyle\sum_{k=0}^{n}\sum_{s=0}^{p-1}\triangle_{sp}a_{m+1,k}\psi_m^s(x)\psi_k^{p-1}(y)\to 0\quad as\min(m,n)\to\infty.$

Also

$$\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\leq\sum_{s=0}^{p-1}\sum_{t=o}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\triangle_{00}a_{m+u+1,n+v+1}|$$

$$\leq\sup_{j>m,k>n}|a_{jk}|\to 0\text{ as }\min(m,n)\to\infty.$$

So $\displaystyle\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\psi_m^s(x)\psi_n^t(y)\to 0$ as $\min(m,n)\to\infty$.

Consequently series (1.1) converges to the function $f(x,y)$ where

$$f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y)\quad\text{and}\quad\lim_{m,n\to\infty}\psi_{mn}(x,y)=f(x,y).$$

Now we will calculate $\|\sum_1\|$, $\|\sum_2\|$, $\|\sum_3\|$ and $\|\sum_4\|$ in the following way:

$$\begin{aligned}
\|\sum_1\| &= \|\sum_{j=0}^{m}\sum_{k=0}^{n} \triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y)\| \\
&\leq \sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|\int_0^{\pi}\int_0^{\pi}|\psi_j^{p-1}(x)\psi_k^{p-1}(y)|dxdy \\
&\leq C_p\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1}\int_0^{\pi}\int_0^{\pi}dxdy \quad \text{(by(2.2))} \\
&\leq C_p\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1}.
\end{aligned}$$

$$\begin{aligned}
\|\sum_2\| &= \|\sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt}a_{j,n+1}\psi_j^{p-1}(x)\psi_n^{t}(y)\| \\
&\leq \sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\Big(\sum_{j=0}^{m}|\triangle_{p0}a_{j,n+v+1}|\Big)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\psi_j^{p-1}(x)\psi_n^{t}(y)|dxdy \\
&\leq C_p\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\ j^{p-1}\Big(\sum_{t=0}^{p-1}n^t\Big) \quad \text{(by(2.2))} \\
&\leq C_p\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\ j^{p-1}k^{p-1}.
\end{aligned}$$

$$\begin{aligned}
\|\sum_3\| &= \|\sum_{s=0}^{p-1}\sum_{k=0}^{n} \triangle_{sp}a_{m+1,k}\psi_m^{s}(x)\psi_k^{p-1}(y)\| \\
&\leq \sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\Big(\sum_{k=0}^{n}|\triangle_{0p}a_{m+u+1,k}|\Big)_m^s\ k^{p-1} \\
&\leq C_p\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\ k^{p-1}\Big(\sum_{s=0}^{p-1}m^s\Big) \\
&\leq C_p\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\ j^{p-1}k^{p-1}.
\end{aligned}$$

$$\begin{aligned}
\|\sum_4\| &= \|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st}a_{m+1,n+1}\psi_m^{s}(x)\psi_n^{t}(y)\| \\
&\leq \sum_{s=0}^{p-1}\sum_{t=o}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\triangle_{00}a_{m+u+1,n+v+1}|m^s\ n^t \\
&\leq C_p\sup_{j>m,k>n}|a_{jk}|\ j^{p-1}k^{p-1}.
\end{aligned}$$

Now let R_{mn} consists of all (j,k) with $j > m$ or $k > n$, that is,

$$\sum\sum\nolimits_{(j,k)\in R_{mn}} = \sum_{j=m+1}^{\infty}\sum_{k=0}^{n} + \sum_{j=0}^{\infty}\sum_{k=n+1}^{\infty} + \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}.$$

Then

$$\|f-\psi_{mn}\| = \left(\int_0^{\pi}\int_0^{\pi}\left|f(x,y)-\psi_{mn}(x,y)\right|dxdy\right)$$

$$\leq \|\sum_{(j,k)}\sum_{\in R_{mn}} \triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y)\| + \|\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}\psi_j^{p-1}(x)\psi_n^{t}(y)\|$$

$$+\|\sum_{k=0}^{n}\sum_{s=0}^{p-1}\triangle_{sp}a_{m+1,k}\psi_m^{s}(x)\psi_k^{p-1}(y)\| + \|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\psi_m^{s}(x)\psi_n^{t}(y)\|$$

$$\leq C_p\Big\{\Big(\sum_{(j,k)\in R_{mn}}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1}\Big) + \Big(\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\ j^{p-1}k^{p-1}\Big)$$

$$+\Big(\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\ j^{p-1}k^{p-1}\Big) + \Big(\sup_{j>m,k>n}|a_{jk}|\ j^{p-1}k^{p-1}\Big)\Big\}$$

(As discussed above)

$\longrightarrow 0$ as $\min(m,n)\to\infty$ (by (1.2) to (1.5))

which proves (ii) part.

Proof of Theorem 1.2

Using the relation (1.8), we find that (1.9) or (1.10) implies

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|S_{11}^{\lambda}(m,n;x,y)|\Big) = 0. \tag{4.1}$$

Assume that $V_{mn}^{\lambda}(x,y)$ converges uniformly on E to $f(x,y)$. Then by Lemma 3.3, we get

$$\overline{\lim_{m,n\to\infty}}\Big(|\sup_{(x,y)\in E}\big(\psi_{mn}(x,y)-V_{mn}^{\lambda}(x,y)\big)|\Big)$$

$$\leq \overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|S_{10}^{\lambda}(m,n;x,y)|\Big)$$

$$+\ \overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|S_{01}^{\lambda}(m,n;x,y)|\Big)$$

$$+\ \overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|S_{11}^{\lambda}(m,n;x,y)|\Big).$$

After taking $\lambda\downarrow 1$ the result follows from (1.9), (1.10) and (4.1).

Proof of Theorem 1.3

Using the Lemma 3.4, we can write the expression for $S_{01}^{\lambda}(m,n;x,y)$ as

$$
\begin{aligned}
S_{01}^{\lambda}(m,n;x,y) &= \sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\,a_{jk}\psi_j(x)\psi_k(y)\\
&= \sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\,\triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y)+\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\frac{\lambda_n-k+1}{\lambda_n-n}\,\triangle_{sp}a_{m+1,k}\psi_m^{s}(x)\psi_k^{p-1}(y)\\
&+\frac{1}{\lambda_n-n}\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,k+1}\psi_j^{p-1}(x)\psi_k^{t}(y)+\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,k+1}\psi_m^{s}(x)\psi_k^{t}(y)\\
&-\sum_{t=0}^{p-1}\sum_{j=0}^{m}\triangle_{pt}a_{j,n+1}\psi_j^{p-1}(x)\psi_n^{t}(y)-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}\psi_m^{s}(x)\psi_n^{t}(y)\\
&=\sum_{11}+\sum_{12}+\sum_{13}+\sum_{14}+\sum_{15}+\sum_{16}.
\end{aligned}
$$

Now by using (1.2)-(1.4) and (1.6) along with estimates of $\psi_j^{p-1}(x)$ etc., as mentioned in [8], we have the following estimates :

$$
\begin{aligned}
\Big|\sum_{11}\Big| &= \Big|\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{pp}a_{jk}\psi_j^{p-1}(x)\psi_k^{p-1}(y)\Big|\\
&\le \sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\left|\triangle_{pp}a_{jk}\right| j^{p-1}k^{p-1}\\
&\to 0 \ \text{ as } \min(m,n)\to\infty.
\end{aligned}
$$

Consequently $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|\sum_{11}|\Big)\to 0$ as $\min(m,n)\to\infty$.

$$
\begin{aligned}
\Big|\sum_{12}\Big| &= \Big|\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{sp}a_{m+1,k}\psi_m^{s}(x)\psi_k^{p-1}(y)\Big|\\
&\le \sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\sum_{k=n+1}^{\lambda_n}|\triangle_{0p}a_{m+u+1,k}|m^s k^{p-1}\\
&\le \sup_{m<j\le m+p}\sum_{k=n+1}^{\lambda_n}|\triangle_{0p}a_{jk}|j^{p-1}k^{p-1}\to 0 \text{ as } \min(m,n)\to\infty.
\end{aligned}
$$

So $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|\sum_{12}|\Big)\to 0$ as $\min(m,n)\to\infty$.

$$
\begin{aligned}
\Big|\sum_{13}\Big| &\le \sup_{n<k\le\lambda_n}\sum_{t=0}^{p-1}\sum_{j=0}^{m}|\triangle_{pt}a_{j,k+1}|j^{p-1}k^{t}\\
&\le \sup_{n<k\le\lambda_n}\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m}|\triangle_{pt}a_{j,k+v+1}|j^{p-1}k^{t}
\end{aligned}
$$

$$\leq \sup_{n<k\leq\lambda_n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|j^{p-1}k^{p-1}\to 0 \text{ as } \min(m,n)\to\infty.$$

which implies $\lim\limits_{\lambda\downarrow 1}\overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|\sum_{13}|\Big)\to 0$ as $\min(m,n)\to\infty$.

Similarly we estimate others in brief

$$|\sum_{14}|\leq \sup_{n<k\leq\lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}|\triangle_{st}a_{m+1,k+1}|j^{p-1}k^{p-1}$$

$$\leq \sup_{j>m,k>n}|a_{jk}|j^{p-1}k^{p-1}\to 0 \text{ as } \min(m,n)\to\infty.$$

Thus $\lim\limits_{\lambda\downarrow 1}\overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|\sum_{14}|\Big)\to 0$ as $\min(m,n)\to\infty$.

$$|\sum_{15}|\leq\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m}|\triangle_{p0}a_{j,n+v+1}|j^{p-1}n^t\leq \sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|j^{p-1}k^{p-1}$$

$\to 0$ as $\min(m,n)\to\infty$.

which implies $\lim\limits_{\lambda\downarrow 1}\overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|\sum_{15}|\Big)\to 0$ as $\min(m,n)\to\infty$.

$$|\sum_{16}|\leq\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\triangle_{00}a_{m+u+1,n+v+1}|m^sn^t$$

$$\leq \sup_{j>m,k>n}|a_{jk}|j^{p-1}k^{p-1}\to 0 \text{ as } \min(m,n)\to\infty.$$

So $\lim\limits_{\lambda\downarrow 1}\overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|\sum_{16}|\Big)\to 0$ as $\min(m,n)\to\infty$.

Thus combining all these, we have

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|S_{01}^{\lambda}(m,n;x,y)|\Big)=0.$$

Similarly (1.2)-(1.4) and (1.7) results in

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|S_{10}^{\lambda}(m,n;x,y)|\Big)=0;$$

Thus first part of theorem follows from Theorem 4.2

Proof of (ii) We have

$\|\psi_{mn}-f\|\leq\|\psi_{mn}-V_{mn}^{\lambda}\|+\|V_{mn}^{\lambda}-f\|$.

By assumption $\|V_{mn}^{\lambda}-f\|\to 0$, so it is sufficient to show that

$\|\psi_{mn}-V_{mn}^{\lambda}\|\to 0$ as $\min(m,n)\to\infty$.

By Lemma 3.3, we have

$$\|\psi_{mn} - V_{mn}^{\lambda}\| \leq \|S_{10}^{\lambda}(m,n;x,y)\| + \|S_{01}^{\lambda}(m,n;x,y)\|$$
$$+\|S_{11}^{\lambda}(m,n;x,y)\|.$$

Now in order to estimate $\|S_{01}^{\lambda}(m,n;x,y)\|$, we first find $\|\sum_{11}\|, \|\sum_{12}\|,$

$\|\sum_{13}\|, \|\sum_{14}\|, \|\sum_{15}\|$ and $\|\sum_{16}\|$, so we have

$$\|\sum_{11}\| = \|\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{pp} a_{jk} \psi_j^{p-1}(x)\psi_k^{p-1}(y)\|$$

$$\leq \sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{pp} a_{jk} j^{p-1} k^{p-1} \int_0^{\pi}\int_0^{\pi} dxdy$$

$$\leq C_p \sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| j^{p-1} k^{p-1}.$$

$$\|\sum_{12}\| = \|\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{sp} a_{m+1,k} \psi_m^{s}(x)\psi_k^{p-1}(y)\|$$

$$\leq C_p \sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{m+u+1,k}| k^{p-1} m^s$$

$$\leq C_p \sup_{m<j\leq m+p}\Big(\sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| k^{p-1}\Big)\Big(\sum_{s=0}^{p-1} m^s\Big)$$

$$\leq C_p \sup_{m<j\leq m+p}\sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| j^{p-1} k^{p-1}.$$

$$\|\sum_{13}\| \leq C_p \sup_{n<k\leq \lambda_n}\sum_{t=0}^{p-1}\sum_{j=0}^{m} |\triangle_{pt} a_{j,k+1}| j^{p-1} k^t$$

$$\leq C_p \sup_{n<k\leq \lambda_n}\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m} |\triangle_{pt} a_{j,k+v+1}| j^{p-1} k^t$$

$$\leq C_p \sup_{n<k\leq \lambda_n+p}\sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1}.$$

$$\|\sum_{14}\| \leq C_p \sup_{n<k\leq \lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} |\triangle_{st} a_{m+1,k+1}| m^s k^t$$
$$\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}.$$

$$\|\sum_{15}\| \leq C_p \sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}| j^{p-1} n^t$$

$$\leq C_p \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1}.$$

$$\|\sum_{16}\| \leq C_p \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{s} \sum_{v=0}^{t} \binom{s}{u}\binom{t}{v} |\triangle_{00} a_{m+u+1,n+v+1}| m^s n^t$$
$$\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}.$$

Thus we can estimate

$$\|S_{01}^{\lambda}(m,n;x,y)\| \leq C_p \sum_{k=n+1}^{\lambda_n} \sum_{j=0}^{m} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| j^{p-1} k^{p-1} + C_p\Big(\sup_{m<j\leq m+p} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| j^{p-1} k^{p-1}\Big)$$
$$+C_p\Big(\sup_{n<k\leq \lambda_n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1}\Big) + C_p\Big(\sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}\Big)$$
$$+C_p\Big(\sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1}\Big) + C_p\Big(\sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}\Big).$$

By (1.2)-(1.4) and (1.6), we conclude that

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \Big(\|S_{01}^{\lambda}(m,n;x,y)\|\Big) = 0.$$

Similarly by conditions (1.2)-(1.4) and (1.7), we get

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \Big(\|S_{10}^{\lambda}(m,n;x,y)\|\Big) = 0.$$

Also by (1.8), we have

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \Big(\|S_{11}^{\lambda}(m,n;x,y)\|\Big) = 0.$$

Thus $\|\psi_{mn} - V_{mn}^{\lambda}\| \to 0$ as $\min(m,n) \to \infty$.

References

[1] N.K. Bary : A treatise on trigonometric series, Vol.II, Pergamon Press, London 1964.
[2] C.P. Chen and Y.W. Chauang : L^1-convergence of double Fourier series, Chinese Journal of Math. 19 (4) (1991), 391-410.
[3] K. Kaur , S. S. Bhatia and B. Ram : L^1-Convergence of complex double trigonometric series, Proc. Indian Acad. Sci., Vol. 113, No. 3 (Nov.2003), 01-09.
[4] A. N. Kolmogorov : Sur l'ordre de grandeur des coefficients de la sėrie de Fourier-Lebesque, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. (1923), 83-86.
[5] Xh. Z. Krasniqi : Integrability of double cosine trigonometric series with coefficients of bounded variation of second order, Commentationes Mathematicae, 51 (2011), 125-139.
[6] F. Móricz : Convergence and integrability of double trigonometric series with coefficients of bounded variation, Proc. Amer. Math. Soc. 102 (1988), 633-640.
[7] F. Móricz : On the integrability and L^1-convergence of double trigonometric series, Studia Math. (1991), 203-225.
[8] T. M. Vukolova : Certain properties of trigonometric series with monotone coefficients (English, Russian original), Mosc. Univ. Math. Bull., 39(6)(1984), 24-30; translation from Vestn. Mosk. Univ., Ser.I, (1984), No.6, 18-23.
[9] W. H. Young : On the Fourier series of bounded functions, Proc. London Math. Soc. 12 (2) (1913), 41-70.
[10] A. Zygmund : Trigonometric series, Vols. I,II, Cambridge University Press (1959).