



Measures of Noncompactness on the Standard Hilbert C^* -Module

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Abstract. We define a measure of noncompactness λ on the standard Hilbert C^* -module $l^2(\mathcal{A})$ over a unital C^* -algebra, such that $\lambda(E) = 0$ if and only if E is \mathcal{A} -precompact (i.e. it is ε -close to a finitely generated projective submodule for any $\varepsilon > 0$) and derive its properties. Further, we consider the known, Kuratowski, Hausdorff and Istrăţescu measure of noncompactness on $l^2(\mathcal{A})$ regarded as a locally convex space with respect to a suitable topology, and obtain their properties as well as some relationship between them and introduced measure of noncompactness λ .

1. Introduction

Measures of noncompactness (MNCs in further) have been studied almost for a century. Roughly speaking, a MNC is a function which assigns a real number to any bounded set in a given metric space, and this real number can be regarded as a characteristic of the extent to which A is not totally bounded, (that is relatively compact when completeness is supposed). There are many different MNCs on metric spaces, among them the most cited are: Kuratowski, Hausdorff and Istrăţescu MNC. Their definitions is given by:

Definition 1.1. Let (M, d) be a metric space, and let $A \subseteq M$ be a bounded set.

a) The Kuratowski measure of noncompactness of A , denoted by $\alpha(A)$, is the infimum of all $d > 0$ such that A admits a partitioning into finitely many subsets whose diameters are less than d ([8], see also [9]).

b) The Hausdorff measure of noncompactness of A , denoted by $\chi(A)$, is the infimum of all $\varepsilon > 0$ such that A has a finite ε -mesh in M . (If we require that this ε -mesh belongs to A , we refer to inner Hausdorff measure of noncompactness, denoted by $\chi_i(A)$.)

c) The Istrăţescu measure of noncompactness, denoted by $I(A)$, is the infimum of all $d > 0$ such that there is an ε -discrete sequence in A , that is, a sequence $x_n \in A$, such that $d(x_m, x_n) \geq \varepsilon$ for all $m \neq n$ ([6]).

Although different in general, these three functions share some common properties, e.g. the following:

Proposition 1.2. Let (M, d) be a metric space, and let μ be any of functions α , χ , I defined above. Then μ has the following properties:

(a) regularity: $\mu(A) = 0$ iff A is relatively compact;

2010 Mathematics Subject Classification. Primary 46L08, 47H08; Secondary 54E15

Keywords. Hilbert module, measures of noncompactness, uniform spaces.

Received: 10 November 2017; Accepted: 01 July 2018

Communicated by Dragan S. Djordjević

The authors was supported in part by the Ministry of education and science, Republic of Serbia, Grant #174034.

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(b) non-singularity: μ is equal to zero on any single-element set;

(c) monotonicity: $\mu(A) \leq \mu(B)$, whenever $A \subseteq B$;

(d) subadditivity: $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$, for all A and B ;

If, in addition, M is a normed space, i.e., M is a linear space and the metric d is defined via $d(x, y) = \|x - y\|$ then μ has additional properties:

(e) algebraic semi-additivity: $\mu(A + B) \leq \mu(A) + \mu(B)$, for all A, B ;

(f) semi-homogeneity: $\mu(tA) = |t|\mu(A)$, for $t \in \mathbf{C}$ and all $A \subseteq M$;

(g) invariance under translations: $\mu(x + A) = \mu(A)$, for all $A \subseteq M$ and $x \in M$;

(h) Lipschitz continuity: $|\mu(A) - \mu(B)| \leq Ld_H(A, B)$, where L is an absolute constant ($L_\alpha = 2, L_\chi = 1, L_I = 2$) and $d(A, B)$ is the so called Hausdorff distance, that is $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$;

(i) invariance to the transition to the closure and to the convex hull: $\mu(A) = \mu(\bar{A}) = \mu(\text{co } A)$;

(j) The functions α, χ and I are equivalent to each other, that is,

$$\chi(A) \leq I(A) \leq \alpha(A) \leq 2\chi(A) \quad \text{for all bounded } A. \quad (1)$$

These properties are well known and their proofs can be found throughout literature, for instance in [16]. (Though, it should be mentioned that inner Hausdorff MNC does not satisfy the equality $\chi_i(\text{co } A) = \chi_i(A)$, see [1, page 9])

Some of the mentioned properties were singled out in order to establish the axiomatic definition of the abstract notion of the MNC, and this was done in various different manners.

For more detailed exposition on MNCs on metric or normed spaces, the reader is referred to [1], [4] or [17].

There is some extension of the MNC theory to the framework of uniform spaces which will be described in Section 3. Among all uniform spaces, we are specially interested in the standard Hilbert module $l^2(\mathcal{A})$ over a unital C^* -algebra \mathcal{A} . It is defined by

$$l^2(\mathcal{A}) = \left\{ x = (\xi_1, \xi_2, \dots) \mid \xi_j \in \mathcal{A}, \sum_{j=1}^{+\infty} \xi_j^* \xi_j \text{ converges in the norm topology} \right\},$$

and it is equipped with the \mathcal{A} -valued inner product

$$l^2(\mathcal{A}) \times l^2(\mathcal{A}) \ni (x, y) \mapsto \sum_{j=1}^{+\infty} \xi_j^* \eta_j \in \mathcal{A}, \quad x = (\xi_1, \xi_2, \dots), \quad y = (\eta_1, \eta_2, \dots).$$

Since an arbitrary \mathcal{A} -linear bounded operator on $l^2(\mathcal{A})$ does not need to have an adjoint, the natural algebra of operators is $B^a(l^2(\mathcal{A}))$ - the algebra of all \mathcal{A} -linear bounded operators on $l^2(\mathcal{A})$ having an adjoint. It is known that $B^a(l^2(\mathcal{A}))$ is a C^* -algebra, as well.

Among all operators in $B^a(l^2(\mathcal{A}))$, those that belong to the linear span of the operators of the form $x \mapsto \Theta_{y,z}(x) = z \langle y, x \rangle$ ($y, z \in l^2(\mathcal{A})$) are called *finite rank operators*. The norm closure of finite rank operators is known as the algebra of all "compact" operators. The quotation marks are usually written in order to emphasize the fact that "compact" operators does not maps bounded sets into relatively compact sets, as it is the case in the framework of Hilbert (and also Banach) spaces, though they share many properties of proper compact operators on a Hilbert space, [12], [13].

(For general literature concerning Hilbert modules over C^* algebras, including the standard Hilbert module, the reader is referred to [10] or [14].)

In our earlier work [7], we construct a locally convex topology τ on $l^2(\mathcal{A})$ such that $T \in B^a(l^2(\mathcal{A}))$ is “compact” implies that its image of the unit ball is totally bounded with respect to τ . The converse is obtained in the special case $\mathcal{A} = B(H)$.

The aim of this note is to introduce a MNC on the standard Hilbert module $l^2(\mathcal{A})$ and to derive its properties. In Section 2, we introduce the MNC λ such that for any bounded set $E \subseteq l^2(\mathcal{A})$ the equality $\lambda(E) = 0$ holds if and only if E is precompact in the sense of [14, Proposition 2.6], which implies that the corresponding MNC of a given operator T is equal to 0 if and only if T is “compact”.

In Section 3 we list some known results on MNCs on uniform spaces and also derive some simple generalizations.

In Section 4, we discuss generalizations of Kuratowski, Hausdorff and Istrăţescu measures of noncompactness on $l^2(\mathcal{A})$ and obtain their relationship with MNC λ .

We use the following basic and also simple facts on Hilbert modules, that can be found throughout the literature. To make proofs more easy we list and prove them.

(F1) Let $z_1 \perp z_2$. Then $\|z_1 + z_2\| \geq \|z_1\|$.

Indeed, we have

$$\langle z_1 + z_2, z_1 + z_2 \rangle = \langle z_1, z_1 \rangle + \langle z_2, z_2 \rangle \geq \langle z_1, z_1 \rangle,$$

in the order defined by the positive cone in A . Therefore

$$\|z_1 + z_2\|^2 = \|\langle z_1 + z_2, z_1 + z_2 \rangle\| \geq \|\langle z_1, z_1 \rangle\| = \|z_1\|^2.$$

(F2) Let M be a projective finitely generated submodule of $l^2(\mathcal{A})$, and let $x \in l^2(\mathcal{A})$ be arbitrary. Then

$$d(x, M) = \|x - P_M x\|,$$

where P_M is orthogonal projection onto M with null-space M^\perp .

Indeed, by [14, Theorem 1.4.5] M is orthogonally complemented, $l^2(\mathcal{A}) = M \oplus M^\perp$, and there is $P_M : l^2(\mathcal{A}) \rightarrow l^2(\mathcal{A})$ such that $P_M^2 = P_M^* = P_M$, the range of P_M is M and the kernel of P_M is M^\perp . Let $y \in M$ be arbitrary. Then

$$\|x - y\| = \|(x - P_M x) + (P_M x - y)\| \geq \|x - P_M x\|,$$

by (F1), because $x - P_M x \in M^\perp$ and $P_M x - y \in M$.

2. Measure of noncompactness λ

Throughout this section, \mathcal{A} will always denote a unital C^* -algebra, and its unit will be denoted by 1. Also, $l^2(\mathcal{A})$ will denote the standard Hilbert C^* -module over \mathcal{A} defined in the introduction.

In [14, Proposition 2.6] \mathcal{A} -precompact sets were defined as those bounded sets E such that for all $\varepsilon > 0$ there is a free finitely generated module $M \cong \mathcal{A}^n$ such that

$$d(E, M) := \sup_{x \in E} d(x, M) = \sup_{x \in E} \inf_{y \in M} d(x, y) < \varepsilon.$$

We generalize this notion in the following way.

Definition 2.1. Let $E \subseteq l^2(\mathcal{A})$ be a bounded set. The measure of noncompactness of E , denoted by $\lambda(E)$ is the greatest lower bound of all $\eta > 0$ for which there is a free finitely generated module $M \leq l^2(\mathcal{A})$ such that

$$d(E, M) := \sup_{x \in E} \inf_{y \in M} d(x, y) < \eta.$$

Proposition 2.2. The measure of noncompactness $\lambda(E)$ can be computed as:

1. $\lambda(E) = \inf_{M \in \mathcal{F}} \sup_{x \in E} d(x, M)$, where \mathcal{F} is the set of all free finitely generated modules;
2. $\lambda(E) = \lim_{n \rightarrow +\infty} \sup_{x \in E} \|x - P_n x\| = \inf_{n \geq 1} \sup_{x \in E} \|x - P_n x\|$, where $P_n : l^2(\mathcal{A}) \rightarrow l^2(\mathcal{A})$ is given by $P_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$.

Proof. The equation (1) is obvious. The sequence $I - P_n$ is decreasing. Hence $\lim_{n \rightarrow +\infty} \sup_{x \in E} \|x - P_n x\| = \inf_{n \geq 1} \sup_{x \in E} \|x - P_n x\|$. Since $P_n l^2(\mathcal{A})$ is a free finitely generated module, we have immediately

$$\lambda(E) \leq \inf_{n \geq 1} \sup_{x \in E} \|x - P_n x\|.$$

To get the opposite inequality, let $\varepsilon > 0$. Then, there is a free finitely generated module M such that $d(x, M) < \lambda(E) + \varepsilon$. Denote the projection on M by Q . By [14, Proposition 2.2.1.], $\|Q - P_n Q\| \rightarrow 0$ as $n \rightarrow +\infty$. Since $P_n Qx \in P_n l^2(\mathcal{A})$, we have by (F2)

$$\begin{aligned} \|x - P_n x\| &= d(x, P_n l^2(\mathcal{A})) \leq \|x - P_n Qx\| \leq \\ &\leq \|x - Qx\| + \|Qx - P_n Qx\| \leq \lambda(E) + \varepsilon + K\|Q - P_n Q\|, \end{aligned}$$

where $\|x\| \leq K$ for all $x \in E$ (E is bounded). Thus

$$\sup_{x \in E} \|x - P_n x\| \leq \lambda(E) + \varepsilon + K\|Q - P_n Q\| \rightarrow \lambda(E) + \varepsilon, \quad \text{as } n \rightarrow +\infty,$$

which finishes the proof. \square

Proposition 2.3. *The measure of noncompactness λ has the following properties*

1. if $E \subseteq F$ then $\lambda(E) \leq \lambda(F)$;
2. $\lambda(E \cap F) \leq \min\{\lambda(E), \lambda(F)\}$;
3. $\lambda(E \cup F) \leq \max\{\lambda(E), \lambda(F)\}$;
4. $\lambda(E + F) \leq \lambda(E) + \lambda(F)$.
5. $\lambda(Ea) \leq \lambda(E)\|a\|$, where $a \in A$. If, in addition, a is invertible, then also $\|a^{-1}\|^{-1}\lambda(E) \leq \lambda(Ea)$. In particular, when a is unitary then $\lambda(Ea) = \lambda(E)$.
6. $\lambda(\text{co } E) = \lambda(E)$, where $\text{co } E = \{\sum_{i=1}^n t_i x_i \mid 0 \leq t_i \in \mathbf{R}, \sum_{i=1}^n t_i = 1, x_i \in E\}$ is the convex hull of E .

Proof. (1) It is obvious.

(2) This follows from (1).

(3) Let $d = \max\{\lambda(E), \lambda(F)\}$. Then for all $x \in E$, as well as for all $x \in F$, we have $\|x - P_n x\| \leq d + \varepsilon$, for all $\varepsilon > 0$ and n large enough. Hence the result.

(4) Let $z \in E + F$. Then $z = x + y$ for some $x \in E, y \in F$. We have

$$\|z - P_n z\| \leq \|x - P_n x\| + \|y - P_n y\| \leq \lambda(E) + \varepsilon + \lambda(F) + \varepsilon,$$

for n large enough.

(5) Any $z \in Ea$ is of the form $z = ya$ for some $y \in E$. Therefore $\|z - P_n z\| \leq \|y - P_n y\| \|a\|$ and the inequality follows by taking a limit. If a has the inverse a^{-1} , then $E = (Ea)a^{-1}$ and by previous $\lambda(E) \leq \lambda(Ea)\|a^{-1}\|$.

(6) Let $x \in \text{co } E$. Then $x = \sum_{i=1}^n t_i x_i$ for some $x_i \in E$, and positive t_i such that $\sum_{i=1}^n t_i = 1$. We have

$$\|x - P_n x\| = \left\| \sum_{i=1}^n t_i (x_i - P_n x_i) \right\| \leq \sum_{i=1}^n t_i \|x_i - P_n x_i\| \leq \sup_{x \in E} \|x - P_n x\|.$$

Thus, $\sup_{x \in \text{co } E} \|x - P_n x\| \leq \sup_{x \in E} \|x - P_n x\|$. The opposite inequality is obvious. The required follows from Proposition 2.2-(2). \square

Proposition 2.4. *Let B denote the unit ball in $l^2(\mathcal{A})$. Then $\lambda(B) = 1$.*

Proof. Any submodule contains the origin. Hence $\lambda(B) \leq 1$. Let $0 < \delta < 1$, and let M be some free finitely generated submodule of $l^2(\mathcal{A})$. Then, there is nontrivial $y \in M^\perp$ and $\delta\|y\|^{-1}y \in B \cap M^\perp$. We have

$$d(B, M) \geq d(\delta\|y\|^{-1}y, M).$$

However, $\delta\|y\|^{-1}y \perp M$ which implies that for all $x \in M$ we have

$$\|\delta\|y\|^{-1}y - x\|^2 = \delta^2 + \|x\|^2 \geq \delta^2.$$

Hence $d(B, M) \geq \delta$. Thus $\lambda(B) \geq \delta$. \square

Corollary 2.5. *If $E \subseteq F + \delta B$ then $\lambda(E) \leq \lambda(F) + \delta$.*

Proof. Immediately follows from Proposition 2.3-(4) and Proposition 2.4. \square

Proposition 2.6. *The measure of noncompactness λ has the following continuity properties:*

1. $|\lambda(E) - \lambda(F)| \leq d_H(E, F) = \max\{d(E, F), d(F, E)\}$ (d_H stands for the so called Hausdorff distance.)
2. $\lambda(E) = \lambda(\bar{E})$ (\bar{E} stands for the norm closure of E);
3. $\lambda(E) = 0$ iff E is \mathcal{A} -precompact;
4. $\lambda(E) \leq \sup_{x \in E} \|x\|$.

Proof. (1) Let $d = d_H(E, F)$. Then $E \subseteq F + dB$ and by Corollary 2.5

$$\lambda(E) \leq \lambda(F) + d, \quad \text{i.e.} \quad \lambda(E) - \lambda(F) \leq d.$$

Similarly, $F \subseteq E + dB$ implying $\lambda(F) - \lambda(E) \leq d$.

- (2) As it is easy to see $d_H(E, \bar{E}) = 0$ we can apply the previous item.
- (3) Follows directly from the definition.
- (4) Follows from $E \subseteq (\sup_{x \in E} \|x\|) \cdot B$. \square

Example 2.7. *In Proposition 2.3-(5) the strict inequalities might hold.*

Indeed, let the algebra \mathcal{A} contain a nontrivial projection, say p , and let $\mathcal{A}(1 - p)$ is isomorphic to \mathcal{A} . (For instance $\mathcal{A} = L^\infty(0, 1)$ and $p = \chi_{[0, 1/2]}$.) Let

$$E = \left\{ (a_1 p + b_1(1 - p), a_2 p + b_2(1 - p), \dots) \mid \sum_{n=1}^{\infty} a_n^* a_n \leq 1, \sum_{n=1}^{\infty} b_n^* b_n \leq 4 \right\}$$

and let $a = p$. Then $\|p\| = 1$ and $\lambda(E) \geq 2$, since E contains a copy of a ball of radius 2 in $l^2(\mathcal{A})$ (when $a_j = 0$). On the other hand $\lambda(Ep) \leq 1$. Indeed, Ep is contained in the unit ball of $l^2(\mathcal{A})$.

Finally, we want to define MNC of an operator $T \in B^a(l^2(\mathcal{A}))$. As it is expected, it will be the MNC of its image of the unit ball.

Definition 2.8. *Let $T \in B^a(l^2(\mathcal{A}))$ be an adjointable operator. We set*

$$\lambda_0(T) = \lambda(T(B_1)),$$

where B_1 is the unit ball in $l^2(\mathcal{A})$.

Proposition 2.9. *The function λ_0 has the following properties:*

(a) λ_0 is subadditive, i.e.

$$\lambda_0(T_1 + T_2) \leq \lambda_0(T_1) + \lambda_0(T_2);$$

(b) λ_0 is positively homogeneous, i.e.

$$\lambda_0(cT) = c\lambda_0(T),$$

for all $c > 0$ and all $T \in B^a(l^2(\mathcal{A}))$;

(c) $\lambda_0(T) \leq \|T\|$, for all $T \in B^a(l^2(\mathcal{A}))$;

Proof. Direct verification. \square

We will be able to say more on the MNC λ_0 in section 4.

3. Measure of noncompactness on uniform spaces - known results

Let us recall some basic definitions and facts concerning uniform spaces. For more details see [18] or [5].

Uniform spaces are those topological spaces in which one can deal with notions such as Cauchy sequence, Cauchy net or uniform continuity. Although it is usual to define them as spaces endowed with a family of sets in $X \times X$ given as some kind of neighborhoods of the diagonal, so called entourages, for our purpose it is more convenient to give an equivalent definition, via a family of semi-metrics.

Definition 3.1. A nonempty set endowed with a family of semi-metrics, functions $d_\alpha: X \times X \rightarrow [0, +\infty)$ satisfying (i) $d_\alpha(x, x) = 0$; (ii) $d_\alpha(x, y) = d_\alpha(y, x)$; (iii) $d_\alpha(x, z) \leq d_\alpha(x, y) + d_\alpha(y, z)$ is called a uniform space.

Remark 3.2. There is some ambiguity in literature; sometimes, functions d_α from the previous definition are called "pseudo-metrics", whereas the term "semi-metric" is reserved for a different notion.

All d_α are metrics except they do not distinguish points, i.e. there might be $d_\alpha(x, y) = 0$ for some $x \neq y$. However it is provided that for all $x \neq y$ there is an α such that $d_\alpha(x, y) > 0$. The family of sets $B_{d_\alpha}(x; \varepsilon) = \{y \in X | d_\alpha(x, y) < \varepsilon\}$ makes a basis for some topology. It is well known that a topological space X is a uniform space if and only if it is completely regular.

Any locally convex topological vector space is a uniform space. Indeed, there is a family of semi-norms generating its topology. This family can be obtained by Minkowski functionals of basic neighborhoods of zero. And an arbitrary semi-norm define a semi-metric in a natural way. Conversely, any family of semi-norms that distinguishes points leads to a locally convex Hausdorff topological vector space.

We point out two generalizations of the notion of MNC to the framework of uniform spaces, i.e, those topological spaces that arise from the family of semi-metrics.

Sadovskii [18] considered a uniform space X and a family P of semi-metrics that are uniformly continuous on $X \times X$. Starting with Kuratowski and Hausdorff MNC on a semi-metric space (which is defined exactly as on a metric space), Sadovskii defined corresponding MNCs $\alpha, \chi: \Upsilon \rightarrow \mathcal{G}$, where Υ denotes the family of all subsets that are bounded with respect to any semi-metric $p \in P$ on the given uniform space, and \mathcal{G} denotes the set of functions $g: P \rightarrow [0, +\infty)$ with uniformity generated by pointwise convergence and the natural partial ordering: $g_1 \leq g_2 \Leftrightarrow (\forall p \in P) g_1(p) \leq g_2(p)$. Their definitions are

$$[\alpha(E)](p) = \inf \left\{ d > 0 : E = \bigcup_{j=1}^m E_j, \text{ for some } E_j, \text{diam}(E_j) < d \right\},$$

$$[\chi(E)](p) = \inf \left\{ \varepsilon > 0 : E \subseteq \bigcup_{j=1}^m B_p(x_j; \varepsilon) \text{ for some } x_j \in X \right\}.$$

For such defined α and χ , in [18, §1.2.3. and §1.2.5.], the following is proved, provided that the family P generates the topology on X .

Theorem 3.3. The Kuratowski and Hausdorff measures of noncompactness ($\mu = \alpha$ or $\mu = \chi$) have the following properties:

- (a) μ is non-singular, that is, they are zero on any single-element set;
- (b) μ is continuous, that is, for all $E \in \Upsilon$, $p \in P$ and $\varepsilon > 0$ there is an entourage V in X such that for all E_1 that is V -close to E there holds $|\mu(E_1)(p) - \mu(E)(p)| < \varepsilon$;
- (c) μ is semi-additive, that is, for all E_1, E_2 we have

$$\mu(E_1 \cup E_2) = \max\{\mu(E_1), \mu(E_2)\};$$

(d) The function is algebraically semi-additive, that is,

$$\mu(E_1 + E_2)(p) \leq \mu(E_1)(p) + \mu(E_2)(p) \quad \text{for all } E_1, E_2 \in \Upsilon;$$

(e) μ is invariant under shifts, that is,

$$\mu(x + E) = \mu(E) \quad \text{for all } E \in \Upsilon, x \in X;$$

(f) μ is invariant under the transition to its closure and to the convex hull of the set, that is,

$$\mu(\text{co } E) = \mu(\bar{E}) = \mu(E) \quad \text{for all } E \in \Upsilon;$$

(g) μ is uniformly continuous, that is, for all $p \in P$ and $\varepsilon > 0$ there is an entourage V in X such that for all V -close E_1 and E_2 there holds $|\mu(E_1)(p) - \mu(E_2)(p)| < \varepsilon$;

(h) The functions α , χ and I are equivalent to each other, that is,

$$\chi(\Omega) \leq I(\Omega) \leq \alpha(\Omega) \leq 2\chi(\Omega) \quad \text{for all } \Omega \in \Upsilon.$$

Although it was not done in [18], one can also define the Istrăţescu MNC in a similar way, i.e.

$$[I(E)](p) = \inf\{\varepsilon > 0 : E \ni x_n, p(x_n, x_m) \geq \varepsilon \text{ for all } m \neq n\}.$$

By (1) one can easily derive

$$[\chi(E)](p) \leq [I(E)](p) \leq [\alpha(E)](p) \leq 2[\chi(E)](p) \quad \text{for all bounded } E. \quad (2)$$

Also, Theorem 3.3 hold for $\mu = I$. Part (a) is obvious, parts (d) and (e) follow from [3, Proposition 1], part (c) follows from [6], parts (b) and (g) from (2) and the corresponding properties of χ , and finally (f) can be derived from [1, Theorem 1.3.4].

Arandjelović [2] dealt with an arbitrary uniform space, and gave the following:

Definition 3.4. Let X be a uniform space, metric space or semi-metric space. Any function Φ defined on the partitive set of X , which satisfies the following:

(1) $\Phi(E) = +\infty$ if and only if E is unbounded;

(2) $\Phi(E) = \Phi(\bar{E})$;

(3) from $\Phi(E) = 0$ follows that E is totally bounded set;

(4) from $E \subseteq F$ it follows $\Phi(E) \leq \Phi(F)$;

(5) if X is complete, and if $\{E_n\}_{n \in \mathbf{N}}$ is a sequence of closed subsets of X such that $E_{n+1} \subseteq E_n$ for each $n \in \mathbf{N}$ and $\lim_{n \rightarrow \infty} \Phi(E_n) = 0$, then $K = \bigcap_{n \in \mathbf{N}} E_n$ is a nonempty compact set.

is called a measure of noncompactness on X .

Remark 3.5. Note that the only nontrivial requirement in (5) is that K is nonempty. Moreover, condition (5) can be replaced by a weaker one - $\Phi(A \cup \{x\}) = \Phi(A)$. It was shown in [15], see also [].

Theorem 3.6. [2, Theorem 3] Let X be a uniform space and let $\{d_i | i \in I\}$ be a family of semi-metrics which defines topology on X . Denote by μ_i arbitrary MNC on the semi-metric space (X, d_i) for each $i \in I$. Then the function $\mu^* : X \rightarrow [0, +\infty]$ defined by

$$\mu^*(E) = \sup_{i \in I} \mu_i(E)$$

for each $E \in X$, is a measure of noncompactness on X .

Uniform spaces make a proper subclass of all topological spaces, but still wide enough. For instance all topological vector spaces are uniform spaces. Hence, we can apply results of Arandjelović to topological vector spaces.

4. Measures of “noncompactness” over standard Hilbert W^* -modul $l^2(B(H))$

In this section we shall discuss standard Hilbert modules over a W^* -algebra \mathcal{A} , a narrower class than that considered in section 2. As it is well known, \mathcal{A} always has a unit.

In our earlier work [7], we construct a locally convex topology τ on $l^2(\mathcal{A})$ such that $T \in B^a(l^2(\mathcal{A}))$ is “compact” implies its image of the unit ball is totally bounded with respect to τ . This topology is defined via the family of semi-norms $p_{\varphi,y}$

$$p_{\varphi,y}(x) = \sqrt{\sum_{j=1}^{\infty} |\varphi(\eta_j^* \xi_j)|^2}, \quad x = (\xi_1, \xi_2, \dots) \in l^2(\mathcal{A}), \tag{3}$$

where $\varphi \in \mathcal{A}_*$ is a normal state and $y = (\eta_1, \eta_2, \dots)$ is a sequence of elements in \mathcal{A} such that

$$\sup_{j \geq 1} \varphi(\eta_j^* \eta_j) = 1. \tag{4}$$

Also, in special case, where $\mathcal{A} = B(H)$ is the full algebra of all bounded operators on a Hilbert space H , the converse is also proved, i.e. that any $T \in B^a(l^2(\mathcal{A}))$ whose image of the unit ball is totally bounded with respect to τ must be “compact”.

Construction described in the previous section endows the space $l^2(\mathcal{A})$ by the corresponding Kuratowski, Hausdorff and Istrăţescu measure of noncompactness, $\alpha, \chi, I : \Upsilon \rightarrow \mathcal{G}$,

$$[\alpha(E)](p_{\varphi,y}) = \inf \left\{ \varepsilon > 0 : E = \bigcup_{i=1}^n S_i, p_{\varphi,y}(x' - x'') < \varepsilon, \forall x', x'' \in S_i \right\},$$

$$[\chi(E)](p_{\varphi,y}) = \inf \left\{ \varepsilon > 0 : E \subset \bigcup_{i=1}^n B_{p_{\varphi,y}}(x_i, \varepsilon), x_i \in l^2(\mathcal{A}) \right\},$$

where $B_{p_{\varphi,y}}(x_i, \varepsilon) = \{y | p_{\varphi,y}(y - x_i) < \varepsilon\}$, and

$$[I(E)](p_{\varphi,y}) = \sup \{ \varepsilon > 0 : \text{there is infinite } S \subset E \text{ such that } p_{\varphi,y}(x' - x'') \geq \varepsilon, \forall x' \neq x'' \in S \}.$$

The function α, χ and I can be regarded as functions depending on two variables, on the bounded set Ω and on the semi-norm $p_{\varphi,y}$. If we want to obtain a MNC that not depends on a particular semi-norm, we can use the functions $\chi^*, \alpha^*, I^* : \Upsilon \rightarrow [0, +\infty)$ defined by

$$\begin{aligned} \chi^*(E) &= \sup_{p_{\varphi,y} \in P} [\chi(E)](p_{\varphi,y}), \\ \alpha^*(E) &= \sup_{p_{\varphi,y} \in P} [\alpha(E)](p_{\varphi,y}), \\ I^*(E) &= \sup_{p_{\varphi,y} \in P} [I(E)](p_{\varphi,y}) \end{aligned} \tag{5}$$

for each $E \in \Upsilon$, where P is the set of all semi-norms of the form (3). Since α, χ and I annihilates all singletons (Theorem 3.3-(a)), they satisfy condition in Remark 3.5, and hence they are measures of noncompactness in the sense of Definition 3.4. By Theorem 3.6 α^*, χ^* and I^* are measures of noncompactness on $(l^2(\mathcal{A}), \tau)$ in the sense of Definition 3.4, as well. Also, properties (c), (d), (e) and (f) in Theorem 3.3 are easily transferred to α^*, χ^* and I^* , by taking a supremum. Finally, by (2), we have

$$\chi^*(E) \leq I^*(E) \leq \alpha^*(E) \leq 2\chi^*(E). \tag{6}$$

Remark 4.1. Note that $l^2(\mathcal{A})$ is rarely complete, due to [7, Proposition 3.3]. Therefore, the condition (5) in Definition 3.4 is vague, unless $l^2(\mathcal{A})' \cong l^2(\mathcal{A})$ which is equivalent to the condition that \mathcal{A} is finite dimensional.

We want to place the MNC λ , discussed in section 2, somewhere in the preceding chain of inequalities.

Proposition 4.2. For any bounded set $E \subseteq \ell^2(\mathcal{A})$, we have

$$\chi^*(E) \leq \lambda(E).$$

Proof. Let E be a bounded set, and let P_n denote the projection to the first n coordinates in $\ell^2(\mathcal{A})$, i.e. $P_n(\xi_1, \xi_2, \dots) = (\xi_1, \xi_2, \dots, \xi_n, 0, \dots)$. Since $E \subseteq P_n E + (I - P_n)E$, and since χ^* is subadditive, we have

$$\chi^*(E) \leq \chi^*(P_n E) + \chi^*((I - P_n)E).$$

However, by ([7, Proposition 3.4.]), the set $P_n E$ is totally bounded, and we have $\chi^*(P_n E) = 0$. Hence

$$\chi^*(E) \leq \chi^*((I - P_n)E) \leq \sup_{x \in E} \|(I - P_n)x\|,$$

for all $n \in \mathbb{N}$. Therefore, by Proposition 2.2-(2), $\chi^*(E) \leq \lambda(E)$. \square

The preceding Proposition establishes a lower bound of λ . Before we obtained an upper bound for λ , in a special case, we introduce balanced sets.

Definition 4.3. Let $E \subset \ell^2(\mathcal{A})$ be a bounded set.

- (a) We say that E is \mathcal{A} -balanced if $x \cdot u \in E$ whenever $x \in E$ and $u \in \mathcal{A}$ is unitary. (This definition is motivated by the notion of balanced sets on topological vector spaces over the field \mathbf{C} , where u unitary is reduced to $|u| = 1$.)
- (b) By \mathcal{A} -balanced hull of E we assume the minimal balanced set containing E , that is $\bigcup Eu$, where the union is taken over all unitaries $u \in \mathcal{A}$.

In Proposition 2.3-(5) we proved $\lambda(Eu) = \lambda(E)$. We give two extensions of this statement, the first of them concerning the balanced hull.

Proposition 4.4. Let $E \subseteq \ell^2(\mathcal{A})$ be a bounded set and let F be its \mathcal{A} -balanced hull. Then $\lambda(F) = \lambda(E)$.

Proof. For all $x \in E$ and all unitaries u , we have $\|u\| = 1$, and hence $\|xu - P_n xu\| \leq \|x - P_n x\|$. Therefore, by Proposition 2.2-(2), we have

$$\lambda(F) = \lim_{n \rightarrow +\infty} \sup_{x \in E, u \text{-unitary}} \|xu - P_n xu\| \leq \lim_{n \rightarrow +\infty} \sup_{x \in E} \|x - P_n x\| = \lambda(E).$$

The opposite inequality $\lambda(E) \leq \lambda(F)$ follows from $E \subseteq F$. \square

Proposition 4.5. Let $E \subseteq \ell^2(\mathcal{A})$ be a bounded set, let $u \in \mathcal{A}$ be a unitary and let μ stands for any of Kuratowski, Hausdorff or Istrătescu MNC. Then

$$\mu^*(Eu) = \mu^*(E).$$

Proof. First, observe that given a normal state φ on \mathcal{A} and a unitary $u \in \mathcal{A}$, the mapping $\varphi^u, \varphi^u(x) = \varphi(u^* x u)$ is also a normal state. Obviously, $\varphi^u(1) = 1$ and $\varphi^u(x) \geq 0$ whenever $x \geq 0$. Hence, it suffices to prove that φ^u is normal. Let x_α be an increasing net with the least upper bound x . Then $u x_\alpha u^*$ is also an increasing net, bounded by $u x u^*$. Thus, its least upper bound is less than $u x u^*$. Moreover, it is equal to $u x u^*$ by interchanging roles. Therefore

$$\sup \varphi^u(x_\alpha) = \sup \varphi(u^* x_\alpha u) = \varphi(u^* x u) = \varphi^u(x).$$

Let $\varepsilon > 0$ be arbitrary. By (5) there is a semi-norm $p_{\varphi,y} \in P$ such that $[\mu(Eu)](p_{\varphi,y}) > \mu^*(Eu) - \varepsilon$. Next, we have $p_{\varphi,y}(xu) = p_{\varphi^u,yu^*}$, where $yu^* = (\eta_1 u^*, \eta_2 u^*, \dots)$. Indeed

$$p_{\varphi,y}(xu)^2 = \sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j u)|^2 = \sum_{j=1}^{+\infty} |\varphi(u^* (\eta_j u^*)^* \xi_j u)|^2 = p_{\varphi^u,yu^*}(x)^2.$$

(Note that the pair (φ^u, yu^*) trivially satisfies (4).) Therefore, $[\mu(E)](p_{\varphi^u,yu^*}) = [\mu(Eu)](p_{\varphi,y}) > \mu^*(Eu) - \varepsilon$ and hence $\mu^*(E) \geq \mu^*(Eu)$. The opposite inequality follows by $E = (Eu)u^{-1}$. \square

Remark 4.6. We don't know whether Kuratowski, Hausdorff and Istrăţescu MNCs are stable with respect to balanced hull.

Now, we are going to derive an upper bound for the MNC λ . Namely, in a special case, where $\mathcal{A} = B(H)$, we can obtain the exact position of λ in the inequality chain (6) for balanced sets.

Theorem 4.7. Let $\mathcal{A} = B(H)$. Let $E \subseteq l^2(\mathcal{A})$ be an \mathcal{A} -balanced set. Then

$$\lambda(E) \leq \sqrt{\|E\|I^*(E)},$$

where $\|E\| = \sup_{x \in E} \|x\|$.

Proof. Let P_k denote the projection to the first k coordinates, i.e. $P_k(\xi_1, \xi_2, \dots) = (\xi_1, \dots, \xi_k, 0, 0, \dots)$. It is well known that all P_k are "compact". If $\inf_{k \geq 1} \|(I - P_k)E\| = 0$, then from Proposition 4.2 and (6), it follows $\lambda^*(E) = \lambda^*(E) = I^*(E) = 0$. So, let

$$\delta = \inf_{k \geq 1} \|(I - P_k)E\| > 0.$$

Then immediately, $\delta \leq \|E\|$. Choose $\varepsilon > 0$ such that $\varepsilon < \delta^2$. Define the sequence of projections $Q_n \in \{P_1, P_2, \dots\}$ and the sequences of vectors x_n and $z_n \in l^2(\mathcal{A})$ in the following way. Let $Q_0 = 0$. If Q_{n-1} is already defined, there is $x_n \in E$ such that $\|x_n\| \geq \|(I - Q_{n-1})x_n\| > C_1$, where $C_1 = \frac{1}{2}(\delta + \sqrt{\delta^2 - \varepsilon/2})$. Since $\lim_{k \rightarrow +\infty} \|(I - P_k)(I - Q_{n-1})x_n\| = 0$, there is a positive integer k_n such that $\|(I - P_{k_n})(I - Q_{n-1})x_n\| < C_2$, where $C_2 = \min\left\{\frac{\varepsilon}{2\|E\|}, \frac{1}{2}(\delta - \sqrt{\delta^2 - \varepsilon/2})\right\}$. Define $Q_n = P_{k_n}$ and

$$z_n = Q_n(I - Q_{n-1})x_n. \tag{7}$$

The sequences x_n and z_n have the following properties:
 Firstly, by definition, there hold the inequalities

$$\|(I - Q_n)(I - Q_{n-1})x_n\| < C_2, \tag{8}$$

$$\|z_n\| \leq \|x_n\| \leq \|E\|, \tag{9}$$

$$\|z_n\| \geq \|(I - Q_{n-1})x_n\| - \|(I - Q_n)(I - Q_{n-1})x_n\| > C_1 - C_2. \tag{10}$$

Secondly,

$$\langle z_n, x_n \rangle = \langle z_n, z_n \rangle. \tag{11}$$

Indeed, since $z_n = Q_n(I - Q_{n-1})x_n$, we have

$$\begin{aligned} \langle z_n, x_n \rangle &= \langle Q_n(I - Q_{n-1})x_n, x_n \rangle = \\ &= \langle Q_n(I - Q_{n-1})x_n, (I - Q_{n-1})Q_n x_n \rangle = \langle z_n, z_n \rangle. \end{aligned}$$

Thirdly, for $m > n$ we have

$$\|\langle z_m, x_n \rangle\| < C_2\|E\|. \tag{12}$$

Indeed, for such m and n we have $Q_{n-1} \leq Q_n \leq Q_{m-1}$, i.e. $I - Q_{m-1} \leq I - Q_n \leq I - Q_{n-1}$, implying $I - Q_{m-1} = (I - Q_{m-1})(I - Q_n)(I - Q_{n-1})$, and thus

$$\begin{aligned} \langle z_m, x_n \rangle &= \langle (I - Q_{m-1})z_m, x_n \rangle = \\ &= \langle z_m, (I - Q_{m-1})(I - Q_n)(I - Q_{n-1})x_n \rangle = \\ &= \langle z_m, (I - Q_n)(I - Q_{n-1})x_n \rangle. \end{aligned}$$

Therefore, by (8) and (10)

$$\| \langle z_m, x_n \rangle \| \leq \| z_m \| \cdot \| (I - Q_n)(I - Q_{n-1})x_n \| \leq C_2 \| E \|.$$

Let us construct a semi-norm p , continuous in τ , and a totally discrete sequence from E . Since by (10) $\| z_n \|^2 = \| \langle z_n, z_n \rangle \| > (C_1 - C_2)^2$, we can choose a normal state φ and $v_j, v_j \in \mathcal{A}$ according to [7, Lemma 4.6.]), such that

$$\varphi(v_n^* \langle z_n, z_n \rangle v_n) > (C_1 - C_2)^2. \tag{13}$$

Consider the semi-norm p given by

$$p(x) = \sqrt{\sum_{j=1}^{+\infty} |\varphi(\langle z_j v_j, x \rangle)|^2}.$$

By (7) there is a sequence $\zeta_j \in \mathcal{A}$ such that

$$z_n = (0, \dots, 0, \zeta_{k_{n-1}+1}, \dots, \zeta_{k_n}, 0, \dots).$$

Define $\omega_j = \zeta_j v_n / \varphi(v_n^* \zeta_j^* \zeta_j v_n)^{1/2}$, for $k_{n-1} + 1 \leq j \leq k_n$. Obviously $\varphi(\omega_j^* \omega_j) = 1$. Also, for $x = (\xi_1, \xi_2, \dots)$ we have

$$\begin{aligned} |\varphi(\langle z_n v_n, x \rangle)|^2 &= \left| \sum_{j=k_{n-1}+1}^{k_n} \varphi(v_n^* \zeta_j^* \zeta_j v_n)^{1/2} \varphi(\omega_j^* \xi_j) \right|^2 \leq \\ &\leq \sum_{j=k_{n-1}+1}^{k_n} \varphi(v_n^* \zeta_j^* \zeta_j v_n) \sum_{j=k_{n-1}+1}^{k_n} |\varphi(\omega_j^* \xi_j)|^2 = \\ &= \varphi(v_n^* \langle z_n, z_n \rangle v_n) \sum_{j=k_{n-1}+1}^{k_n} |\varphi(\omega_j^* \xi_j)|^2. \end{aligned}$$

Including (9) we obtain $\varphi(v_n^* \langle z_n, z_n \rangle v_n) \leq \| v_n^* \langle z_n, z_n \rangle v_n \| = \| z_n \|^2 \leq \| E \|^2$ and hence

$$p(x)^2 = \sum_{n=1}^{+\infty} |\varphi(\langle z_n v_n, x \rangle)|^2 \leq \| E \|^2 \sum_{j=1}^{+\infty} |\varphi(\omega_j^* \xi_j)|^2 = \| E \|^2 p_{\varphi, \omega_1, \dots, \omega_n, \dots}(x)^2.$$

Thus, we conclude that p is well defined and also that it is continuous with respect to τ .

Also, E is \mathcal{A} -balanced, so $x_n v_n \in E$. Finally we shall prove that $x_n v_n$ is a totally discrete sequence. Indeed, for $m > n$ we have

$$\begin{aligned} p(x_m v_m - x_n v_n) &\geq |\varphi(\langle z_m v_m, x_m v_m - x_n v_n \rangle)| \geq \\ &\geq |\varphi(v_m^* \langle z_m, x_m \rangle v_m)| - |\varphi(v_m^* \langle z_m, x_n \rangle v_n)|. \end{aligned}$$

However, by (11) and (13),

$$|\varphi(v_m^* \langle z_m, z_m \rangle v_m)| > (C_1 - C_2)^2$$

and, by (12)

$$|\varphi(v_m^* \langle z_m, x_n \rangle v_n)| \leq \| \langle z_m, x_n \rangle \| < C_2 \| E \|.$$

Therefore

$$p(x_m v_m - x_n v_n) > (C_1 - C_2)^2 - C_2 \| E \| \geq \delta^2 - \varepsilon$$

and

$$p_{\varphi, \omega_1, \dots, \omega_n, \dots}(x_m v_m - x_n v_n) > \frac{\delta^2 - \varepsilon}{\| E \|}.$$

For $\varepsilon \in (0, \delta^2)$, we have $I^*(E) \geq \frac{\delta^2 - \varepsilon}{\| E \|}$ and hence $I^*(E) \geq \frac{\delta^2}{\| E \|} = \frac{\lambda(E)^2}{\| E \|}$. Thus, $\lambda(E) \leq \sqrt{\| E \| I^*(E)}$. \square

Remark 4.8. The preceding proof is adapted proof of our earlier result [7, Theorem 4.10].

Corollary 4.9. On the standard Hilbert module $l^2(B(H))$ over the algebra $B(H)$ of all bounded operators on a Hilbert space H , there holds

$$\chi^*(E) \leq \lambda(E) \leq \sqrt{\|E\|I^*(E)} \leq \sqrt{\|E\|\alpha^*(E)} \leq \sqrt{2\|E\|\chi^*(E)},$$

for any balanced set, i.e. a set for which $x \in E$, u -unitary implies $xu \in E$.

Finally, we discuss some relationship between the MNC of an arbitrary adjointable operator λ_0 introduced in Definition 2.8, and corresponding MNCs derived from α^* , χ^* and I^* .

Definition 4.10. Let \mathcal{A} is an arbitrary W^* -algebra and let $T \in B^a(l^2(\mathcal{A}))$ be an adjointable operator. The functions α_0^* , χ_0^* , $I_0^* : B^a(l^2(\mathcal{A})) \rightarrow [0, +\infty)$ defined by

$$\alpha_0^*(T) = \alpha^*(T(B_1)), \quad \chi_0^*(T) = \chi^*(T(B_1)), \quad I_0^*(T) = I^*(T(B_1))$$

are called, respectively, Kuratowski, Hausdorff and Istrăţescu measure of noncompactness of the operator T .

Proposition 4.11. Let \mathcal{A} be an arbitrary W^* -algebra, let $T, S \in B^a(l^2(\mathcal{A}))$ and let μ stands for any of MNCs α, χ, I . Then

(a) All α_0^* , χ_0^* and I_0^* are subadditive and positively homogeneous, i.e. there holds

$$\mu_0^*(T + S) \leq \mu_0^*(T) + \mu_0^*(S), \quad \mu_0^*(cT) = c\mu^*(T), \quad \text{for all } c > 0.$$

(b) The functions α_0^* , χ_0^* and I_0^* are equivalent to each other, that is,

$$\chi_0^*(T) \leq I_0^*(T) \leq \alpha_0^*(T) \leq 2\chi_0^*(T).$$

Also, there holds $\chi_0^*(T) \leq \lambda_0(T)$.

(c) $\chi_0^*(T)$, $\lambda_0(T) \leq \|T\|$ and $\alpha_0^*(T)$, $I_0^*(T) \leq 2\|T\|$.

(d) If T is "compact", i.e. T belongs to the closed linear space generated by $x \mapsto z \langle y, x \rangle$, then $\lambda_0(T) = \chi_0(T) = \alpha_0(T) = I_0(T) = 0$. In general, the converse might not hold.

(e) $\mu_0^*(T + K) = \mu_0^*(T)$, as well as $\lambda_0(T + K) = \lambda_0(T)$ for all "compact" operators K .

Proof. Part (a) follows easily from Theorem 3.3-(d), whereas part (b) follows from (6) and Proposition 4.2.

Since $T(B_1) \subseteq B(0; \|T\|) = \|T\|B_1$, it follows $\lambda_0(T) \leq \|T\|$ according to Proposition 2.4. Other inequalities in part (c) follows from part (a).

If T is "compact", then T is norm limit of finite rank operators. Hence $T(B_1)$ is \mathcal{A} -precompact. Therefore $\lambda_0(T) = 0$. This, together with (b) proves (d). The converse does not always hold due to [7, Example 5.1.].

Finally, (e) follows from (d). \square

In the case $\mathcal{A} = B(H)$, we can obtain more.

Proposition 4.12. Let $\mathcal{A} = B(H)$ and let $T \in B^a(l^2(B(H)))$. Then

(a) There holds

$$\chi_0^*(T) \leq \lambda_0(T) \leq \sqrt{\|T\|I_0^*(T)} \leq \sqrt{\|T\|\alpha_0^*(T)} \leq \sqrt{2\|T\|\chi_0^*(T)}.$$

(b) $\mu_0^*(T) = 0$, $\mu \in \{\alpha, \chi, I\}$ iff $\lambda_0(T) = 0$ iff T is a "compact" operator.

Proof. Part (a) follows from Proposition 4.11-(b) and Theorem 4.7, since $T(B_1)$ is a balanced set ($y = Tx \in T(B_1)$ implies $yu = T(xu) \in T(B_1)$).

Part (b) follows from part (a) and Proposition 4.11-(d). Indeed, if any of four MNCs annihilate T , then, by part (a), $\alpha_0^*(T) = 0$, and hence, $T(B_1)$ is totally bounded in the topology τ . By [7, Theorem 4.10.], T is "compact". \square

5. Three questions

Question 5.1. *It is easy to obtain the following: Let $E \subseteq l^2(\mathcal{A})$ be bounded. For any $\varepsilon > 0$ there is an \mathcal{A} -precompact set C_ε such that $E \subseteq C_\varepsilon + (\lambda(E) + \varepsilon)B$. (For instance $C_\varepsilon = E \cap M$ for a suitable free finitely generated M .)*

Is it possible to get something stronger: There is an \mathcal{A} -precompact set C such that $E \subseteq C + \lambda(E) \cdot B$?

Question 5.2. *Among all properties of MNCs on a Banach space, it turns out that the most important is $\mu(\text{co } E) = \mu(E)$. This was proved in this note for λ if we $\text{co } E$ regard as a real field convex hull, i.e. $\text{co } E = \{\sum_1^n c_j x_j \mid \sum c_j = 1, c_j \in \mathbf{R}^+, x_j \in E\}$. However, there is a notion of \mathcal{A} -convex hull (see for instance [11])*

$$\text{co}_{\mathcal{A}} E = \left\{ \sum_{j=1}^n a_j^* x_j a_j \mid x_j \in E, a_j \in \mathcal{A}, \sum a_j^* a_j = 1 \right\}.$$

Is it possible to obtain $\lambda(\text{co}_{\mathcal{A}} E) = \lambda(E)$?

Question 5.3. *As it was mentioned in Remark 4.6, we ask for the following: Is it true $\mu^*(E) = \mu^*(F)$, where F denotes the balanced hull of E , i.e. $F = \bigcup_u Eu$ (the union runs through all unitaries u), and μ denotes any of Kuratowski, Hausdorff or Istrăţescu MNC?*

References

- [1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii. *Measures of Noncompactness and Condensing Operators*. Birkhäuser Verlag, Basel, Boston, Berlin, 1992. Translated from Russian.
- [2] Ivan D. Arandjelović. Measure of noncompactness on uniform spaces. *Math. Morav.*, 2:1–8, 1998.
- [3] Ivan D. Arandjelović. Some properties of Istratescu's measure of noncompactness. *Filomat*, 13:99–104, 1999.
- [4] J. M. Ayerbe Toledano, T. Dominguez Benavides, and G. Lopez Acedo. *Measures of Noncompactness in Metric Fixed Point Theory*. Springer Basel AG, 1997.
- [5] N. Bourbaki. *General Topology, chapters 1–4*. Springer, 1987.
- [6] Vasile Ion Istrăţescu. On a measure of noncompactness. *Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S)*, 16:195–197, 1972.
- [7] Dragoljub J. Kečkić and Zlatko Lazović. Compact and "compact" operators on the standard Hilbert module over a W^* algebra. *Ann. Funct. Anal.*, 9(2):258–270, 2018.
- [8] C. Kuratowski. Sur les espaces complets. *Fund. Math.*, 15:301–309, 1930.
- [9] C. Kuratowski. *Topologie I*. Monografie Matematyczne, Warsaw, 4th edition, 1958.
- [10] K. L. Lance. *Hilbert C^* -Modules: A toolkit for operator algebraists*. Cambridge University Press, 1995.
- [11] Bojan Magajna. C^* -convex sets and completely positive maps. *Integral Eq. Op. Th.*, 85:37–62, 2016.
- [12] V. M. Manuilov. Diagonalization of compact operators in Hilbert modules over finite W^* -algebras. *Ann. Global Anal. Geom.*, 13(3):207–226, 1995.
- [13] V. M. Manuilov. Diagonalization of compact operators on Hilbert modules over C^* -algebras of real rank zero. *Math. Notes*, 62(6):726–730, 1997.
- [14] V. M. Manuilov and E. V. Troitsky. *Hilbert C^* -modules*. Translations of mathematical monographs Vol. 226. AMS, Providence, Rhode Island, 2005.
- [15] Marina M. Milovanović-Arandjelović. Measures of noncompactness on uniform spaces—the axiomatic approach. *Filomat*, 15:221–225, 2001.
- [16] Mohammad Mursaleen, Syed M. H. Rizvi, and Bessem Samet. Measures of noncompactness and their applications. In J. Banas, M. Jleli, M. Mursaleen, B. Samet, and Vetro C., editors, *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, chapter 2, pages XIII, 487. Springer, 2017.
- [17] Vladimir Rakočević. Measures of noncompactness and some applications. *Filomat*, 12(2):87–120, 1998.
- [18] A. I. Sadovskii. Limit-compact and condensing operators. *Russian Math. Surveys*, 27(1):85–155, 1972.