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Well-Posedness Results for a Sixth-Order Logarithmic Boussinesq Equation

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Abstract. The main goal of this paper is to study for a sixth-order logarithmic Boussinesq equation. We obtain several results: Firstly, by using Feado-Galerkin method and a logaritmic Sobolev inequality, we proved global existence of solutions. Later, we proved blow up property in infinity time of solutions. Finally, we showed the decay estimates result of the solutions.

1. Introduction

In this work, we consider the following sixth-order logarithmic Boussinesq equation

$$\begin{cases} u_{tt} - u_{xx} - u_{xxtt} + u_{xxxxtt} + u_{xxxx} + u_{xxxxxx} + \left(u_x \log |u_x|^k\right)_x = 0, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = u(l,t) = 0, u_{xx}(x,t) = u_{xx}(l,t) = 0, u_{xxx}(x,t) = u_{xxx}(l,t) = 0\ , & t \geq 0 \end{cases} \tag{1}$$

where $\Omega = (0,l)$, $k \geq 1$, $u(x,t)$ denotes the unknown function.

In 1872, Boussinesq [1] derived the following classical Boussinesq equations (BE)

$$u_{tt} - u_{xx} + \mu u_{xxxx} = \left(u^2\right)_{xx},$$

$$u_{tt} - u_{xx} - u_{xxtt} = \left(u^2\right)_{xx}.$$

The Boussinesq equation to describe the propagation of small amplitude long waves on the surface of shallow water. In [3] Daripa derived the higher-order Boussinesq equation

$$u_{tt} - u_{xx} - \alpha(u^2)_{xx} \mp \alpha u_{xxxx} - \varepsilon^2 \alpha u_{xxxxxx} = 0 \tag{2}$$

for two-way propagation of shallow water waves. For contributions related to (2), we refer to [4, 5, 10–14].

Wazwaz [15] studied the logarithmic Boussinesq equation (log-BE) the following form

$$u_{tt} - u_{xx} + u_{xxxx} + \left(u \log |u|^k\right)_{xx} = 0.$$

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Zhang et al. [16] looked into the following equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} + \left(u_x \log |u_x|^k\right)_x = 0,$$

and proved global existence, growth and decay results. Also, some authors interested logarithmic Boussinesq equation (see [8, 9]).

Motivated by the above studies, in this paper we consider global existence, blow up property in infinity time and decay estimates of solutions for (1).

In our paper we orginazed as follows: In Section 2 we give some notations and lemmas which are essential for our proofs. In Section 3, we prove global existence of problems by using Faedo-Galerkin methods. In Section 4, we study growth of solutions. In Section 5, we add linear damping terms and we consider the decay estimates of the energy.

2. Preliminaries

In this section, we will give some notations and lemmas needed for the proof of our results.
We denote

$$\|.\|_p = \|.\|_{L^p(\Omega)}, \ \|.\| = \|.\|_{L^2(\Omega)}$$

and

$$(u, v) = \int_\Omega u(x) v(x) dx.$$

Now, to obtain the energy equation of the problem (1), we multiply the equation by u_t and integrate it in the Ω region

$$\begin{aligned}
& \int_\Omega u_{tt}u_t dx - \int_\Omega u_{xx}u_t dx + \int_\Omega u_{xxxx}u_t dx - \int_\Omega u_{xxtt}u_t dx \\
& + \int_\Omega u_{xxxxtt}u_t dx + \int_\Omega u_{xxxxxx}u_t dx + \int_\Omega \left(\log |u|^k u_x\right)_x u_t dx \\
= \ & 0,
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 - \int_\Omega u_x^2 \log |u|^k dx + \frac{k}{2}\|u_x\|^2\right) = 0$$

so, the energy functional associated with problem (1) is

$$\begin{aligned}
E(t) = \ & \frac{1}{2}\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) \\
& -\frac{1}{2}\int_\Omega u_x^2 \log |u|^k dx + \frac{k}{4}\|u_x\|^2,
\end{aligned} \tag{3}$$

and differentiation of (3), using (1), leads to

$$E(t) \leq E(0), \text{ for all } t \in [0, T). \tag{4}$$

Lemma 2.1. *[7] (Logarithmic Sobolev inequality). Let u be any function $u \in H_0^1(\Omega)$ and $a > 0$ be any number. Then,*

$$2\int_\Omega |u|^2 \log \frac{|u|}{\|u\|} dx \leq \frac{a^2}{\pi}\int_\Omega |u_x|^2 dx - (1 + \log a)\|u\|^2. \tag{5}$$

Lemma 2.2. *[2] (Logarithmic Gronwall Inequality). Assume that* $\phi(t)$ *is nonnegative function,* $\phi(t) \in L^{\infty}(0,T)$, $\phi(0) \geq 0$, *and it satisfies*

$$\phi(t) \leq \phi(0) + \alpha \int_0^t \phi(s) \log\left[\alpha + \phi(s)\right] ds, \ t \in [0,T],$$

where $\alpha > 1$ *is a positive constant. Then we have*

$$\phi(t) \leq \left(\alpha + \phi(0)\right) e^{\alpha t}, \ t \in [0,T].$$

Now we define two functionals $J(u)$ and $I(u)$;

$$J(u) = \frac{1}{2}\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) - \frac{1}{2}\int_\Omega u_x^2 \log|u_x|^k dx + \frac{k}{4}\|u_x\|^2, \tag{6}$$

$$I(u) = \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 - \int_\Omega u_x^2 \log|u_x|^k dx. \tag{7}$$

From the above definitions, it is clear that

$$J(u) = \frac{1}{2}I(u) + \frac{k}{4}\|u_x\|^2, \tag{8}$$

$$E(t) = \frac{1}{2}\left(\|u_{xt}\|^2 + \|u_{xxt}\|^2 + \|u_{xxxt}\|^2\right) + J(u). \tag{9}$$

According the Logarithmic Sobolev inequality , $I(u)$ and $J(u)$ are well defined on H_0^3.

Lemma 2.3. *For any* $u \in H_0^3(\Omega) / \{0\}$, *then*

i) $I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u)$ *and* $\lim_{\lambda \to 0} J(\lambda u) = 0$, $\lim_{\lambda \to \infty} J(\lambda u) = -\infty$.

ii) There exists a unique $\lambda^* = \lambda^*(u)$ *such that* $\frac{d}{d\lambda} J(\lambda u)\mid_{\lambda=\lambda^*} = 0$ *and* $J(\lambda u)$ *is increasing on* $0 < \lambda \leq \lambda^*$, *decreasing on* $\lambda^* \leq \lambda < \infty$ *and takes maksimum at* $\lambda^* = \lambda$. *On the other hands, there exists a unique* $\lambda^* \in (0,\infty)$ *such that*

$$I(\lambda u) = \begin{cases} > 0, & 0 < \lambda \leq \lambda^* \\ = 0, & \lambda^* = \lambda \\ < 0, & \lambda > \lambda^* \end{cases} \tag{10}$$

where

$$\lambda^* = \exp\left(\frac{\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 - \int_\Omega u_x^2 \log|u_x|^k dx}{k\|u_x\|^2}\right). \tag{11}$$

Proof. By the defination of $J(u)$, we obtain

$$\begin{aligned} J(\lambda u) &= \frac{1}{2}\left(\|\lambda u_x\|^2 + \|\lambda u_{xx}\|^2 + \|\lambda u_{xxx}\|^2 - \int_\Omega (\lambda u_x)^2 \log|\lambda u_x|^k dx + \frac{k}{2}\|\lambda u_x\|^2\right), \\ &= \frac{\lambda^2}{2}\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) - \frac{k\lambda^2}{2}\left(\int_\Omega u_x^2 \log|u_x| dx + \int_\Omega \log \lambda u_x^2 dx\right) + \frac{k\lambda^2}{4}\|u_x\|^2, \\ &= \frac{\lambda^2}{2}\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) + \frac{k\lambda^2}{4}\|u_x\|^2 - \frac{k\lambda^2}{2}\int_\Omega u_x^2 \log|u_x| dx - \frac{k\lambda^2}{2}\log\lambda\|u_x\|^2. \end{aligned}$$

Since $\|u\| \neq 0$, $\lim\limits_{\lambda\to 0} g(\lambda) = 0$, $\lim\limits_{\lambda\to\infty} g(\lambda) = -\infty$. Now, differentiating $g(\lambda)$ with respect to λ, we have

$$\begin{aligned}
\frac{dJ(\lambda u)}{d\lambda} &= \lambda\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) + \frac{k\lambda}{2}\|u_x\|^2 \\
&\quad -\frac{k\lambda}{2}\int\limits_\Omega u_x^2 \log|u_x|\,dx - k\lambda\log\lambda\|u_x\|^2 - \frac{k\lambda}{2}\|u_x\|^2 \\
&= \lambda\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) - \frac{\lambda}{2}\int\limits_\Omega u_x^2\log|u_x|^k\,dx - k\lambda\log\lambda\|u_x\|^2
\end{aligned}$$

and

$$I(\lambda u) = \lambda^2\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) - \frac{\lambda^2}{2}\int\limits_\Omega u_x^2\log|u_x|^k\,dx - k\lambda^2\log\lambda\|u_x\|^2$$

We can see clearly that

$$\lambda\frac{dJ(\lambda u)}{d\lambda} = I(\lambda u).$$

We can derive $I(\lambda u) = 0$, when

$$\lambda^* = \exp\left(\frac{\|u_{xxx}\|^2 + \|u_{xx}\|^2 + \|u_x\|^2 - \int\limits_\Omega u_x^2\log|u_x|^k\,dx}{k\|u_x\|^2}\right).$$

Thus, we have

$$I(\lambda u) = \begin{cases} >0, & 0 \le \lambda < \lambda^*, \\ =0, & \lambda = \lambda^*, \\ <0, & \lambda < \lambda^* < \infty. \end{cases}$$

□

The potential well depth is defined as

$$0 < d = \inf_u\left\{\sup_{\lambda\ge 0} J(\lambda u) : u \in H_0^3(\Omega)/\{0\}\right\} \tag{12}$$

and the well-known Nehari manifold

$$N = \left\{u : u \in H_0^3(\Omega)/\{0\}, I(u) = 0\right\},$$

$$0 < d = \inf_{u\in N} J(u). \tag{13}$$

Then, we define two subset of H_0^3 related to (1). We introduce,

$$W = \left\{u \in H_0^3 : J(u) < d, I(u) > 0\right\} \cup \{0\}, \tag{14}$$

$$V = \left\{u \in H_0^3 : J(u) < d, I(u) < 0\right\}. \tag{15}$$

Lemma 2.4. *Assume that $u \in H_0^3$ and define $r = \left(\frac{2\pi}{k}\right)^{\frac{1}{4}} e^{\frac{1}{2}}$,*
i) If $0 < \|u_x\| < r$, then $I(u) > 0$,
ii) If $I(u) < 0$, then $\|u_x\| > r$,
iii) If $I(u) = 0$, and $\|u_x\| \neq r$, i.e. $u \notin N$, then $\|u_x\| \ge r$.

Proof. If we use the Logarithmic Sobolev inequality to the last term of (7) for any $\alpha > 0$, we have

$$\begin{aligned} I(u) &= \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 - k\int_\Omega u_x^2\left(\log\frac{|u_x|}{\|u_x\|} + \log\|u_x\|\right)dx \\ &\geq \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 - k(\left[\frac{\alpha^2}{2\pi}\|u_{xx}\|^2 + \frac{(1+\log\alpha)}{2}\|u_x\|^2 - \|u_x\|^2\log\|u_x\|\right] \\ &\geq \left(1 - \frac{k\alpha^2}{2\pi}\right)\left(\|u_{xx}\|^2 + \|u_{xxx}\|^2\right) + k\left(\frac{(1+\log\alpha)}{2} - \log\|u_x\|\right)\|u_x\|^2. \end{aligned} \tag{16}$$

If we take $\alpha = \sqrt{\frac{2\pi}{k}}$ in (16), we have

$$I(u) \geq k\left(\frac{2+\log\frac{2\pi}{k}}{4} - \log\|u_x\|\right)\|u_x\|^2. \tag{17}$$

i) If $0 < \|u_x\| < r$, we have

$$1 < \log\|u_x\| < \frac{2+\log\frac{2\pi}{k}}{4},$$

so that from (17) we gain $I(u) > 0$.

ii) If we take $I(u) < 0$ from (16) we can write;

$$\begin{aligned} \left(\frac{2+\log\frac{2\pi}{k}}{4} - \log\|u_x\|\right) &< 0, \\ \frac{1}{2} + \left(\frac{2\pi}{k}\right)^{\frac{1}{4}} &< \log\|u_x\|, \\ \left(\frac{2\pi}{k}\right)^{\frac{1}{4}} e^{\frac{1}{2}} &< \|u_x\|, \\ r &< \|u_x\|. \end{aligned}$$

iii) If $I(u) = 0$ and and $\|u_x\| \neq r$, i.e. $u \notin N$, then $\|u_x\| \geq r$. □

Lemma 2.5. *i)* $d \geq \frac{k}{4}\left(\frac{2\pi}{k}\right)^{\frac{1}{2}} e = \frac{k}{4}r^2$ *and* $2\pi e^2 \geq k$,

ii) If $u \in H_0^3$ *and* $I(u) < 0$, *we can obtain*

$$I(u) < 2(J(u) - d). \tag{18}$$

Proof. i) If $I(u) = 0$ and $\|u_x\| \neq 0$, then by Lemma 2.4, we have $\|u_x\| \geq r = \left(\frac{2\pi}{k}\right)^{\frac{1}{4}} e^{\frac{1}{2}}$. From the definition of $J(u)$, we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}I(u) + \frac{k}{4}\|u_x\|^2 \\ &\geq \frac{1}{2}I(u) + \frac{k}{4}r^2 \\ &= \frac{1}{2}I(u) + \frac{k}{4}\left(\frac{2\pi}{k}\right)^{\frac{1}{4}} e^{\frac{1}{2}}, \end{aligned}$$

because of (12), we have

$$d \geq \frac{k}{4}\left(\frac{2\pi}{k}\right)^{\frac{1}{2}} e = \frac{k}{4}r^2.$$

and

$$\begin{aligned} r &\geq 1, \\ \left(\frac{2\pi}{k}\right)^{\frac{1}{4}} e^{\frac{1}{2}} &\geq 1, \\ \frac{2\pi}{k} e^{2} &\geq 1, \\ 2\pi e^{2} &\geq k. \end{aligned}$$

ii) If $u \in H_0^3$ and $I(u) < 0$, then from Lemma 2.2 it follows that there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. From the (12), we have

$$\begin{aligned} d &\leq J(\lambda^* u) = \frac{1}{2} I(\lambda^* u) + \frac{k}{4} \|\lambda^* u_x\|^2, \\ &= \frac{k}{4} (\lambda^*)^2 \|u_x\|^2, \\ &< \frac{k}{4} \|u_x\|^2, \end{aligned} \tag{19}$$

By the (6) and (7), we get

$$d < \frac{k}{4} \|u_x\|^2 = J(u) - \frac{1}{2} I(u),$$

so that

$$I(u) < 2(J(u) - d).$$

□

Lemma 2.6. *Let $u_0 \in H_0^3, u_1 \in H^1$ such that*

$$0 < E(0) < d,$$

then
i) $u \in W$ if $I(u_0) > 0$,
ii) $u \in V$ if $I(u_0) < 0$.

Proof. By the definition of weak solution and (4), we get

$$\frac{1}{2}\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\right) + J(u) \leq \frac{1}{2}\left(\|u_1\|^2 + \|u_{1x}\|^2 + \|u_{1xx}\|^2\right) + J(u_0) < d, \forall t \in [0, T). \tag{20}$$

i) Then, we claim that $u(t) \in W$ for all $t \in [0, T)$. If it is not true, then there exists a $t_0 \in [0, T)$ such that $u(t_0) \in \partial W$, so we have

(a) either $I(u(t_0)) = 0$ and $\|u_x(t_0)\| \neq 0$, or (b) $J(u(t_0)) = d$.

By (20), (b) is impossible, thus we have $I(u(t_0)) > 0$ and $\|u_x(t_0)\| \neq 0$. So, at least one $J(u(t_0)) \geq d$ exists if $0 < d = \inf_{u \in N} J(u)$. Because of this contradiction, $u(t) \in W$ is found for $\forall t \in [0, T)$.

ii) If there is a $t_0 \in [0, T)$ such that $u(x, t) \in W$ for $0 \leq t < t_0$, and $u(x, t_0) \in \partial W$, thus

(a) either $I(u(t_0)) = 0$, or (b) $J(u(t_0)) = d$.

By (20), (b) is false. If $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $0 < t < t_0$, then $\|u_x\| > r$ for $0 < t \leq t_0$ by Lemma 2.4 (ii). However, at least one $J(u(t_0)) \geq d$ exists if $0 < d = \inf_{u \in N} J(u)$. Because of this contradiction, $u(t) \in V$ is found for $\forall t \in [0, T)$. □

3. Existence of global solution

In this section, we investigated the global existence result for problem (1). The proof is based Faedo-Galerkin method. Also, we used Logarithmic Sobolev inequality and Logarithmic Gronwall inequality.

Theorem 3.1. *Let $u_0 \in H_0^3(\Omega)$, $u_1 \in H^1(\Omega)$, then the problem (1) has a global weak solution $u \in L^\infty\left(0,\infty,H_0^3\right)$ with $u \in L^\infty\left(0,\infty,H^1\right)$.*

Proof. We will use the Faedo-Galerkin method to construct approximate solutions. Let $\left\{w_j\right\}_{j=1}^{\infty}$ be an orthogonal basis of the "separable" space $H_0^3(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let

$$V_m = \operatorname{span}\{w_1, w_2, ..., w_m\}$$

and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$\begin{aligned} u_m(0) &= u_{m0}(x) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0 \text{ in } H_0^3(\Omega), \\ u_{mt}(0) &= u_{m1}(x) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1 \text{ in } H^1(\Omega), \end{aligned}$$

for $j = 1, 2, ..., m$.

We look for the approximate solutions

$$u_m(x,t) = \sum_{j=1}^{m} h_{jm}(t) w_j(x)$$

of the approximate problem in V_m

$$\begin{cases} (u_{mtt}, w_s) + (u_{mx}, w_{sx}) + (u_{mxtt} w_{sx}) + (u_{mxx}, w_{sxx}) + (u_{mxxtt}, w_{sxx}) \\ \qquad + (u_{mxxx}, w_{sxxx}) - \left(u_{mx} \log|u_{mx}|^k, w_{sx}\right) = 0, \\ \qquad\qquad w \in V_m, s = 1, 2, ...m, \\ u_0^m(x) = \sum_{j=1}^{m} a_j w_j(x) \to u_0 \text{ in } H_0^3(\Omega), \\ u_1^m(x) = \sum_{j=1}^{m} b_j w_j(x) \to u_1 \text{ in } H^1(\Omega). \end{cases} \tag{21}$$

This leads to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j : [0, t_m) \to R, \quad j = 1, 2, ..., m,$$

which satisfy (21) in a maximal interval $[0, t_m)$, $0 < t_m \le T$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_{mt} in (21) and integrate by parts we obtain

$$\frac{d}{dt} E_m(t) = 0 \tag{22}$$

where

$$\begin{aligned} E_m(t) &= \frac{1}{2}\left(\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 + \|u_{mx}\|^2 + \|u_{mxx}\|^2 + \|u_{mxxx}\|^2\right) \\ &\quad -\frac{1}{2}\int_0^l u_{mx}^2 \log|u_{mx}|^k \, dx + \frac{k}{4}\|u_{mx}\|^2. \end{aligned} \tag{23}$$

Integrating (22) with respect to t from 0 to t, we obtain

$$E_m(t) = E_m(0). \tag{24}$$

Since $u_m(0) \to u_0$ in the space H_0^3, we obtain that the sequence $u_m(0)$ is bounded in H_0^3. By the following inequality

$$\left|t^2 \log t\right| \leq C\left(1 + t^3\right), t > 0,$$

and the Sobolev embedding $H_0^3 \hookrightarrow L^\infty$, we have

$$\int_\Omega u_{0mx}^2 \log |u_{0mx}|^k \, dx \leq C\left(1 + \|u_{0m}\|^3\right) \leq C\left(1 + \|u_{0m}\|_{H_0^3}^3\right) \leq C_1$$

where $C_1 = C\left(\|u_0\|_{H_0^3}, \|u_1\|\right)$ is a positive constant. Subsequently, from (24) we can write

$$\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 + \|u_{mx}\|^2 + \|u_{mxx}\|^2 + \|u_{mxxx}\|^2 \leq C + 2\int_0^l u_{mx}^2 \log |u_{mx}|^k \, dx. \tag{25}$$

The last term on the righthand side of (25) inequality and the Logarithmic Sobolev Inequality leads to

$$\begin{aligned} 2\int_0^l u_{mx}^2 \log |u_{mx}|^k \, dx &= 2k\int_0^l u_{mx}^2 \left(\log \frac{|u_x|}{\|u_{mx}\|} + \log \|u_{mx}\|\right) dx \\ &\leq k\left(\frac{\alpha^2}{\pi}\int_0^l u_{mxx}^2 dx - (1 + \log \alpha)\|u_{mx}\|^2 + \log \|u_{mx}\| \|u_{mx}\|^2\right). \end{aligned} \tag{26}$$

By combining of (26) and (25), we obtain

$$\begin{aligned} &\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 + \|u_{mx}\|^2 + \left(1 - \frac{k\alpha^2}{\pi}\right)\|u_{mxx}\|^2 + \|u_{mxxx}\|^2 \\ \leq\ & C + k\left(\frac{\alpha^2}{\pi}\int_0^l u_{mxx}^2 dx - (1 + \log \alpha)\|u_{mx}\|^2 + \log \|u_{mx}\| \|u_{mx}\|^2\right), \end{aligned} \tag{27}$$

From (27), we have

$$\begin{aligned} &\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 + (1 + k + k\log \alpha)\|u_{mx}\|^2 \\ &+\left(1 - \frac{k\alpha^2}{\pi}\right)\|u_{mxx}\|^2 + \|u_{mxxx}\|^2 \\ \leq\ & C + \left(1 + \log \|u_{mx}\| \|u_{mx}\|^2\right). \end{aligned} \tag{28}$$

By taking

$$\alpha = \sqrt{\frac{\pi}{2k}}$$

in (28), we obtain

$$\begin{aligned} &\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 + \|u_{mx}\|^2 + \|u_{mxx}\|^2 + \|u_{mxxx}\|^2 \\ \leq\ & C\left(1 + \log \|u_{mx}\| \|u_{mx}\|^2\right). \end{aligned} \tag{29}$$

Noting that

$$u_{mx}(t) = u_{mx}(0) + \int_0^t u_{mxt}(s)\, ds.$$

We make use of the following Cauchy-Schwarz inequality

$$(a+b)^2 \le 2\left(a^2+b^2\right),$$

we obtain

$$\begin{aligned}
\|u_{mx}(t)\|^2 &= \left\| u_{mx}(0) + \int_0^t u_{mxt}^2(s)\, ds \right\|^2 \\
&\le 2\|u_{mx}(0)\|^2 + 2T\int_0^t \|u_{mxt}\|^2(s)\, ds \\
&\le 2\|u_{mx}(0)\|^2 + \max\{1, 2T\}\frac{1+C}{C}\int_0^t \|u_{mxt}\|^2(s)\, ds.
\end{aligned} \tag{30}$$

So if we write inequality (30) instead of inequality (29), and If we put

$$\begin{aligned}
X &= 2\|u_{mx}(0)\|^2 + \max\{1, 2T\}(1+C)T, \\
Y &= \max\{1, 2T\}(1+C),
\end{aligned}$$

(30) leads to

$$\|u_{mx}\|^2 \le X + Y\int_0^t \log\|u_{mx}\|\,\|u_{mx}\|^2\, ds.$$

Taking $B \ge 1$, then by the Logarithmic Gronwall inequality, we get

$$\|u_{mx}\|^2 \le X + Ye^{Yt} \le C_t. \tag{31}$$

Hence, from inequality (31) and (29), it follows that

$$\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 + \|u_{mx}\|^2 + \|u_{mxx}\|^2 + \|u_{mxxx}\|^2 \le C. \tag{32}$$

From the guess of (32) we $T_{\max} = T$ and we have that u_{mtt} is uniformly bounded in $L^\infty\left(0, T; H^{-2}\right)$ by the standart way. It follows that there exists a subsequence of $\{u_m\}$, denote by $\{u_m\}$, such that

$$\begin{cases}
u_m \to u, \text{ weakly* in } L^\infty\left(0, T; H_0^3\right), \\
u_{mt} \to u_t, \text{ weakly* in } L^\infty\left(0, T; H^2\right), \\
u_{mtt} \to u_{tt}, \text{ weakly in } L^2\left(0, T; H^{-2}\right).
\end{cases}$$

Then by using Aubin–Lions' lemma and Lebesgue dominated convergence theorem as [6], we have

$$u_{mx}\log|u_{mx}|^k \to u_x \log|u_x|^k \text{ in } L^\infty\left(0, T; L^2\right) \text{ weakly star.}$$

We integrate (21) over $(0,t)$ and letting $m \to \infty$ for the fixed s, we get

$$
\begin{aligned}
& (u_t, w_s) + (u_{xt}, w_{sx}) + (u_{xxt}, w_{sxx}) + \int_0^t \left[(u_x w_{sx}) + (u_{xx}, w_{sxx}) + (u_{xxx}, w_{sxxx}) \right] ds \\
& - \int_0^t \left(u_x \log |u_x|^k, w_{sx} \right) ds \\
= \; & (u_1, w_s) + (u_{1x}, w_{sx}) + (u_{1xx}, w_{sxx})
\end{aligned}
$$

and (1) in definition of the solution. On the other hand, (21) give $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$. From (24), we know that the above u satisfies (4). So that $u(x,t)$ is a global weak solution of problem (1). This comleted the proof of the theorem. □

Theorem 3.2. *Let $u_0 \in H_0^3$, $u_1 \in H^1$, assume that $E(0) < d$, and $I(u_0) > 0$ or $\|u_0\| = 0$, then we accept the problem (1) has a global weak solution $u \in L^\infty\left(0, \infty, H_0^3\right)$ with $u \in L^\infty\left(0, \infty, H^1\right)$ and $u(t) \in W$ for $0 \leq t < \infty$.*

Proof. By the Theorem we know that the problem (1) admits a global weak solution. From Lemma 2.6 we get $u(t) \in W$ for $0 \leq t < T$. □

4. Blow up property in infinity time

In this section, we proved blow up property in infinity time of problem (1). In fact, we try to show H_0^3 norm of the solution will grow up as an exponential function when time goes to infinity for suitable initial data conditions.

Theorem 4.1. *Let $u_0 \in H_0^3$, $u_1 \in H^1$, assume that $u_0 \in V$, $0 < E(0) < d$ and*

$$(u_0, u_1) + (u_{0x}, u_{1x}) + (u_{0xx}, u_{1xx}) > 0,$$

then the solution of (1) goes to the ∞.

Proof. Let $u(t,x)$ is a weak solution of problem (1) with $J(u_0) < E(0) < d$, $I(u_0) < 0$. Then we take the function $\phi(t) : [0, \infty) \to R^+$ defined by

$$\phi(t) = \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 \tag{33}$$

Now, we take derivative of (33), we have

$$\phi'(t) = 2 \int_\Omega u\, u_t + 2 \int_\Omega u_x\, u_{xt} + 2 \int_\Omega u_{xx}\, u_{xxt}, \tag{34}$$

then from the derivative of (34), because of (1), and (7) and using of partial integral formulas; we have

$$\begin{aligned}
\phi''(t) &= 2\int_\Omega u\, u_{tt} + 2\int_\Omega u_x\, u_{xtt} + 2\int_\Omega u_{xx}\, u_{xxtt} + 2\int_\Omega u_t^2 + 2\int_\Omega u_{xt}^2 + 2\int_\Omega u_{xxt}^2, \\
&= +2\int_\Omega u\, \left(u_{xx} + u_{xxtt} - u_{xxxxtt} - u_{xxxx} - u_{xxxxxx} - \left(u_x \log|u_x|^k\right)_x\right) \\
&\quad +2\int_\Omega u_x\, u_{xtt} + 2\int_\Omega u_{xx}\, u_{xxtt} + 2\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\right), \\
&= 2\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\right) - 2\|u_x\|^2 - 2\int_\Omega u_x\, u_{xtt} - 2\int_\Omega u_{xx}\, u_{xxtt} \\
&\quad -2\|u_{xx}\|^2 - 2\|u_{xxx}\|^2 + 2\int_\Omega u_x^2 \log|u_x|^k + 2\int_\Omega u_x\, u_{xtt} + 2\int_\Omega u_{xx}\, u_{xxtt}, \\
&= 2\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\right) - 2\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 - \int_\Omega u_x^2 \log|u_x|^k\right), \\
&= 2\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\right) - 2I(u).
\end{aligned} \tag{35}$$

We make use of the following Cauchy-Schwarz inequality

$$(ax + by + cz)^2 \le 2\left(a^2 + b^2 + c^2\right)\left(x^2 + y^2 + z^2\right),$$

it follows from (34) that for $t \in [0, \infty)$

$$\begin{aligned}
\left|\phi'(t)\right|^2 &\le 4\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\right) 2\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right), \\
&= 4\,\phi(t)\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right)
\end{aligned} \tag{36}$$

Then we have for each $t \in [0, \infty)$ that, by (33), (35), (36) and using of (4), (9)

$$\begin{aligned}
\phi''(t)\,\phi(t) - \left[\phi'(t)\right]^2 &\ge 2\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2 - I(u)\right)\phi(t) \\
&\quad -4\,\phi(t)\left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\right) \\
&= -2\phi(t)\left(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2 + I(u)\right) \\
&\ge -2\phi(t)\left[E(0) - J(u) + I(u)\right].
\end{aligned} \tag{37}$$

Because of $u_0 \in V,\ 0 < E(0) < d$, it follows from Lemma 2.6 that $I(u) < 0$. Thus, using Lemma 2.4, we obtain that

$$\begin{aligned}
E(0) - J(u) + I(u) &< d - J(u) + 2(J(u) - d), \\
&= J(u) - d, \\
&< 0.
\end{aligned}$$

ıf we use that for (37), we have

$$\phi''(t)\,\phi(t) - \left[\phi'(t)\right]^2 > 0. \tag{38}$$

In other words, by using rule of derivate, we can write

$$\left(\log\left|\phi(t)\right|\right)' = \frac{\phi'(t)}{\phi(t)}, \tag{39}$$

By (39) and (38) we have

$$\left(\log\left|\phi(t)\right|\right)'' = \frac{\phi''(t)\,\phi(t) - \left[\phi'(t)\right]^2}{\phi^2(t)} > 0 \tag{40}$$

From (40), we can say that

$$\left(\log\left|\phi(t)\right|\right)' = \frac{\phi'(t)}{\phi(t)}$$

is increasing with respect to t, using this fact, integrating (39) from t_0 to t, we obtain

$$\begin{aligned}
\log\left|\phi(t)\right| - \log\left|\phi(t_0)\right| &= \int_{t_0}^{t} \left(\log\left|\phi(\tau)\right|\right)' d\tau \\
&= \int_{t_0}^{t} \frac{\phi'(\tau)}{\phi(\tau)} d\tau \\
&\geq \frac{\phi'(t_0)}{\phi(t_0)}(t - t_0),
\end{aligned}$$

where $0 \leq t_0 < t$. Then

$$\begin{aligned}
\log\frac{\left|\phi(t)\right|}{\left|\phi(t_0)\right|} &\geq \frac{\phi'(t_0)}{\phi(t_0)}(t - t_0) \\
\left|\phi(t)\right| &\geq \left|\phi(t_0)\right| e^{\left(\frac{\phi'(t_0)}{\phi(t_0)}(t-t_0)\right)}.
\end{aligned} \tag{41}$$

If we choose t_0 sufficiently small such that $\phi'(t_0) > 0,\ \phi(t_0) > 0$, then, from (41), we have

$$\lim_{t\to\infty} \phi(t) = \infty$$

so that we can say the solution of (1) have a exponential grow up. □

5. Energy decay for the problem (1) with linear damping terms

In this section, we will prove decay estimates of solutions to problem (1). That is to say, we will add linear damping terms to equation (1). So that we can write problem as

$$\begin{cases}
u_{tt} - u_{xx} - u_{xxt} + u_{xxxxt} - u_{xxtt} + u_{xxxxtt} + u_{xxxx} + u_{xxxxxx} + \left(u_x \log|u_x|^k\right)_x = 0, & x \in \Omega,\ t > 0 \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\
u(x,t) = u(l,t) = 0, u_{xx}(x,t) = u_{xx}(l,t) = 0, u_{xxx}(x,t) = u_{xxx}(l,t) = 0, & t \geq 0.
\end{cases} \tag{42}$$

Then all the results in Section 2 and Section 3 hold and we have

$$E'(t) = -\|u_{xt}\|^2 - \|u_{xxt}\|^2 \tag{43}$$

hence we can say that $E(t)$ is nonincreasing.

Theorem 5.1. *Let $u_0 \in W, u_1 \in H^1$. Assume further that $0 < E(0) < \frac{\alpha k}{4} r^2 < d$, where α is a positive constant satisfying $0 < k^4\alpha^4 \frac{2\pi}{k} e^2 < 1$, then there exists two positive constants M and η independent of t such that:*

$$0 < E(t) < Me^{-\eta t}, \quad t \geq 0.$$

Proof. Let $u(t,x)$ be a weak solution of problem (42). Since $u_0 \in W, u_1 \in H^1$, by Theorem 3.2, we have $u \in W$ for $\forall t \in [0,\infty)$ and then $0 < E(t) < d$ and $I(u) > 0$.

For this purpose, we use the Lyapunov functional

$$\Phi(t) = E(t) + \varepsilon \left[\int_\Omega u u_t dx + \int_\Omega u_x u_{xt} dx + \int_\Omega u_{xx} u_{xxt} dx \right] \tag{44}$$

where ϵ is a positive constant. We will show the $\Phi(t)$ and $E(t)$ are equivalent. For $\epsilon > 0$ small enough, the relation

$$\gamma_1 \Phi(t) \le E(t) \le \gamma_2 \Phi(t) \tag{45}$$

holds for two positive constants γ_1 and γ_2. We can choose ε small enough such that $\Phi \sim E$.

By taking the time derivative of the function $\Phi(t)$, using (42) and performing several integration by parts, we get

$$\begin{aligned}
\Phi'(t) &= E'(t) + \varepsilon \left[\int_\Omega u u_{tt} dx + \int_\Omega u_t^2 dx + \int_\Omega u_x u_{xtt} dx + \int_\Omega u_{xt}^2 dx + \int_\Omega u_{xx} u_{xxtt} dx + \int_\Omega u_{xxt}^2 dx \right] \\
&= -\|u_{xt}\|^2 - \|u_{xxt}\|^2 + \varepsilon \Bigg[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2 + \int_\Omega u u_{xx} dx + \int_\Omega u u_{xxt} dx \\
&\quad - \int_\Omega u u_{xxxxt} dx + \int_\Omega u u_{xxtt} dx - \int_\Omega u u_{xxxx} dx - \int_\Omega u u_{xxxxxx} dx \\
&\quad - \int_\Omega u u_{xxxxtt} dx - \int_\Omega u \left(u_x \log |u_x|^k \right)_x dx + \int_\Omega u_x u_{xtt} dx + \int_\Omega u_{xx} u_{xxtt} dx \Bigg] \\
&= -\|u_{xt}\|^2 - \|u_{xxt}\|^2 + \varepsilon \Big[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2 - \|u_x\|^2 \\
&\quad - \int_\Omega u_x u_{xt} dx - \int_\Omega u_{xx} u_{xxt} dx - \int_\Omega u_x u_{xtt} dx - \|u_{xx}\|^2 - \|u_{xxx}\|^2 \\
&\quad - \int_\Omega u_{xx} u_{xxtt} dx + \int_\Omega u_x^2 \log |u_x|^k dx + \int_\Omega u_x u_{xtt} dx + \int_\Omega u_{xx} u_{xxtt} dx \Big] \\
&= (\varepsilon - 1) \left(\|u_{xt}\|^2 + \|u_{xxt}\|^2 \right) + \varepsilon \|u_t\|^2 - \varepsilon \left(\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 \right) \\
&\quad - \varepsilon \left(\int_\Omega u_x u_{xt} dx + \int_\Omega u_{xx} u_{xxt} dx \right) + \varepsilon \int_\Omega u_x^2 \log |u_x|^k dx
\end{aligned} \tag{46}$$

If we use Young Inequality for fifth term in the right hand side of (46) for $\delta > 0$; we can write

$$\left| \int_\Omega u_x u_{xt} dx + \int_\Omega u_{xx} u_{xxt} dx \right| \le \frac{1}{4\delta} \left(\|u_{xt}\|^2 + \|u_{xxt}\|^2 \right) + \delta \left(\|u_x\|^2 + \|u_{xx}\|^2 \right), \tag{47}$$

Then we insert (47) into (46), we obtain

$$\begin{aligned}
\Phi'(t) &\le \left(\varepsilon + \frac{\varepsilon}{4\delta} - 1 \right) \left(\|u_{xt}\|^2 + \|u_{xxt}\|^2 \right) + \varepsilon \|u_t\|^2 - \varepsilon \|u_{xxx}\|^2 \\
&\quad + \varepsilon (\delta - 1) \left(\|u_x\|^2 + \|u_{xx}\|^2 \right) + \varepsilon \int_\Omega u_x^2 \log |u_x|^k dx.
\end{aligned} \tag{48}$$

Adding and subtracting $\varepsilon\theta E(t)$ into (48) where $0<\theta<1$ is a positive constant, we obtain

$$\begin{aligned}\Phi'(t) &\le -\varepsilon\theta E(t)+\left(\frac{\varepsilon\theta}{2}+\varepsilon-\frac{\varepsilon}{4\delta}-1\right)\left(\|u_{xt}\|^2+\|u_{xxt}\|^2\right)\\ &\quad+\varepsilon\left(1+\frac{\theta}{2}\right)\|u_t\|^2+\varepsilon\left(\frac{\theta}{2}-1\right)\|u_{xxx}\|^2\\ &\quad+\varepsilon\left(\frac{\theta}{2}+\delta-1\right)\left(\|u_x\|^2+\|u_{xx}\|^2\right)\\ &\quad+\varepsilon\frac{\theta k}{4}\|u_x\|^2+\varepsilon\left(1-\frac{\theta}{2}\right)\int_\Omega u_x^2\log|u_x|^k\,dx. \qquad (49)\end{aligned}$$

If we take $0<\theta<2$ and using of (5) to the last term of the (49) and because of Sobolev embedding theorems; we can edit (49)

$$\begin{aligned}\Phi'(t) &\le -\varepsilon\theta E(t)+\left(\frac{\varepsilon\theta}{2}+\varepsilon-\frac{\varepsilon}{4\delta}-1\right)\left(\|u_{xt}\|^2+\|u_{xxt}\|^2\right)\\ &\quad+\varepsilon\left(1+\frac{\theta}{2}\right)\|u_t\|^2+\varepsilon\left(\frac{\theta}{2}-1\right)\|u_{xxx}\|^2\\ &\quad+\varepsilon\left(\frac{\theta}{2}+\delta-1\right)\left(\|u_x\|^2+\|u_{xx}\|^2\right)\\ &\quad+\varepsilon\frac{\theta k}{4}\|u_x\|^2+\varepsilon\left(1-\frac{\theta}{2}\right)k\left[\frac{\alpha^2}{2\pi}\left\|u_{xx}^2\right\|-\frac{(1+\log\alpha)}{2}\|u_x\|^2+\log\|u_x\|^2\|u_x\|^2\right],\\ &\le -\varepsilon\theta E(t)+\left[\left(\frac{\varepsilon\theta}{2}+\varepsilon-\frac{\varepsilon}{4\delta}-1\right)(1+C_0)+C_1\left(1+\frac{\theta}{2}\right)\right]\left(\|u_{xt}\|^2+\|u_{xxt}\|^2\right)\\ &\quad+\varepsilon\left(\frac{\theta}{2}-1\right)\|u_{xxx}\|^2+\varepsilon\left[\left(\frac{\theta}{2}+\delta-1\right)+\left(1-\frac{\theta}{2}\right)\frac{k\alpha^2}{2\pi}\right]\left(\|u_{xx}\|^2\right)\\ &\quad+\varepsilon\left[\left(\frac{\theta}{2}+\delta-1\right)-\left(1-\frac{\theta}{2}\right)k\frac{(1+\log\alpha)}{2}+\left(1-\frac{\theta}{2}\right)k\log\|u_x\|^2+\frac{\theta k}{4}\right]\|u_x\|^2. \qquad (50)\end{aligned}$$

Because of $0<\theta<2$ we note $1-\frac{\theta}{2}>0$ also we know from (8)

$$\begin{aligned}J(u) &> \frac{k}{4}\|u_x\|^2,\\ \log\frac{4J(u)}{k} &> \log\|u_x\|^2. \qquad (51)\end{aligned}$$

If we write (51) in (50) we have

$$\begin{aligned}\Phi'(t) &\le -\varepsilon\theta E(t)+\left[\left(\frac{\varepsilon\theta}{2}+\varepsilon-\frac{\varepsilon}{4\delta}-1\right)(1+C_0)+C_1\left(1+\frac{\theta}{2}\right)\right]\left(\|u_{xt}\|^2+\|u_{xxt}\|^2\right)\\ &\quad+\varepsilon\left(\frac{\theta}{2}-1\right)\|u_{xxx}\|^2+\varepsilon\left[\left(\frac{\theta}{2}+\delta-1\right)+\left(1-\frac{\theta}{2}\right)\frac{k\alpha^2}{2\pi}\right]\left(\|u_{xx}\|^2\right)\\ &\quad+\varepsilon\left[\left(\frac{\theta}{2}+\delta-1\right)-\left(1-\frac{\theta}{2}\right)k\frac{(1+\log\alpha)}{2}+\left(1-\frac{\theta}{2}\right)k\log\frac{4J(u)}{k}+\frac{\theta k}{4}\right]\|u_x\|^2 \qquad (52)\end{aligned}$$

Because of our accept in Theorem 5.1, we know $0<\theta<2$ and $J(u)<E(0)<\frac{\alpha k}{4}r^2<d$, taking of α according $k^4\alpha^4\frac{2\pi}{k}e^2<\alpha^2<\frac{2\pi}{k}$, where α is suitable by the assumption in Theorem 5.1 and choose $\delta>0$ small enough, such that

$$\frac{\theta}{2}-1<0,$$

$$
\begin{aligned}
\left(\frac{\theta}{2}+\delta-1\right)+\left(1-\frac{\theta}{2}\right)\frac{k\alpha^2}{2\pi} &< 0, \\
\left(\frac{\theta}{2}-1\right)\left(1-\frac{k\alpha^2}{2\pi}\right) &< 0, \\
\left(1-\frac{k\alpha^2}{2\pi}\right) &> 0,
\end{aligned}
$$

$$
\left(\frac{\theta}{2}+\delta-1\right)-\left(1-\frac{\theta}{2}\right)k\left(\frac{(1+\log\alpha)}{2}-\log\frac{4J(u)}{k}\right)+\frac{\theta k}{4}<0,
$$

so that we have,

$$
\Phi'(t) \le -\varepsilon\theta E(t) + \left[\left(\frac{\varepsilon\theta}{2}+\varepsilon-\frac{\varepsilon}{4\delta}-1\right)(1+C_0)+C_1\left(1+\frac{\theta}{2}\right)\right]\left(\|u_{xt}\|^2+\|u_{xxt}\|^2\right). \tag{53}
$$

If we take $\varepsilon > 0$ small enough such that

$$
\left(\frac{\varepsilon\theta}{2}+\varepsilon-\frac{\varepsilon}{4\delta}-1\right)(1+C_0)+C_1\left(1+\frac{\theta}{2}\right)<0,
$$

$$
\begin{aligned}
1+C_0 &> 0 \\
C_0 &> -1,
\end{aligned}
$$

Consequently, inequlatiy (53) becomes

$$
\Phi'(t) \le -\varepsilon\theta E(t).
$$

By (45), we write

$$
\Phi'(t) \le -\varepsilon\theta\gamma_2\Phi(t) \tag{54}
$$

setting $\eta = \varepsilon\theta\gamma_2 > 0$ and integrating (54) between $(0,t)$ gives the following estimate

$$
\Phi(t) \le Me^{-\eta t}
$$

Consequently, by using (45) once again. This completes the proof. □

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